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The classifying space of the monoid of self-homotopy equivalences of a space and the Kedra-McDuff μ-classes

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ABSTRACT. In the forthcoming paper [10], we shall discuss a rational visibility problem of a Lie group in the monoid of self-homotopy equivalences of a homogeneous space. Moreover the Kedra-McDuff μ-classes of the classifying space of such the monoid of a c-symplectic manifold are considered. This article surveys main results in [10]. The tool for the study is an elaborate rational model in [9] and [6] for the evaluation map of a function space, which is described in terms of the function space model due to Brown and Szczarba [2] and due to Haefliger [5]. We recall these algebraic models in the appendix.

1. RATIONAL VISIBILITY OF A LIE GROUP IN A MONOID

Let $f : X \to Y$ be a map between connected spaces X and Y whose fundamental groups are abelian. We say that X is *rationally visible* in Y with respect to the map f if the induced map $f_* \otimes 1 : \pi_i(X) \otimes \mathbb{Q} \to \pi_i(Y) \otimes \mathbb{Q}$ is injective for any $i \geq 1$. Let $\mathrm{aut}_1(X)$ denote the identity component of the monoid of self-homotopy equivalences of a space X. Let G be a connected Lie group and M a homogeneous space of the form G/U for which U is a connected subgroup of G. Then the left translation of G on M gives rise to a map of the monoids

$$\lambda_{G,M} : G \to \mathrm{aut}_1(M)$$

defined by $\lambda_{G,M}(g)(x) = gx$ for $g \in G$ and $x \in M$. The purpose of this section is to survey the results in [10] concerning rational visibility of a Lie group G in the monoid $\mathrm{aut}_1(M)$ with respect to $\lambda_{G,M}$.

We observe that the map $\lambda_{G,M} : G \to \mathrm{aut}_1(M)$ factors not only through the identity component $\mathrm{Homeo}_1(M)$ of the monoid of homeomorphisms of M but also through the identity component $\mathrm{Diff}_1(M)$ of the space of diffeomorphisms of M. Therefore the rational visibility of G in $\mathrm{aut}_1(M)$ implies that of G in the groups $\mathrm{Homeo}_1(M)$ and $\mathrm{Diff}_1(M)$. This fact is very interesting because very little is known about the rational homotopy of such groups. Then one might expect a criterion for a given Lie group G to be rationally visible in $\mathrm{aut}_1(M)$.

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Theorem 1.1. [10] *Let G be a connected Lie group and U a connected subgroup of G. Let $B\iota : BU \to BG$ be the map between the classifying spaces induced by the inclusion $\iota : U \to G$. Suppose that $H^*(BG;\mathbb{Q}) \cong \mathbb{Q}[c_1, ..., c_k]$ and that $(B\iota)^*(c_{i_1})$, ..., $(B\iota)^*(c_{i_s})$ are decomposable for some elements $c_{i_1}, ..., c_{i_s} \in \{c_1, ..., c_k\}$, where $(B\iota)^* : H^*(BG;\mathbb{Q}) \to H^*(BU;\mathbb{Q})$ denotes the map induced by $B\iota$. Then there exists a map $\rho : \times_{j=1}^{s} S^{\deg c_{i_j}-1} \to G$ such that $\times_{j=1}^{s} S^{\deg c_{i_j}-1}$ is rationally visible in* $\mathrm{aut}_1(G/U)$ *with respect to the map $(\lambda_{G,G/U}) \circ \rho$.*

Corollary 1.2. *Under the same assumption and notations as in Theorem 1.1, there exist elements with infinite order in $\pi_l(\mathrm{Diff}_1(G/U))$ and $\pi_l(\mathrm{Homeo}_1(G/U))$ for $l =$* $\deg c_{i_1} - 1, ..., \deg c_{i_s} - 1$.

Let $ev : \mathrm{aut}_1(G/U) \times G/U \to G/U$ be the evaluation map defined by $ev(\varphi, x) = \varphi(x)$. The key device for the study of rational visibility is a rational model for the evaluation map ev which is constructed in [9]; see Proposition 3.1. In fact, this model allows us to construct explicitly a rational model for the map $\lambda_{G,G/U}$; see Theorem 3.3 and [10, Theorem 4.2]. By analyzing such models in detail, we have

Theorem 1.3. [10] *Let M be the flag manifold $U(m)/U(m_1) \times \cdots \times U(m_l)$. Then $SU(m)$ is rationally visible in* $\mathrm{aut}_1(M)$ *with respect to the map $\lambda_{SU(m),M}$ given by the left translations. In particular, the localized map*

$$(\lambda_{SU(m),U(m)/U(m-1)\times U(1)})_{\mathbb{Q}} : SU(m)_{\mathbb{Q}} \to \mathrm{aut}_1(\mathbb{C}P^{m-1})_{\mathbb{Q}}$$

is a homotopy equivalence. Here $X_{\mathbb{Q}}$ denotes the localization of a nilpotent space X; see Appendix.

The result is not new because the first assertion follows from [8, Proposition 4.8] due to Kedra and McDuff. The latter half is a particular case of the main theorem in [18]. We here emphasize that not only does our machinery developed in [6] and [10] work well to prove Theorem 1.3 but also it leads us to an unifying way of looking at the visibility problem. In fact, the same argument as in the proof of Theorem 1.3 enables us to deduce the following result.

Theorem 1.4. [10] *Let M be the flag manifold $Sp(m)/Sp(m_1) \times \cdots \times Sp(m_l)$. Then the 3-connected cover $Sp(m)\langle 3\rangle$ is rationally visible in* $\mathrm{aut}_1(M)$ *with respect to $\lambda_{Sp(m),M} \circ \pi$, where $\pi : Sp(m)\langle 3\rangle \to Sp(m)$ is the projection.*

We mention that Notbohm and Smith [15] proved the rational visibility of a Lie group G in $\mathrm{aut}_1(G/T)$, in which T is a maximal torus of G, in order to show the uniqueness of a certain fake Lie group [14]. Moreover Kedra and McDuff [8] proved the rational visibility of $SU(m+1)$ in $\mathrm{Symp}(\mathbb{C}P^m, \omega)$ by showing the non-triviality of μ-classes of the classifying space $B\mathrm{Symp}(\mathbb{C}P^m, \omega)$ in $H^*(SU(m+1))$. Here $(\mathbb{C}P^m, \omega)$ denotes the manifold $\mathbb{C}P^m$ with a symplectic form ω and $\mathrm{Symp}(\mathbb{C}P^m, \omega)$ is the topological group of the symplectomorphisms; namely diffeomorphisms which fix the form ω. These results also motivates us to investigate visibility problems of Lie groups.

Remark 1.5. Suppose that M is a homogeneous space of the form G/H with $\mathrm{rank}\, G = \mathrm{rank}\, H$. The main theorem in [19] due to Shiga and Tezuka implies that $\pi_{2i}(\mathrm{aut}_1(M))\otimes$

$\mathbb{Q} = 0$ for any i. Thus the cohomology $H^*(B\mathrm{aut}_1(M);\mathbb{Q})$ is a polynomial algebra generated by the graded vector space $(sV)^\sharp$, where $(sV)_l = \pi_{l-1}(\mathrm{aut}_1(M))\otimes\mathbb{Q}$. This fact yields that the dual map to the Hurewicz homomorphism

$$\Theta^\sharp : H^*(B\mathrm{aut}_1(M);\mathbb{Q}) \to \mathrm{Hom}(\pi_*(B\mathrm{aut}_1(M)),\mathbb{Q})$$

induces an isomorphism on the vector space of indecomposable elements of the cohomology algebra $H^*(B\mathrm{aut}_1(M);\mathbb{Q})$. Therefore the commutative diagram

$$\begin{tikzcd}
H^*(BG;\mathbb{Q}) \arrow[d, "\Theta^\sharp"'] & H^*(B\mathrm{aut}_1(M);\mathbb{Q}) \arrow[l, "(B\lambda_{G,M})^*"'] \arrow[d, "\Theta^\sharp"] \\
\mathrm{Hom}(\pi_*(BG),\mathbb{Q}) & \mathrm{Hom}(\pi_*(B\mathrm{aut}_1(M)),\mathbb{Q}) \arrow[l, "(B\lambda_{G,M*})^\sharp"]
\end{tikzcd}$$

enables us to conclude that the map $(B\lambda_{G,M})^*$ is surjective if G is rationally visible in $\mathrm{aut}_1(M)$. In this case, we can extend familiar characteristic classes of an appropriate Lie group G, for example, Chern classes and Pontrjagin classes, to cohomology classes of $B\mathrm{aut}_1(M)$.

We conclude this section with an application of Theorem 1.1 to *the group cohomology* of $\mathrm{Diff}_1(M)$. Given a space X, let X^δ denote the space with the discrete topology whose underlying set is the same as that of X. Let $j : G^\delta \to G$ stand for the natural map.

We consider a real semi-simple connected Lie group G with finitely many components and let $h : G \to G_{\mathbb{C}}$ be the complexification of G. One has a commutative diagram

$$\begin{tikzcd}
H^*(BG_{\mathbb{C}}) \arrow[r, "h^*"] & H^*(BG) \arrow[r, "j^*"] & H^*(BG^\delta) \\
H^*(B\mathrm{aut}_1(G/U)) \arrow[r, "B\lambda^*"'] \arrow[ur, "B\lambda^*"] & H^*(B\mathrm{Diff}_1(G/U)) \arrow[r, "j^*"'] \arrow[u] & H^*(B(\mathrm{Diff}_1(G/U))^\delta) \arrow[u]
\end{tikzcd},$$

where U is a connected subgroup of G and $H^*(\ )$ denotes for the rational cohomology. The result [12, THEOREM 2] asserts that the kernel of j^* is equal to the ideal generated by the positive dimensional elements in $\mathrm{Im}\ h^*$. Consider the case where $G = SL(2m;\mathbb{R})$. Then the Euler class χ of $H^*(BSL(2m;\mathbb{R}))$ survives in $H^*(BSL(2m;\mathbb{R})^\delta)$ though the Pontrjagin classes vanish in $H^*(BSL(2m;\mathbb{R})^\delta)$ via j^*; see [13]. Suppose that $m > 1$ and that U is a maximal rank subgroup of $SO(2m)$ with $(QH^*(BU;\mathbb{Q}))^{2m} = 0$. A maximal torus of $SO(2m)$ is an example of such a subgroup. By virtue of Theorem 1.1 and Remark 1.5, we see that the induced map $(B\lambda)^* : H^i(B\mathrm{aut}_1(G/U)) \to H^i(BG)$ is surjective for $i = 2m$. Therefore the Euler class in $H^*(B(G^\delta))$ extends to an element $\widetilde{\chi}$ of $H^*(B(\mathrm{Diff}_1(G/U))^\delta)$. This implies that the rational cohomology algebra $H^*(B(\mathrm{Diff}_1(G/U))^\delta)$ contains the polynomial algebra $\mathbb{Q}[\widetilde{\chi}]$ generated by the extended element $\widetilde{\chi}$. In particular, it follows that

$$H^{2mi}(B(\mathrm{Diff}_1(G/U))^\delta)) \neq 0$$

for $i \geq 0$.

2. Cohomologically symplectic manifolds and Kedra-McDuff μ-classes

We turn our attention to generators of the cohomology of the classifying space $B\mathrm{aut}_1(X)$, for which X is a c-symplectic manifold.

In what follows, we write $H^*(-)$ for the cohomology with coefficients in the rational field. Let M be a $2m$-dimensional c-symplectic manifold; that is, there is a class $\omega \in H^2(M)$ such that $\omega^m \neq 0$. Let $\mathcal{H}_\omega$ denote the group of diffeomorphisms of M that fix ω. In [8, Section 3], Kedra and McDuff defined characteristic classes, which are called μ-classes, of the classifying space of $B\mathcal{H}_\omega$ provided $H^1(M) = 0$. By the same way, we define below characteristic classes μ_k of $B\mathrm{aut}_1(M)$ for $2 \leq k \leq m+1$, which are also referred to μ-classes.

Let (M, ω) be a $2m$-dimensional c-symplectic manifold and $\mathcal{G}$ denote the monoid $\mathcal{H}_\omega$ or $\mathrm{aut}_1(M)$. Let $M \xrightarrow{i} M_\mathcal{G} \xrightarrow{\pi} B\mathcal{G}$ be the universal M-fibration; see [11, Proposition 7.9]. Proposition 2.1 below follows from the proofs of [7, Proposition 2.4.2] and [8, Proposition 3.1].

Proposition 2.1. *Suppose that $H^1(M) = 0$, then the element $\omega \in H^2(M)$ extends to an element $\overline{\omega} \in H^2(M_\mathcal{G})$. Moreover, there exists a unique element $\widetilde{\omega} \in H^2(M_\mathcal{G})$ that restricts to $\omega \in H^2(M)$ and such that $\pi!(\widetilde{\omega}^{m+1}) = 0$. In fact the element $\widetilde{\omega}$ has the form*

$$\widetilde{\omega} = \overline{\omega} - \frac{1}{n+1}\pi^*\pi!(\overline{\omega}^{m+1}).$$

The class $\widetilde{\omega}$ in Proposition 2.1 is called *the coupling class.*

Definition 2.2. [8, Section 3.1] [7, Section 2.4] [17] We define $\mu_k \in H^{2k}(B\mathcal{G})$, which is called kth μ-class, by

$$\mu_k := \pi!(\widetilde{\omega}^{m+k}),$$

where $\widetilde{\omega}$ is the coupling class and $\pi! : H^{p+k}(M_\mathcal{G}) \to H^p(B\mathcal{G})$ denotes the integration along the fibre.

In the case where the cohomology algebra $H^*(M)$ is generated by a single element, we can relate the μ-classes to generators of $H^*(B\mathrm{aut}_1(M))$, which are determined algebraically by means of the function space model due to Brown and Szczarba [2]; see Example 3.2 in Appendix for the generators of the model.

Theorem 2.3. [10] *Let (M, ω) be a nilpotent connected c-symplectic manifold whose cohomology is isomorphic to $\mathbb{Q}[\omega]/(\omega^{m+1})$. Then, as an algebra,*

$$H^*(B\mathrm{aut}_1(M); \mathbb{Q}) \cong \mathbb{Q}[\mu_2, \ldots, \mu_{m+1}],$$

where $\deg \mu_k = 2k$.

We here give a computational example. Consider the real Grassmann manifold M of the form $SO(2m+1)/SO(2) \times SO(2m-1)$ and the map

$$\lambda_{SO(2m+1),M} : SO(2m+1) \to M$$

arising from the left translation of $SO(2m+1)$ on M. Since $H^*(M) \cong \mathbb{Q}[\chi]/(\chi^{2m})$ as an algebra, it follows from Theorem 2.3 that

$$H^*(B\mathrm{aut}_1(M)) \cong \mathbb{Q}[\mu_2, \mu_3, \mu_4, \ldots, \mu_{2m}].$$

Observe that $\chi \in H^2(M)$ is the element which comes from the Euler class $\chi \in H^2(BSO(2))$ via the induced map

$$j^* : H^*(B(SO(2)\times SO(2m-1)) \cong \mathbb{Q}[\chi, p'_1, \ldots, p'_{m-1}] \to H^*(M),$$

where j is the fibre inclusion of the fibration

$$M \xrightarrow{j} B(SO(2)\times SO(2m-1) \xrightarrow{Bi} BSO(2m+1).$$

The rational cohomology of $BSO(2m+1)$ is a polynomial algebra generated by Pontrjagin classes; that is, $H^*(BSO(2m+1)) \cong \mathbb{Q}[p_1, \ldots, p_m]$, where $\deg p_i = 4i$. We relate the Pontrjagin classes to the μ-classes with the induced map by $\lambda_{SO(2m+1),M}$. More precisely, we have

Proposition 2.4. [10] $(B\lambda_{SO(2m+1),M})^*(\mu_{2i}) \equiv p_i$ *modulo decomposable elements.*

Theorem 2.3 and the latter half of Theorem 1.3 allow us to deduce that the image of the kth μ-class by the induced map $(B\lambda_{SU(m+1),\mathbb{C}P^m})^* : H^*(B\mathrm{aut}_1(\mathbb{C}P^m)) \to H^*(BSU(m+1))$ coincides with kth Chern class modulo decomposable elements; compare [8, Proposition 1.7] and see its proof.

Example 2.5. Consider a c-symplectic manifold M whose underlying manifold is rationally homotopy equivalent to the product $\mathbb{C}P^m \times \mathbb{C}P^n$. Then $H^*(B\mathrm{aut}_1(M))$ has elements which are not detected by the Kedra-McDuff μ-classes. In fact we have $\dim H^4(B\mathrm{aut}_1(M)) \geq 2$; see [10].

3. Appendix: How to construct models for the evaluation map and for the map $\lambda_{G,M}$

Before describing models for a function space and the evaluation map, we recall briefly rational homotopy theory and fix terminology.

One might hope a category of appropriate algebraic objects which controls a full subcategory of the category of topological spaces. As one of algebraic categories for the study of spaces, we can exhibit that of differential graded algebras related to a topological category which appears in rational homotopy theory due to Quillen [16] and Sullivan [20].

Let C be a category with a family of *weak equivalences* and $h(C)$ denote the homotopy category obtained by giving formal inverses of weak equivalences. The correspondences between "spaces" and "algebras" in rational homotopy theory are roughly summarized as follows:

Rational Homotopy Theory, see also [1] and [4]. The functor $A_{PL}(\cdot)$ of rational polynomial differential forms on a space and the realization functor $|\cdot|$ give equivalences

$$h\begin{pmatrix}\text{the category of connected nilpotent rational spaces}\\ \text{of finite } \mathbb{Q}\text{-type}\end{pmatrix}$$

$$|\cdot|\Big\uparrow \cong \Big\downarrow A_{PL}(\cdot)$$

$$h\Big(\text{the category of differential graded algebras over } \mathbb{Q}\Big).$$

Thus we can deal with topological spaces and continuous maps algebraically via the functors $A_{PL}(\cdot)$ and $|\cdot|$. Observe that functors $A_{PL}(\cdot)$ and $|\cdot|$ are contravariant and that $H(A_{PL}(X)) \cong H^*(X;\mathbb{Q})$ as an algebra.

Let X be a nilpotent space of finite $\mathbb{Q}$-type; that is, the fundamental group $\pi_1(X)$ acts on $\pi_i(X)$ nilpotently for any i and $H^j(X;\mathbb{Q})$ is of finite dimension for any j. The space $|A_{PL}(X)|$, denoted $X_{\mathbb{Q}}$, is called *a localization* of X. In rational homotopy theory, an important object is a tractable free differential graded algebra (DGA) called a Sullivan algebra. We denote by $\wedge W$ the free algebra generated by a graded vector space W. The theory ensures in particular that for a given space X there exists a Sullivan algebra $(\wedge V, d)$ which is quasi-isomorphic (weak equivalent) to the DGA $A_{PL}(X)$ and is minimal in the sense that dv is decomposable in $\wedge V$ for $v \in V$. Such a DGA is called a minimal model for X.

Let $(\wedge V, d) \xrightarrow{\simeq} A_{PL}(X)$ and $(\wedge W, d') \xrightarrow{\simeq} A_{PL}(Y)$ be minimal models for X and for Y, respectively. A morphism $\varphi : (\wedge W, d') \to (\wedge V, d)$ of DGA's is referred to *a Sullivan representative* for a map $f : X \to Y$ if φ is homotopic to $A_{PL}(f)$ up to quasi-isomorphisms. In this case, the realization $|\varphi|$ coincides with f on rational homotopy and homology.

In what follows, for spaces X and Y, we denote by $\mathcal{F}(X,Y)$ the space of maps from X to Y. Let $f : X \to Y$ be a map and $\mathcal{F}(X,Y;f)$ the connected component of $\mathcal{F}(X,Y)$ containing f. We observe that $\mathrm{aut}_1(X)$ is the identity component of the function space $\mathcal{F}(X,X)$.

We here recall a model of the evaluation map $ev : \mathcal{F}(X,Y) \times X \to Y$ defined by $ev(\varphi, x) = \varphi(x)$. Let $(\wedge V, d)$ be a minimal DGA and (B, d_B) a connected, locally finite DGA. Let B_* denote the differential graded coalgebra defined by $B_q = \mathrm{Hom}(B^{-q},\mathbb{Q})$ for $q \le 0$ together with the coproduct D and the differential d_{B*} which are dual to the multiplication of B and to the differential d_B, respectively. We denote by I the ideal of the free algebra $\wedge(\wedge V \otimes B_*)$ generated by $1 \otimes 1_* - 1$ and all elements of the form

$$a_1a_2 \otimes \beta - \sum_i (-1)^{|a_2||\beta_i'|}(a_1 \otimes \beta_i')(a_2 \otimes \beta_i''),$$

where $a_1, a_2 \in \mathbb{Q}[V]$, $\beta \in B_*$ and $D(\beta) = \sum_i \beta_i' \otimes \beta_i''$. Observe that $\wedge(\wedge V \otimes B_*)$ is a DGA with the differential $d := d_A \otimes 1 \pm 1 \otimes d_{B*}$.

The result [2, Theorem 3.5] implies that the composite $\rho : \wedge(V \otimes B_*) \hookrightarrow \wedge(\wedge V \otimes B_*) \to \wedge(\wedge V \otimes B_*)/I$ is an isomorphism of graded algebras.

Thus $(\wedge(V \otimes B_*), \delta = \rho^{-1}d\rho)$ is a DGA. Observe that, if $d(v) = v_1 \cdots v_m$ with $v_i \in V$ and $D^{(m-1)}(e_j) = \sum_j e_{j_1} \otimes \cdots \otimes e_{j_m}$, then

$$\delta(v \otimes e) = \textstyle\sum_j \pm(v_1 \otimes e_{j_1}) \cdots (v_m \otimes e_{j_m}). \tag{3.1}$$

Here the sign is determined by the Koszul rule; that is, $ab = (-1)^{\deg a \deg b} ba$ in a graded algebra.

We choose a basis $\{a_k, b_k, c_j\}_{k,j}$ for B_* so that $d_{B_*}(a_k) = b_k$, $d_{B_*}(c_j) = 0$ and $c_0 = 1$. Moreover we take a basis $\{v_i\}_{i\geq 1}$ for V such that $\deg v_i \leq \deg v_{i+1}$ and $d(v_{i+1}) \in \wedge V_i$, where V_i is the subspace spanned by the elements $v_1, ..., v_i$. The result [2, Lemma 5.1] implies that there exist free algebra generators w_{ij}, u_{ik} and v_{ik} such that

(3.2) $w_{i0} = v_i \otimes 1$ and $w_{ij} = v_i \otimes c_j + x_{ij}$, where $x_{ij} \in \wedge(V_{i-1} \otimes B_*)$,

(3.3) δw_{ij} is decomposable and in $\wedge(\{w_{sl}; s < i\})$,

(3.4) $u_{ik} = v_i \otimes a_k$ and $\delta u_{ik} = v_{ik}$.

We then have a inclusion

$$\gamma : E := (\wedge(w_{ij}), \delta) \hookrightarrow (\wedge(V \otimes B_*), \delta), \tag{3.5}$$

which is a homotopy equivalence with a retract

$$r : (\wedge(V \otimes B_*), \delta) \to E; \tag{3.6}$$

see [2, Lemma 5.2] for example. Let $A = (\wedge V, d) \xrightarrow{\simeq} A_{PL}(Y)$ and $(B, d) \xrightarrow{\simeq} A_{PL}(X)$ be minimal models for Y and for X, respectively. Applying the construction above, we have a DGA $\wedge(V \otimes B_*) \cong \wedge(\wedge V \otimes B_*)/I$.

Let q be a Sullivan representative for a map $f : X \to Y$; that is, we have a homotopy commutative diagram

$$\begin{tikzcd} \wedge W \arrow[r, "\simeq"'] & A_{PL}(X) \\ \wedge V \arrow[u, "q"] \arrow[r, "\simeq"'] & A_{PL}(Y). \arrow[u, "A_{PL}(f)"'] \end{tikzcd}$$

We define a DGA map $u : \wedge(\wedge V \otimes B_*)/I \to \mathbb{Q}$ by

$$u(a \otimes e) = (-1)^{\tau(|a|)} e(q(a)),$$

where $a \in \wedge V$ and $e \in B_*$, where $\tau(n) = [(n+1)/2]$, the greatest integer in $(n+1)/2$.

Let F be the ideal of $E = \wedge(V \otimes B_*)$ generated by the set $(\oplus_{i\leq 0} E^i) \cup \delta(E^0)$ and M_u the ideal generated by

$$(\oplus_{i\leq 0} E^i) \cup \delta(E^0) \cup \{\eta - u(\eta) \mid \deg \eta = 0\},$$

respectively. Then E/F is a free algebra and $(E/F, \delta)$ is a Sullivan algebra (not necessarily connected). Moreover the realization $|(E/F, \delta)|$ is homotopy equivalent to $\mathcal{F}(X, Y_{\mathbb{Q}})$; see [2, Theorem 1.3]. In view of [2, Theorem 6.1], it follows from [6, Theorem 3.3] that $(E/M_u, \delta)$ is a model for a connected component of the function space $\mathcal{F}(X, Y)$ containing f; see also [3].

The proof of [9, Proposition 4.3] and [6, Remark 3.4] allow us to deduce the following proposition.

Proposition 3.1. *Define a map $m(ev) : A = (\wedge V, d) \to (E/M_u, \delta) \otimes B$, by*

$$m(ev)(x) = \sum_j (-1)^{\tau(|b_j|)} \pi \circ r(x \otimes b_{j*}) \otimes b_j,$$

for $x \in A$. Then $m(ev)$ is a model for the evaluation map $ev : \mathcal{F}(X,Y;f) \times X \to Y$; that is, there exists a homotopy commutative diagram

$$\begin{tikzcd} A_{PL}(Y) \arrow[r, "A_{PL}(ev)"] & A_{PL}(\mathcal{F}(X,Y;f)) \otimes A_{PL}(X) \\ A \arrow[u, "\simeq"] \arrow[r, "m(ev)"'] & (E/M_u, \delta) \otimes B, \arrow[u, "\simeq"'] \end{tikzcd}$$

in which vertical arrows are quasi-isomorphism.

Example 3.2. Let (M, ω) be a c-symplectic manifold as in Theorem 2.3. We take a minimal model of the form $(\wedge V, d) = (\wedge(y, \omega), d)$ with $d(y) = \omega^{m+1}$. Then we have a Sullivan model $(E/M_u, \delta)$ for $\mathrm{aut}_1(M)$ in which

$$E/M_u = \wedge(\omega \otimes 1_*, y \otimes (\omega^s)_*; 0 \leq s \leq m),$$

$\delta(x \otimes 1_*) = 0$ and $\delta(y \otimes (\omega^s)_*) = (-1)^s \begin{pmatrix} m+1 \\ s \end{pmatrix} (\omega \otimes 1_*)^{m+1-s}$, where $\deg(y \otimes (\omega^s)_*) = 2m - 2s + 1$; see [6, Example 3.6] for more details. Therefore it follows that

$$H^*(\mathrm{aut}_1(M)) \cong \wedge(y \otimes 1_*, y \otimes (\omega^1)_*, \dots, y \otimes (\omega^m)_*)$$

and that

$$H^*(B\mathrm{aut}_1(M)) \cong \wedge([y \otimes 1_*], [y \otimes (\omega^1)_*], \dots, [y \otimes (\omega^m)_*]),$$

where $\deg([y \otimes (\omega^s)_*]) = \deg(y \otimes (\omega^s)_*) + 1$. We can give a function space model description of the Kedra-McDuff μ-classes. In fact the proof of [10, Theorem 1.8] implies that

$$\mu_k \equiv \pm [y \otimes (\omega^{m-k+1})_*]$$

modulo decomposable elements.

We are ready to describe a model for the map $\lambda_{G,M}$. Let G be a connected Lie group and K a connected subgroup of G. Then letting U be a subgroup of K, we define a map $q : G \times G/U \to G/K$ by the composite of the left translation $G \times G/U \to G/U$ and the projection $p : G/U \to G/K$. The map

$$\lambda_{G,G/K} : G \to \mathcal{F}(G/U, G/K, p)$$

defined by $\lambda_{G,G/K}(g)([x]) = p[gx]$ is the adjoint of the map $q : G \times G/U \to G/K$; that is, we have the commutative diagram below.

$$\begin{tikzcd} G \times G/U \arrow[rr, "(\lambda_{G,G/K}) \times 1"] \arrow[dr, "q"'] & & \mathcal{F}(G/U, G/K) \times (G/U) \arrow[dl, "ev"] \\ & (G/K). & \end{tikzcd} \tag{3.7}$$

We then obtain a model for $\lambda_{G,G/K}$ if a Sullivan representative ζ' for q is constructed explicitly. In fact, the procedure is as follows: If we have a DGA map $\widetilde{\mu}$ between models of $\mathcal{F}(G/U, G/K, p)$ and G which fits in the commutative diagram

$$\begin{tikzcd} \wedge V_G \otimes \wedge W' & & E/F \otimes \wedge W' \arrow[ll, "\widetilde{\mu}\otimes 1"'] \\ & \wedge W \arrow[ul, "\zeta'"] \arrow[ur, "m(ev)"'] & \end{tikzcd} \tag{3.8}$$

in the category of DGA's, then the map $\widetilde{\mu}$ is a model for $\lambda_{G,G/K}$. This fact follows from the adjointness of $\lambda_{G,G/K}$ and the equivalent correspondence in rational homotopy theory mentioned above.

We shall construct a model for $\lambda_{G,G/K}$ by using a Sullivan representative $\zeta' : \wedge W \to \wedge V_G \otimes \wedge W'$ for the map $q : G \times G/U \to G/K$. Let A, B and C be DGA's. Let $\{b_j\}_{j\in J}$ and $\{b_{j*}\}_{j\in J}$ denote a basis for B and its dual basis, respectively. Recall from [2, Section 3] the bijection $\Psi : (A \otimes B_*, C)_{DG} \xrightarrow{\cong} (A, C \otimes B)_{DG}$ defined by

$$\Psi(w)(a) = \sum_j (-1)^{\tau(|b_j|)} w(a \otimes b_{j*}) \otimes b_j.$$

Here $(\ ,\ )_{DG}$ stands for the homset in the category of differential graded *modules*.

Consider the case where $A = (\wedge W, d)$, $B = (\wedge W', d)$ and $C = (\wedge V_G, d)$. Define a map $\widetilde{\mu} : \wedge(A \otimes B_*) \to \wedge V_G$ by

$$\widetilde{\mu}(y \otimes b_{j*}) = (-1)^{\tau(|b_j|)} \langle \zeta'(y), b_{j*} \rangle, \tag{3.2}$$

where $\langle\ , b_{j*}\rangle : \wedge V_G \otimes \wedge W' \to \wedge V_G$ is a map defined by $\langle x \otimes a, b_{j*}\rangle = x \cdot \langle a, b_{j*}\rangle$. Then we see that $\Psi(\widetilde{\mu}) = \zeta'$. It follows from [2, Theorem 3.3] that

$$\widetilde{\mu} : E := \wedge(A \otimes B_*)/I \to \wedge V_G$$

is a well-defined DGA map. Moreover we can verify the commutativity of the diagram (3.8). This allows us to conclude that $\widetilde{\mu}$ is a model for $\lambda := \lambda_{G,G/K} : G \to \mathcal{F}(G/U, G/K, p)$. Thus we have

Theorem 3.3. [10] *Let $\mathbb{Q}\{x_1, ..., x_s\}$ be a subspace of the image of the induced map*

$$H(Q(\widetilde{\mu})) : H_*(Q(E/F), \delta_0) \to H_*(Q(\wedge V_G), d_0) = V_G.$$

Then there exists a map $\rho : \times_{j=1}^{s} S^{\deg x_i} \to G$ such that the map

$$(\lambda_{\mathbb{Q}} \circ \rho_{\mathbb{Q}})_* : \pi_*((\times_{j=1}^{s} S^{\deg x_i})_{\mathbb{Q}}) \to \pi_*(\mathcal{F}(G/U, (G/K)_{\mathbb{Q}}), e \circ p)$$

is injective.

By applying theorem 3.3, we can prove Theorem 1.1; see [10, Section 2] for more details.

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