数理解析研究所講究録
第 1581 巻 2008 年 142-151

THE DRINFELD CENTER OF THE CATEGORY OF MACKEY FUNCTORS

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We determine the center of the tensor category of Mackey functors for a finite group. Details are in [5].

1. The center of a tensor category

The center of a tensor category was defined by Drinfeld, Joyal and Street ([3]), and Magid ([4]). We review the definition. Let $\mathcal{A}$ be a tensor category over a field. The tensor product of objects $A, B \in \mathcal{A}$ is denoted by $A \otimes B$, and the unit object of $\mathcal{A}$ is denoted by I.

The center $\mathbb{Z}(\mathcal{A})$ is a category defined as follows. An object of $\mathbb{Z}(\mathcal{A})$ is a pair (A, θ), where $A \in \mathcal{A}$ and θ is a family of isomorphisms $\theta_B \colon B \otimes A \to A \otimes B$ for all $B \in \mathcal{A}$ satisfying the conditions

$$\theta_{B\otimes B'} = (\theta_B \otimes 1) \circ (1 \otimes \theta_{B'}) \quad \text{for all } B, B' \in \mathcal{A},$$
$$\theta_I = 1.$$

A morphism $(A, \theta) \to (A', \theta')$ of $\mathbb{Z}(\mathcal{A})$ is a morphism $f \colon A \to A'$ of $\mathcal{A}$ satisfying

$$(f \otimes 1) \circ \theta_B = \theta'_B \circ (1 \otimes f) \quad \text{for all } B \in \mathcal{A}.$$

2. Mackey functors

We review the definition of a Mackey functor ([1], [2]). Let G be a finite group. Denote by $\mathcal{S}$ the category of finite G-sets. For $X, Y \in \mathcal{S}$, we write the direct product $X \times Y$ as XY, and the disjoint sum of X and Y as $X + Y$. Let k be a field. Denote by $\mathcal{V}$ the category of vector spaces over k.

A Mackey functor M on $\mathcal{S}$ consists of k-vector spaces $M(X)$ for all G-sets X and linear maps $f_* \colon M(X) \to M(Y)$ and $f^* \colon M(Y) \to M(X)$ for all G-maps $f \colon X \to Y$ satisfying the following conditions:

(i) $M(X)$ and f_* form a functor $\mathcal{S} \to \mathcal{V}$.

(ii) $M(X)$ and f^* form a functor $\mathcal{S}^{\mathrm{op}} \to \mathcal{V}$.

(iii) For a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[q]{} & Y' \end{array}$$

in $\mathcal{S}$, the diagram

$$\begin{array}{ccc} M(X) & \xleftarrow{p^*} & M(X') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ M(Y) & \xleftarrow[q^*]{} & M(Y') \end{array}$$

is commutative.

(iv) Let $i_1: X_1 \to X_1 + X_2$ and $i_2: X_2 \to X_1 + X_2$ be the inclusion maps in $\mathcal{S}$. Then the maps

$$(i_{1*}, i_{2*}): M(X_1) \oplus M(X_2) \to M(X_1 + X_2),$$
$$(i_1^*, i_2^*): M(X_1 + X_2) \to M(X_1) \oplus M(X_2)$$

are inverse to each other.

(v) $M(\emptyset) = 0$.

The category of Mackey functors on $\mathcal{S}$ is denoted by $\mathbb{M}(\mathcal{S})$.

We use the following fact later. If M is a Mackey functor and $i: Y \to X$ is a monomorphism in $\mathcal{S}$, then the composite

$$M(Y) \xrightarrow{i_*} M(X) \xrightarrow{i^*} M(Y)$$

is the identity. So the composite

$$M(X) \xrightarrow{i^*} M(Y) \xrightarrow{i_*} M(X)$$

is an idempotent endomorphism.

The category $\mathbb{M}(\mathcal{S})$ is a tensor category. Its tensor product is defined as follows. Let $M, M', M'' \in \mathbb{M}(\mathcal{S})$. A bilinear morphism ϕ: $(M, M') \to M''$ is a family of linear maps $\phi_{X,Y}: M(X) \otimes M'(Y) \to M''(XY)$ which commute with f_* and f^* for the both variables X, Y. Given $M, M' \in \mathbb{M}(\mathcal{S})$, there exists a bilinear morphism $(M, M') \to M_0$ which is universal among all bilinear morphisms $(M, M') \to M''$. We define $M_0 = M \otimes M'$.

If $\mathcal{C}$ is a category with pullbacks and sums, Mackey functors on $\mathcal{C}$ are similarly defined. We denote by $\mathbb{M}(\mathcal{C})$ the category of Mackey functors on $\mathcal{C}$.

3. The main result

We define a category $\mathcal{T}_{c*}$. An object of $\mathcal{T}_{c*}$ is a pair (X, a) of $X \in \mathcal{S}$ and an automorphism $a\colon X \to X$ of $\mathcal{S}$ such that a leaves all G-orbits in X stable. A morphism $(X, a) \to (X', a')$ of $\mathcal{T}_{c*}$ is a morphism $f\colon X \to X'$ of $\mathcal{S}$ such that $f \circ a = a' \circ f$.

The category $\mathcal{T}_{c*}$ has pullbacks and sums, so the category $\mathbb{M}(\mathcal{T}_{c*})$ is defined.

A construction of pullback in $\mathcal{T}_{c*}$ is as follows. Given a diagram

$$\begin{array}{ccc} & & (Y, b) \\ & & \downarrow \\ (X, a) & \longrightarrow & (Z, c) \end{array}$$

in $\mathcal{T}_{c*}$, form a pullback

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

in $\mathcal{S}$. The maps $a : X \to X$ and $b\colon Y \to Y$ induce $d\colon W \to W$. Put

$$V = \bigcup\{U \mid U \text{ is a } G\text{-orbit in } W,\ d(U) = U\}$$

and $e = d|V$. Then

$$\begin{array}{ccc} (V, e) & \longrightarrow & (Y, b) \\ \downarrow & & \downarrow \\ (X, a) & \longrightarrow & (Z, c) \end{array}$$

is a pullback in $\mathcal{T}_{c*}$.

Our result is

Theorem. *An equivalence of categories* $\mathbb{Z}(\mathbb{M}(\mathcal{S})) \simeq \mathbb{M}(\mathcal{T}_{c*})$.

By definition an object of $\mathbb{Z}(\mathbb{M}(\mathcal{S}))$ is a pair (M, θ) of $M \in \mathbb{M}(\mathcal{S})$ and a family θ of isomorphisms $\theta_{M'}\colon M' \otimes M \to M \otimes M'$ for all $M' \in \mathbb{M}(\mathcal{S})$ satisfying certain conditions. We may also regard an object of $\mathbb{Z}(\mathbb{M}(\mathcal{S}))$ as a pair (M, ω) of $M \in \mathbb{M}(\mathcal{S})$ and a family ω of isomorphisms $\omega_{X,Y}\colon M(XY) \to M(YX)$ for all $X, Y \in \mathcal{S}$ satisfying (i)–(iii):

(i) $\omega_{X,Y}$ is natural in X, Y.

(ii) The diagram

$$\begin{array}{ccc} M(XYZ) & \xrightarrow{\omega_{X,YZ}} & M(YZX) \\ & \omega_{XY,Z} \searrow & \downarrow \omega_{Y,ZX} \\ & & N(ZXY) \end{array}$$

commutes for all $X, Y, Z \in \mathcal{S}$.

(iii) $\omega_{\mathbf{1},X} = 1$ for a one-element G-set $\mathbf{1}$.

The equivalence of the theorem is given as follows. Let $(M,\omega) \in \mathbb{Z}(\mathbb{M}(\mathcal{S}))$. For an object $(X,a) \in \mathcal{T}_{c*}$, define $L(X,a)$ as the pullback

$$\begin{array}{ccc} L(X,a) & \to & M(X) \\ & & \downarrow{\scriptstyle (a,1)_*} \\ \downarrow & & M(XX) \\ & & \downarrow{\scriptstyle \omega_{X,X}} \\ M(X) & \underset{(1,1)_*}{\longrightarrow} & M(XX) \end{array}$$

where $(a,1)\colon X \to XX$ is the map $x \mapsto (a(x),x)$, and $(1,1)\colon X \to XX$ is the diagonal map. Then the assignment $(X,a) \mapsto L(X,a)$ becomes a Mackey functor L on $\mathcal{T}_{c*}$. The functor $(M,\omega) \mapsto L$ gives the equivalence $\mathbb{Z}(\mathbb{M}(\mathcal{S})) \simeq \mathbb{M}(\mathcal{T}_{c*})$.

4. Outline of the proof

The equivalence of the theorem is obtained as the composite of equivalences

$$\mathbb{Z}(\mathbb{M}(\mathcal{S})) \underset{(1)}{\simeq} {}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}} \underset{(2)}{\simeq} \mathbb{M}_0(\mathcal{W}') \underset{(3)}{\simeq} \mathbb{M}(\mathcal{W}_{ic*}) \underset{(4)}{\simeq} \mathbb{M}(\mathcal{T}_{c*}).$$

We will sketch each equivalence in order.

(1) $\mathbb{Z}(\mathbb{M}(\mathcal{S})) \simeq {}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}}$.

A *bi-Mackey functor* N on $\mathcal{S}$ consists of vector spaces $N(X,Y)$ for all G-sets X and Y, and linear maps

$$\langle f,g\rangle_*\colon N(X,Y) \to N(X',Y'),$$
$$\langle f,g\rangle^*\colon N(X',Y') \to N(X,Y)$$

for all G-maps $f\colon X \to X'$ and $g\colon Y \to Y'$ satisfying (i)–(ix):

(i) The collection of $N(X,Y)$ and $\langle f,g\rangle_*$ forms a functor $\mathcal{S} \times \mathcal{S} \to \mathcal{V}$.

(ii) The collection of $N(X,Y)$ and $\langle f,g\rangle^*$ forms a functor $\mathcal{S}^{\mathrm{op}} \times \mathcal{S}^{\mathrm{op}} \to \mathcal{V}$.

(iii) For G-maps $f\colon X \to X'$ and $g\colon Y \to Y'$, the diagrams

$$\begin{array}{ccc} N(X,Y) & \xrightarrow{\langle f,1\rangle_*} & N(X',Y) \\ {\scriptstyle \langle 1,g\rangle^*}\uparrow & & \uparrow{\scriptstyle \langle 1,g\rangle^*} \\ N(X,Y') & \underset{\langle f,1\rangle_*}{\longrightarrow} & N(X',Y') \end{array} \qquad \begin{array}{ccc} N(X,Y) & \xleftarrow{\langle f,1\rangle^*} & N(X',Y) \\ {\scriptstyle \langle 1,g\rangle_*}\downarrow & & \downarrow{\scriptstyle \langle 1,g\rangle_*} \\ N(X,Y') & \underset{\langle f,1\rangle^*}{\longleftarrow} & N(X',Y') \end{array}$$

are commutative.

(iv) If

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_1' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X_2 & \xrightarrow[f_2]{} & X_2' \end{array}$$

is a pullback diagram, then

$$\begin{array}{ccc} N(X_1,Y) & \xrightarrow{\langle f_1,1\rangle_*} & N(X_1',Y) \\ {\scriptstyle \langle p,1\rangle^*}\uparrow & & \uparrow{\scriptstyle \langle p',1\rangle^*} \\ N(X_2,Y) & \xrightarrow[\langle f_2,1\rangle_*]{} & N(X_2',Y) \end{array}$$

is commutative.

(v) The analogue of (iv) for the second variable.

(vi) Let $i_1\colon X_1 \to X_1+X_2$, $i_2\colon X_2 \to X_1+X_2$ denote the inclusion maps. Then the maps

$$(\langle i_1,1\rangle_*, \langle i_2,1\rangle_*)\colon N(X_1,Y)\oplus N(X_2,Y) \to N(X_1+X_2,Y),$$
$$(\langle i_1,1\rangle^*, \langle i_2,1\rangle^*)\colon N(X_1+X_2,Y) \to N(X_1,Y)\oplus N(X_2,Y)$$

are inverse to each other.

(vii) The analogue of (vi) for the second variable.

(viii) $N(\emptyset,Y)=0$.

(ix) $N(X,\emptyset)=0$.

A *bi-Mackey functor on $\mathcal{S}$ with two-sided action* is a bi-Mackey functor N on $\mathcal{S}$ equipped with maps

$$Z!\colon N(X,Y)\to N(ZX,ZY),$$
$$!Z\colon N(X,Y)\to N(XZ,YZ)$$

for $X,Y,Z\in\mathcal{S}$ satisfying (i)–(ix):

(i) For G-maps $f\colon X\to X'$ and $g\colon Y\to Y'$, the diagrams

$$\begin{array}{ccc} N(X,Y) & \xrightarrow{\langle f,g\rangle_*} & N(X',Y') \\ {\scriptstyle !Z}\downarrow & & \downarrow{\scriptstyle !Z} \\ N(XZ,YZ) & \xrightarrow[\langle 1f,1g\rangle_*]{} & N(X'Z,Y'Z) \end{array}$$

and

$$\begin{array}{ccc} N(X,Y) & \xleftarrow{\langle f,g\rangle^*} & N(X',Y') \\ {\scriptstyle !Z}\downarrow & & \downarrow{\scriptstyle !Z} \\ N(XZ,YZ) & \xleftarrow[\langle 1f,1g\rangle^*]{} & N(X'Z,Y'Z) \end{array}$$

are commutative.

(ii) For G-map $h\colon Z \to Z'$, the diagrams

$$\begin{array}{ccc} N(X,Y) & \xrightarrow{!Z} & N(XZ,YZ) \\ {\scriptstyle !Z'}\downarrow & & \downarrow{\scriptstyle \langle 1,1h\rangle_*} \\ N(XZ',YZ') & \xrightarrow[\langle 1h,1\rangle^*]{} & N(XZ,YZ') \end{array}$$

and

$$\begin{array}{ccc} N(X,Y) & \xrightarrow{!Z} & N(XZ,YZ) \\ {\scriptstyle !Z'}\downarrow & & \downarrow{\scriptstyle \langle 1h,1\rangle_*} \\ N(XZ',YZ') & \xrightarrow[\langle 1,1h\rangle^*]{} & N(XZ',YZ) \end{array}$$

are commutative.

(iii) The diagram

$$\begin{array}{ccc} N(X,Y) & \xrightarrow{!Z} & N(XZ,YZ) \\ & {\scriptstyle !(ZZ')}\searrow & \downarrow{\scriptstyle !Z'} \\ & & N(XZZ',YZZ') \end{array}$$

is commutative.

(iv) For a one-element G-set $\mathbf{1}$,

$$!\mathbf{1}\colon N(X,Y) \to N(X\mathbf{1},Y\mathbf{1})$$

is the identity.

(v)–(viii) The analogue of (i)–(iv) for $Z!$.

(ix) The diagram

$$\begin{array}{ccc} N(X,Y) & \xrightarrow{Z!} & N(ZX,ZY) \\ {\scriptstyle !W}\downarrow & & \downarrow{\scriptstyle !W} \\ N(XW,YW) & \xrightarrow[Z!]{} & N(ZXW,ZYW) \end{array}$$

is commutative.

The category of bi-Mackey functors on $\mathcal{S}$ with two-sided action is denoted by ${}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}}$.

Proposition. *We have an equivalence* $\mathbb{Z}(\mathbb{M}(\mathcal{S})) \simeq {}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}}$.

This equivalence takes an object $(M,\omega) \in \mathbb{Z}(\mathbb{M}(\mathcal{S}))$ to an object $N \in {}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}}$ defined as follows. For $X, Y \in \mathcal{S}$

$$N(X,Y) = M(XY).$$

The operation

$$!Z\colon N(X,Y) \to N(XZ,YZ)$$

is the composite

$$M(XY) \xrightarrow{(11p)^*} M(XYZ) \xrightarrow{(11\Delta)_*} M(XYZZ) \xrightarrow{(1\tau 1)_*} M(XZYZ),$$

where $\Delta\colon Z \to ZZ$ is the diagonal map and $\tau\colon YZ \to ZY$ is the transposition. The operation

$$Z!\colon N(X,Y) \to N(ZX,ZY)$$

is the composite

$$M(XY) \xrightarrow{(p11)^*} M(ZXY) \xrightarrow{(\Delta 11)_*} M(ZZXY) \xrightarrow{\omega_{Z,ZXY}} M(ZXYZ) \xrightarrow{(11\tau)_*} M(ZXZY).$$

(2) ${}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}} \simeq \mathbb{M}_0(\mathcal{W}')$.

Let $\mathcal{W}'$ be the category whose objects are diagrams

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array}$$

of G-sets such that the induced maps $U \to XY$, $V \to XY$ are injective, and morphisms are natural ones. This has pullbacks and sums, so one has the category $\mathbb{M}(\mathcal{W}')$ of Mackey functors on $\mathcal{W}'$.

Suppose that $(M,\omega) \in \mathbb{Z}(\mathbb{M}(\mathcal{S}))$ corresponds to $N \in {}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}}$ under the equivalence (1). Let

$$\mathbf{X} = \left(\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array} \right) \in \mathcal{W}'.$$

As noted after the definition of a Mackey functor, the injection $V \to XY$ determines an idempotent endomorphism on $M(XY)$. As $M(XY) = N(X,Y)$, this is an idempotent endomorphism on $N(X,Y)$, which we denote by

$$e^R(X \leftarrow V \to Y).$$

Similarly the injection $U \to YX$ determines an idempotent endomorphism on $M(YX)$. Through the isomorphism $\omega_{X,Y}\colon M(XY) \to M(YX)$ and $M(XY) = N(X,Y)$, this yields an idempotent endomorphism on $N(X,Y)$, which we denote by

$$e^L(X \leftarrow U \to Y).$$

Lemma. *The idempotent endomorphisms* $e^L(X \leftarrow U \rightarrow Y)$ *and* $e^R(X \leftarrow V \rightarrow Y)$ *on* $N(X,Y)$ *commute with each other.*

We set

$$H(\mathbf{X}) = \operatorname{Im} e^L(X \leftarrow U \rightarrow Y) \cap \operatorname{Im} e^R(X \leftarrow V \rightarrow Y)$$

Then the assignment $\mathbf{X} \mapsto H(\mathbf{X})$ becomes a Mackey functor H on $\mathcal{W}'$. We thus obtain a functor

$$\begin{aligned} {}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}} &\rightarrow \mathbb{M}(\mathcal{W}') \\ N &\mapsto H. \end{aligned}$$

This is fully faithful. To describe its image, we define a full subcategory $\mathbb{M}_0(\mathcal{W}')$ of $\mathbb{M}(\mathcal{W}')$.

An object of $\mathbb{M}_0(\mathcal{W}')$ is an object H of $\mathbb{M}(\mathcal{W}')$ which satisfies (i)–(viii):

(i) Suppose that

$$\mathbf{X} = \left(\begin{array}{ccc} & U_1+U_2 & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array}\right)$$

is an object of $\mathcal{W}'$. Put

$$\mathbf{X}_1 = \left(\begin{array}{ccc} & U_1 & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array}\right), \quad \mathbf{X}_2 = \left(\begin{array}{ccc} & U_2 & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array}\right)$$

and let $\mathbf{i}_1 \colon \mathbf{X}_1 \rightarrow \mathbf{X}$, $\mathbf{i}_2 \colon \mathbf{X}_2 \rightarrow \mathbf{X}$ be the natural injections. Then the maps

$$\begin{aligned} (\mathbf{i}_{1*}, \mathbf{i}_{2*}) &\colon H(\mathbf{X}_1) \oplus H(\mathbf{X}_2) \rightarrow H(\mathbf{X}), \\ (\mathbf{i}_1^*, \mathbf{i}_2^*) &\colon H(\mathbf{X}) \rightarrow H(\mathbf{X}_1) \oplus H(\mathbf{X}_2) \end{aligned}$$

are inverse to each other.

(ii)

$$H\left(\begin{array}{ccc} & \emptyset & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array}\right) = 0.$$

(iii) The analogue of (i) for the V-component.

(iv) The analogue of (ii) for the V-component.

(v) Let

$$\mathbf{X}_1 = \left(\begin{array}{ccc} & U_1 & \\ \swarrow & & \searrow \\ X_1 & & Y \\ \nwarrow & & \nearrow \\ & V_1 & \end{array}\right), \quad \mathbf{X}_2 = \left(\begin{array}{ccc} & U_2 & \\ \swarrow & & \searrow \\ X_2 & & Y \\ \nwarrow & & \nearrow \\ & V_2 & \end{array}\right)$$

be objects of $\mathcal{W}'$. Put

$$\mathbf{X} = \left(\begin{array}{ccccc} & & U_1 + U_2 & & \\ & \swarrow & & \searrow & \\ X_1 + X_2 & & & & Y \\ & \nwarrow & & \nearrow & \\ & & V_1 + V_2 & & \end{array} \right)$$

and let $\mathbf{j}_1 \colon \mathbf{X}_1 \to \mathbf{X}$, $\mathbf{j}_2 \colon \mathbf{X}_2 \to \mathbf{X}$ be the natural injections. Then the maps

$$(\mathbf{j}_{1*}, \mathbf{j}_{2*}) \colon H(\mathbf{X}_1) \oplus H(\mathbf{X}_2) \to H(\mathbf{X}),$$
$$(\mathbf{j}_1^*, \mathbf{j}_2^*) \colon H(\mathbf{X}) \to H(\mathbf{X}_1) \oplus H(\mathbf{X}_2)$$

are inverse to each other.

(vi) The analogue of (v) for the Y-component.

(vii) Let

$$\mathbf{X} = \left(\begin{array}{ccccc} & & U & & \\ & \overset{a}{\swarrow} & & \overset{b}{\searrow} & \\ X & & & & Y \\ & \underset{c}{\nwarrow} & & \underset{d}{\nearrow} & \\ & & V & & \end{array} \right)$$

be an object of $\mathcal{W}'$. Let

$$\begin{array}{ccc} V_1 & \xrightarrow{(c_1, d_1)} & UU \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle ab} \\ V & \xrightarrow[(c,d)]{} & XY \end{array}$$

be a pullback. Put

$$\mathbf{U} = \left(\begin{array}{ccccc} & & U & & \\ & \overset{1}{\swarrow} & & \overset{1}{\searrow} & \\ U & & & & U \\ & \underset{c_1}{\nwarrow} & & \underset{d_1}{\nearrow} & \\ & & V_1 & & \end{array} \right)$$

and

$$\mathbf{a} = \left(\begin{array}{ccc} & 1 & \\ a & & b \\ & e & \end{array} \right) \colon \mathbf{U} \to \mathbf{X}.$$

Then the maps

$$\mathbf{a}_* \colon H(\mathbf{U}) \to H(\mathbf{X}),$$
$$\mathbf{a}^* \colon H(\mathbf{X}) \to H(\mathbf{U})$$

are inverse to each other.

(viii) The analogue of (vii) for the V-component.

The functor ${}_S\mathbb{M}(S, S)_S \to \mathbb{M}(\mathcal{W}')$ constructed before has the image $\mathbb{M}_0(\mathcal{W}')$, and yields

Proposition. *An equivalence* ${}_{\mathcal{S}}\mathbb{M}(\mathcal{S},\mathcal{S})_{\mathcal{S}} \simeq \mathbb{M}_0(\mathcal{W}')$.

(3) $\mathbb{M}_0(\mathcal{W}') \simeq \mathbb{M}(\mathcal{W}_{\mathrm{ic}*})$.

Let $\mathcal{W}_{\mathrm{ic}*}$ be the full subcategory of $\mathcal{W}'$ consisting of finite sums of diagrams

$$\begin{array}{ccc} & U & \\ \swarrow & & \searrow \\ X & & Y \\ \nwarrow & & \nearrow \\ & V & \end{array}$$

in which X, Y, U, V are transitive G-sets and the four arrows are isomorphisms.

Lemma. *The inclusion functor* $\mathcal{W}_{\mathrm{ic}*} \to \mathcal{W}'$ *has a right adjoint.*

Denote the inclusion $\mathcal{W}_{\mathrm{ic}*} \to \mathcal{W}'$ by i and a right adjoint by R.

Proposition. *We have an equivalence* $\mathbb{M}_0(\mathcal{W}') \simeq \mathbb{M}(\mathcal{W}_{\mathrm{ic}*})$.

Under the equivalence objects $H \in \mathbb{M}_0(\mathcal{W}')$ and $K \in \mathbb{M}(\mathcal{W}_{\mathrm{ic}*})$ correspond if

$$H \cong K \circ R, \quad K \cong H \circ i.$$

(4) $\mathbb{M}(\mathcal{W}_{\mathrm{ic}*}) \simeq \mathbb{M}(\mathcal{T}_{\mathrm{c}*})$.

An object of the category $\mathcal{T}_{\mathrm{c}*}$ is a pair (X, a) of $X \in \mathcal{S}$ and an automorphism $a \colon X \to X$ such that a leaves all G-orbits stable. The functor

$$(X,a) \mapsto \left(\begin{array}{ccc} & X & \\ {\scriptstyle 1}\swarrow & & \searrow{\scriptstyle a} \\ X & & X \\ {\scriptstyle 1}\nwarrow & & \nearrow{\scriptstyle 1} \\ & X & \end{array} \right)$$

gives an equivalence $\mathcal{T}_{\mathrm{c}*} \simeq \mathcal{W}_{\mathrm{ic}*}$. This yields

Proposition. *An equivalence* $\mathbb{M}(\mathcal{W}_{\mathrm{ic}*}) \simeq \mathbb{M}(\mathcal{T}_{\mathrm{c}*})$.

Combining (1)–(4), we obtain $\mathbb{Z}(\mathbb{M}(\mathcal{S})) \simeq \mathbb{M}(\mathcal{T}_{\mathrm{c}*})$.

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