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SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY

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Given a finite group G of automorphisms of a compact Riemann surface, we discuss a relation between Mumford-Morita-Miller classes of odd indices and the homology representation of G. Since most participants were group theorists rather than topologists, I separate the algebraic and the topological ingredients and explain the former in detail.

1. Surface symmetries

1.1. **The Grieder group of a finite group.** Let G be a finite group and $\hat{\gamma}$ the conjugacy class of $\gamma \in G$. We denote by $\langle \hat{\gamma}_1, \hat{\gamma}_2, \dots, \hat{\gamma}_q \rangle$ an unordered q-tuple ($q \geq 0$) of conjugacy classes of nontrivial elements of G satisfying $\gamma_1\gamma_2 \cdots \gamma_q \in [G, G]$, and $\mathcal{M}_G$ the set of all such q-tuples. We can define an abelian monoid structure on $\mathcal{M}_G$ by

$$\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle + \langle \hat{\gamma}_{q+1}, \dots, \hat{\gamma}_r \rangle = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q, \hat{\gamma}_{q+1}, \dots, \hat{\gamma}_r \rangle.$$

The identity element is the empty tuple $\langle \rangle$. We call $\mathcal{M}_G$ the *Grieder monoid* of G. Now let $\mathcal{M}'_G$ be the submonoid generated by $\langle \hat{\gamma}, \hat{\gamma}^{-1} \rangle$ ($\gamma \in G$) and set $\mathcal{A}_G := \mathcal{M}_G / \mathcal{M}'_G$. The quotient $\mathcal{A}_G$ is an abelian group. The inverse element is given by

$$-\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle = \langle \hat{\gamma}_1^{-1}, \dots, \hat{\gamma}_q^{-1} \rangle \text{ in } \mathcal{A}_G.$$

We call $\mathcal{A}_G$ the *Grieder group* of G. As the names suggest, $\mathcal{M}_G$ and $\mathcal{A}_G$ were introduced and studied by Grieder [5, 6] to study surface symmetries. First of all, $\mathcal{A}_G$ is finitely generated:

Proposition 1 ([5]). *$\mathcal{A}_G \cong \mathbb{Z}^m \oplus \mathbb{Z}_2^n$ for some $m, n \geq 0$.*

A homomorphism $f : H \to G$ of groups induces a homomorphism $f_* : \mathcal{A}_H \to \mathcal{A}_G$ of abelian groups by $f_*(\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle) = \langle \widehat{f(\gamma_1)}, \dots, \widehat{f(\gamma_q)} \rangle$ so that the assignment $G \mapsto \mathcal{A}_G$ is a covariant functor. In addition, for an inclusion $i : H \hookrightarrow G$, one can also define the restriction $i^* : \mathcal{A}_G \to \mathcal{A}_H$ via surface symmetries. Grieder [5] verified the double coset formula and hence proved the following proposition:

Proposition 2. *The assignment $G \mapsto \mathcal{M}_G$ is a Mackey functor.*

1.2. **Ramification data.** By a *surface symmetry* we mean a pair (G,C), where C is a compact Riemann surface of genus $g \geq 2$, and G is a finite group of automorphisms of C. For each $x \in C$, let G_x be the isotropy subgroup at x. Note that G_x is necessary cyclic. Set $\mathcal{S} = \{x \in C \mid G_x \neq 1\}$, and let $\mathcal{S}/G = \{x_1, x_2, \dots, x_q\}$ be a set of representatives of G-orbits of elements of $\mathcal{S}$. For each $x_i \in \mathcal{S}/G$, choose a generator γ_i of G_{x_i} such that γ_i acts on the holomorphic tangent space $T_{x_i}C$ by $z \mapsto \exp(2\pi\sqrt{-1}/|G_{x_i}|)z$ with respect to a suitable local coordinate z at x_i. The *ramification data* of (G,C), abbreviated by $\delta(G,C)$, is the unordered q-tuple $\langle \hat{\gamma}_1, \hat{\gamma}_2, \cdots, \hat{\gamma}_q \rangle$. It satisfies $\gamma_1\gamma_2\cdots\gamma_q \in [G,G]$, and hence $\delta(G,C)$ is an element of the Grieder monoid $\mathcal{M}_G$. Conversely, we have the following proposition.

Proposition 3 (see [5]). *For any element $\mu \in \mathcal{M}_G$, there exists a surface symmetry (G,C) whose ramification data coincides with μ.*

2. Group cohomology

2.1. **The first Chern class.** Let $\langle\gamma\rangle$ be a cyclic group of order m generated by γ and $\rho_\gamma : \langle\gamma\rangle \to \mathbb{C}^\times$ a linear character defined by $\gamma \mapsto \exp(2\pi i/m)$. For any finite group G, we have natural isomorphisms

$$\mathrm{Hom}(G,\mathbb{C}^\times) \cong H^1(G,\mathbb{C}^\times) \cong H^2(G,\mathbb{Z}).$$

Here, the latter isomorphism is the connecting homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$. Define $c(\gamma) \in H^2(\langle\gamma\rangle,\mathbb{Z})$ to be the image of ρ_γ under the isomorphism $\mathrm{Hom}(\langle\gamma\rangle,\mathbb{C}^\times) \cong H^2(\langle\gamma\rangle,\mathbb{Z})$. The cohomology class $c(\gamma)$ is sometimes called the *first Chern class* of ρ_γ.

2.2. **MMM classes (algebra).** For each element $\mu = \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle$ of $\mathcal{M}_G$, define a series of cohomology classes $e_k(\mu) \in H^{2k}(G,\mathbb{Z})$ $(k \geq 1)$ by

$$e_k(\mu) := \sum_{i=1}^{q} \mathrm{Tr}^G_{\langle\gamma_i\rangle}(c(\gamma_i)^k) \in H^{2k}(G,\mathbb{Z}),$$

where $\mathrm{Tr}^G_{\langle\gamma\rangle} : H^*(\langle\gamma\rangle,\mathbb{Z}) \to H^*(G,\mathbb{Z})$ is the transfer. We call $e_k(\mu)$ the *k-th Mumford-Morita-Miller class* of μ (MMM class in short). The definition of $e_k(\mu)$ is motivated by topology, as will be explained in the next subsection. Observe that the assignment $\mu \mapsto e_k(\mu)$ defines a well-defined homomorphism $\mathcal{M}_G \to H^{2k}(G,\mathbb{Z})$ of abelian monoids. For k odd, it induces a well-defined homomorphism $\mathcal{A}_G \to H^{2k}(G,\mathbb{Z})$ of abelian groups, for we have $c(\gamma^{-1}) = -c(\gamma)$. In addition, we can prove the following proposition:

Proposition 4. *For odd $k \geq 1$, the homomorphism $\mathcal{A}_G \to H^{2k}(G,\mathbb{Z})$ is a natural transformation of Mackey functors.*

2.3. **MMM classes (topology).** The definition of $e_k(\mu)$ is inspired by a result of Kawazumi and Uemura [8] concerning of characteristic classes of oriented surface bundles. Let Σ_g be the closed oriented surface of genus $g \geq 2$. Let $\pi : E \to B$ an oriented Σ_g-bundle, $T^v E$ the tangent bundle along the fiber of π, and $e \in H^2(E;\mathbb{Z})$ the Euler class of $T^v E$. Define $e_k^{\mathrm{top}}(\pi) \in H^{2k}(B;\mathbb{Z})$ by $e_k^{\mathrm{top}}(\pi) := \pi_!(e^{k+1})$ where $\pi_! : H^*(E;\mathbb{Z}) \to H^{*-2}(B;\mathbb{Z})$ is the Gysin homomorphism (the superscript "top" stands for "topology"). $e_k^{\mathrm{top}}(\pi)$ is called the *k-th Mumford-Morita-Miller class* of π, as it was introduced in [11, 10, 9].

Now let (G,C) be a surface symmetry as in Section 1.2. Associated with (G,C), there is an oriented surface bundle $\pi : EG \times_G C \to BG$ called the *Borel construction,* where $EG \to BG$ is the universal G-bundle. We denote by $e_k^{\mathrm{top}}(G,C) \in H^{2k}(G,\mathbb{Z})$ the k-th MMM class of the Borel construction π. A result of Kawazumi and Uemura [8] implies the following result:

Theorem 5. *We have $e_k^{\mathrm{top}}(G,C) = e_k(\delta(G,C))$ where $\delta(G,C)$ is the ramification data of (G,C).*

3. Homology representations

3.1. **Algebra.** In what follows, we denote by $R(G)$ the complex representation ring (or the character ring) of a finite group G. Let $\langle\gamma\rangle$ be a cyclic group of order m generated by γ and $\rho_\gamma : \langle\gamma\rangle \to \mathbb{C}^\times$ a linear character as in Section 2.1. Define $\Delta_\gamma \in R(\langle\gamma\rangle) \otimes \mathbb{Q}$ by

$$\Delta_\gamma := 2\sum_{k=1}^{m-1}\left(\frac{k}{m} - \frac{1}{2}\right)\rho_\gamma^{\otimes k} = \frac{2}{m}\sum_{k=1}^{m-1} k\rho_\gamma^{\otimes k} - r_{\langle\gamma\rangle} + 1_{\langle\gamma\rangle},$$

where $r_{\langle\gamma\rangle}$ is the regular representation and $1_{\langle\gamma\rangle}$ is the trivial 1-dimensional representation of $\langle\gamma\rangle$. Now, for each element $\mu = \langle\hat{\gamma}_1,\ldots,\hat{\gamma}_q\rangle$ of $\mathcal{M}_G$, define the *G-signature* $\sigma(\mu)$ of μ by

$$\sigma(\mu) := \sum_{k=1}^{q} \mathrm{Ind}_{\langle\gamma_k\rangle}^{G}(\Delta_{\gamma_k}) \in R(G) \otimes \mathbb{Q}.$$

Proposition 6. $\sigma(\mu) \in R(G)$ *for every G and $\mu \in \mathcal{M}_G$.*

See the next section for the proof. Note that, in case $\mu \in \mathcal{M}_G$ consists of a single conjugacy class ($\mu = \langle\hat{\gamma}\rangle$ for $\gamma \in [G,G]$), Proposition 6 was proved by T. Yoshida [13].

The assignment $\mu \mapsto \sigma(\mu)$ yields a homomorphism $\mathcal{M}_G \to R(G)$ of monoids, which induces a well-defined homomorphism $\mathcal{A}_G \to R(G)$ of abelian groups. In addition, we can prove the following proposition:

Proposition 7. *$\mathcal{A}_G \to R(G)$ is a natural transformation of Mackey functors.*

3.2. **Topology.** Let (G,C) be a surface symmetry, and H_C the space of holomorphic 1-forms on C. Note that $\dim_{\mathbb{C}} H_C = g$ where g is the genus of the Riemann surface C. Then G acts on H_C and hence H_C is a complex representation of G. A virtual representation $\sigma^{\text{top}}(G,C) := H_C - \overline{H}_C \in R(G)$ is called the *G-signature* of (G,C), where $\overline{H}_C$ is the complex conjugate.

Proposition 8. *We have $\sigma^{\text{top}}(G,C) = \sigma(\delta(G,C))$ where $\delta(G,C)$ is the ramification data of (G,C).*

The character of $\sigma^{\text{top}}(G,C)$ is given by the Eichler trace formula (see [4] for instance). The proposition can be verified by comparing characters of $\sigma^{\text{top}}(G,C)$ and $\sigma(\delta(G,C))$. An alternative proof was given by N. Kawazumi (unpublished manuscript). Since every $\mu \in \mathcal{M}_G$ can be realized as a ramification data of a surface symmetry, Proposition 6 follows from the last proposition. The following fact is an easy consequence of Proposition 8.

Corollary 9. *If all the complex characters of G are $\mathbb{R}$-valued, then $\sigma(\mu) = 0$ for all $\mu \in \mathcal{M}_G$.*

Proof. Choose a surface symmetry (G,C) with $\delta(G,C) = \mu$. Then we have $\sigma(\mu) = \sigma^{\text{top}}(G,C) = H_C - \overline{H}_C = 0$ since $H_C = \overline{H}_C$ by the assumption. □

4. A RELATION OF $e_k(\mu)$ AND $\sigma(\mu)$

Theorem 10. *Let G be a finite group and $\mu, \nu \in \mathcal{M}_G$.*

(1) *If $\sigma(\mu) = \sigma(\nu)$ then $e_k(\mu) = e_k(\nu)$ for all odd $k \geq 1$.*
(2) *If $\sigma(\mu) = 0$ then $e_k(\mu) = 0$ for all odd $k \geq 1$.*

Since $R(G)$ is free as an abelian group, the homomorphism $\mathcal{A}_G \to R(G)$ in Section 3.1 induces $\phi_1 : \mathcal{A}_G/\text{Tor}(\mathcal{A}_G) \to R(G)$, where $\text{Tor}(\mathcal{A}_G)$ is the torsion subgroup of $\mathcal{A}_G$. For odd $k \geq 1$, let $\phi_2 : \text{Tor}(\mathcal{A}_G) \to H^{2k}(G,\mathbb{Z})$ be the restriction of the homomorphism $\mathcal{A}_G \to H^{2k}(G,\mathbb{Z})$ in Section 2.2. The proof of Theorem 10 is based on the following two facts:

Theorem 11. *For any finite group G and any odd $k \geq 1$,*

(1) *The homomorphism $\phi_1 : \mathcal{A}_G/\text{Tor}(\mathcal{A}_G) \to R(G)$ is injective.*
(2) *The homomorphism $\phi_2 : \text{Tor}(\mathcal{A}_G) \to H^{2k}(G,\mathbb{Z})$ is trivial.*

The first statement is proved by using a result of Edmonds and Ewing [3], while the second statement is proved by considering the cohomology of metacyclic 2-groups. The detail will appear elsewhere. Theorem 10 and Corollary 9 imply the following corollary:

Corollary 12. *If all the complex characters of G are $\mathbb{R}$-valued, then $e_k(\mu) = 0$ for all $\mu \in \mathcal{M}_G$ and odd $k \geq 1$.*

Define $\mathcal{R}_G$ to be the image of $\phi_1 : \mathcal{A}_G/\mathrm{Tor}(\mathcal{A}_G) \to R(G)$. In view of Theorem 11, there exists a series of homomorphisms $\Phi_k : \mathcal{R}_G \to H^{2k}(G,\mathbb{Z})$ (k odd) which assigns $e_k(\mu)$ to $\sigma(\mu)$. Let $c : \mathrm{Hom}(G,\mathbb{C}^\times) \to H^2(G,\mathbb{Z})$ be the natural isomorphism as in Section 2.1 and $\det : R(G) \to \mathrm{Hom}(G,\mathbb{C}^\times)$ the determinant homomorphism (see [13] for precise). Then the homomorphism Φ_1 is determined by the following proposition:

Proposition 13. $e_1(\mu) = 6 \cdot c(\det(\sigma(\mu)))$ *for all $\mu \in \mathcal{M}_G$.*

The proposition follows from the Grothendieck-Riemann-Roch theorem and a result of Harer [7] (see also [1, Proposition 6]). Proposition 13 can be generalized to larger k, provided G is cyclic. Recall that, for every finite group G, there is a series of homomorphisms $s_k : R(G) \to H^{2k}(G,\mathbb{Z})$ $(k \geq 0)$ of abelian groups, which satisfies the following properties:

(1) $s_1(\rho) = c(\det\rho)$ for all $\rho \in R(G)$.
(2) If ρ is a linear character, then $s_k(\rho) = c(\rho)^k$.

$s_k(\rho)$ is called the *k-th Newton class* of $\rho \in R(G)$. See [12] for further details. Let B_{2k} be the $2k$-th Bernoulli number and N_{2k}, D_{2k} coprime integers satisfying $B_{2k}/k = N_k/D_k$. Then a result of the author and Kawazumi [2] implies the following result:

Theorem 14. *If G is cyclic, then $N_{2k} \cdot e_{2k-1}(\mu) = D_{2k} \cdot s_{2k-1}(\sigma(\mu))$ holds for all $\mu \in \mathcal{M}_G$ and $k \geq 1$.*

Now let G be a cyclic group of order m, and suppose that N_{2k} is prime to m. Choose an integer N_{2k}^* satisfying $N_{2k} \cdot N_{2k}^* \equiv 1 \pmod{m}$. Under these assumptions, we have

$$e_{2k-1}(\mu) = N_{2k}^* D_{2k} \cdot s_{2k-1}(\sigma(\mu))$$

for all $\mu \in \mathcal{M}_G$, and hence determining Φ_{2k-1} for these cases. In particular, we have $e_1(\mu) = 6 \cdot s_1(\sigma(\mu))$, $e_3(\mu) = -60 \cdot s_3(\sigma(\mu))$, $e_5(\mu) = 126 \cdot s_5(\sigma(\mu))$, $e_7(\mu) = -120 \cdot s_7(\sigma(\mu))$ for *any* cyclic group G and $\mu \in \mathcal{M}_G$, since $N_{2k} = 1$ for $1 \leq k \leq 4$.

REFERENCES

1. T. Akita, Nilpotency and triviality of mod p Morita-Mumford classes of mapping class groups of surfaces, Nagoya Math. J. **165** (2002) 1-22.
2. T. Akita, N. Kawazumi, Integral Riemann-Roch formulae for cyclic subgroups of mapping class groups, Math. Proc. Cambridge Phil. Soc. (to appear).
3. A. L. Edmonds, J. H. Ewing, Remarks on the cobordism group of surface diffeomorphisms, Math. Ann. **259** (1982) 497–504.
4. H. M. Farkas, I. Kra, *Riemann Surfaces* (2nd ed.), GTM Vol. 71, Springer-Verlag, New York–Heidelberg–Berlin, 1992.
5. R. Grieder, Geometric representation theory and G-signature, arXiv:math/9811102.
6. ———, G-actions on Riemann surfaces and the associated group of singular orbit data, arXiv:math/9902048.
7. J. L. Harer, The second homology group of the mapping class group of an oriented surface, Invent. Math. **72** (1983) 221–239.
8. N. Kawazumi, T. Uemura, Riemann-Hurwitz formula for Morita-Mumford classes and surface symmetries, Kodai Math. J. **21** (1998) 372–380.
9. E. Y. Miller, The homology of the mapping class group, J. Differential Geom. **24** (1986) 1–14.
10. S. Morita, Characteristic classes of surface bundles, Invent. Math. **90** (1987) 551–577.
11. D. Mumford, Towards an enumerative geometry of the moduli space of curves, in *Arithmetic and Geometry* Vol. II, Birkhäuser, Boston (1983) 271–328.
12. C. B. Thomas, *Characteristic Classes and Cohomology of Finite Groups*, Cambridge Studies in Advanced Mathematics 9, Cambridge University Press, Cambridge, 1986.
13. T. Yoshida, Character-theoretic transfer (II), J. Algebra **118** (1988) 498–527.

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