数理解析研究所講究録
第 1582 巻 2008 年 108-117

Translation Formulae and Its Applications

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1 Introduction

The purpose of the present paper is to establish translation formulae and to give a new representation of solutions to the periodic linear differential equation of the form

$$\frac{d}{dt}x(t) = A(t)x(t) + f(t), \quad x(0) = w \in \mathbb{C}^p \tag{1}$$

where $A(t)$ is a continuous $p \times p$ matrix function with period $\tau > 0$ and $f : \mathbb{R} \to \mathbb{C}^p$ a τ-periodic continuous function.

In [1], [3] and [4], we gave representations of solutions to the linear difference equation of the form

$$x_{n+1} = Bx_n + b, \quad x_0 = w \in \mathbb{C}^p, \tag{2}$$

$$x_{n+1} = e^{\tau A}x_n + b, \quad x_0 = w \in \mathbb{C}^p, \tag{3}$$

respectively, where A and B are complex $p \times p$ matrices and $b \in \mathbb{C}^p$. If $B = e^{\tau A}, \tau > 0$, then, comparing two representations of solutions to the equations (2) and (3), translation formulae between $A - \lambda E, \lambda \in \sigma(A)$ and $B - \mu E, \mu = e^{\tau\lambda}$ are naturally derived. These are related to the binomial coefficients, the Bernoulli numbers and the Stirling numbers.

Let μ be a characteristic multiplier for homogeneous equation associated with the equation (1). If $\mu \neq 1$, then representations of some component of solutions to the equation (1) were given in [4]. However, it is not yet solved this problem for the case where $\mu = 1$.

In this paper, we present a representation of some component of solutions to the equation (1) for the case where $\mu = 1$ by using translation formulae, Floque's representation and a result in [1].

2 Translation Formulae

Let E be the unit $p \times p$ matrix. For a complex $p \times p$ matrix H we denote by $\sigma(H)$ the set of all eigenvalues of H, and by $h_H(\eta)$ the geometric multiplicity of $\eta \in \sigma(H)$. Let $M_H(\eta) = \mathrm{N}((H - \eta E)^{h_H(\eta)})$ be the generalized eigenspace corresponding to $\eta \in \sigma(H)$. Let $Q_\eta(H) : \mathbb{C}^p \to M_H(\eta)$ be the projection corresponding to the direct sum decomposition $\mathbb{C}^p = \oplus_{\eta \in \sigma(H)} M_H(\eta)$.

Throughout this section, we assume that two $p \times p$ matrices A and B are related as $B = e^{\tau A}$. Put

$$P_\lambda = Q_\lambda(A), h(\lambda) = h_A(\lambda) \ (\lambda \in \sigma(A)), \quad Q_\mu = Q_\mu(B), h(\mu) = h_B(\mu) \ (\mu \in \sigma(B)).$$

By using a spectral mapping theorem, we get

$$\sigma_\mu(A) := \{\lambda \in \sigma(A) \ : \ \mu = e^{\tau\lambda}\} \neq \emptyset$$

for every $\mu \in \sigma(B)$. Moreover, the following relations hold true : $h(\mu) = \max\{h(\lambda) : \lambda \in \sigma_\mu(A)\}$,

$$BP_\lambda = P_\lambda B, \quad P_\lambda Q_\mu = P_\lambda \ (\lambda \in \sigma_\mu(A)), Q_\mu = \sum_{\lambda \in \sigma_\mu(A)} P_\lambda.$$

Let $\omega = 2\pi/\tau, \varepsilon(z) = (e^z - 1)^{-1}, a(z) = (z-1)^{-1}$, and $B_i (i = 0, 1, \cdots)$ be Bernoulli's numbers. For $\mu \in \sigma(B)$ and $\lambda \in \sigma(A)$, vectors $\alpha_\lambda(w, b), \beta_\lambda(w, b), \gamma_\mu(w, b)$ and $\delta(w, b)$ are defined as follows :

$$\alpha_\lambda(w, b) := \alpha_\lambda(w, b; A) = P_\lambda w + X_\lambda(A) P_\lambda b \quad (\lambda \notin i\omega\mathbb{Z}),$$

$$\beta_\lambda(w, b) := \beta_\lambda(w, b; A) = \tau(A - \lambda E) P_\lambda w + Y_\lambda(A) P_\lambda b \quad (\lambda \in i\omega\mathbb{Z}),$$

$$\gamma_\mu(w, b) := \gamma_\mu(w, b; B) = Q_\mu w + Z_\mu(B) Q_\mu b \quad (\mu \neq 1),$$

$$\delta(w, b) := \delta(w, b; B) = (B - E) Q_1 w + Q_1 b \quad (\mu = 1),$$

where

$$X_\lambda(A) = \sum_{i=0}^{h(\lambda)-1} \varepsilon^{(i)}(\tau\lambda) \frac{\tau^i}{i!} (A - \lambda E)^i, \quad Y_\lambda(A) = \sum_{i=0}^{h(\lambda)-1} B_i \frac{\tau^i}{i!} (A - \lambda E)^i,$$

$$Z_\mu(B) = \sum_{k=0}^{h(\mu)-1} \frac{a^{(k)}(\mu)}{k!}(B-\mu E)^k = -\sum_{k=0}^{h(\mu)-1} \frac{1}{(1-\mu)^{k+1}}(B-\mu E)^k,$$

Let $x \in \mathbb{R}$ and $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We define the well known factorial function $(x)_k$ as

$$(x)_k = \begin{cases} 1, & (k=0), \\ x(x-1)(x-2)\cdots(x-k+1), & (k \in \mathbb{N}). \end{cases}$$

The Stirling numbers of the first kind $\left[\begin{matrix} j \\ k \end{matrix} \right]$ and the Stirling numbers of the second kind $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ are introduced as the coefficients of the transform of bases of polynomials as follows $(x)_j = \sum_{k=0}^{j} \left[\begin{matrix} j \\ k \end{matrix} \right] x^k$, $x^k = \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} (x)_j$ for $j,k \in \mathbb{N}_0$. Set

$$B_{k,\mu} = \frac{1}{k!\mu^k}(B-\mu E)^k \quad (\mu \in \sigma(B)), \quad A_{k,\lambda} = \frac{\tau^k}{k!}(A-\lambda E)^k \quad (\lambda \in \sigma(A)).$$

Representations of solutions to the equations (2) and (3) are given as follows, respectively.

Theorem 2.1 *Let $B = e^{\tau A}$ and $\lambda \in \sigma_\mu(A)$. Then the component $P_\lambda x_n(w,b)$ of the solution $x_n(w,b)$ of the equation (2) is expressed as follows :*

1) *If $\mu = e^{\tau\lambda} \neq 1$, then*

$$\begin{aligned} P_\lambda x_n(w,b) &= B^n P_\lambda \gamma_\mu(w,b) - Z_\mu(B) P_\lambda b, \\ &= \mu^n \sum_{k=0}^{h(\mu)-1} (n)_k B_{k,\mu} P_\lambda \gamma_\mu(w,b) - Z_\mu(B) P_\lambda b. \end{aligned}$$

2) *If $\mu = e^{\tau\lambda} = 1$, then*

$$P_\lambda x_n(w,b) = \sum_{k=0}^{h(1)-1} (n)_{k+1} \frac{1}{k+1} B_{k,1} P_\lambda \delta(w,b) + P_\lambda w.$$

Theorem 2.2 [1],[2] *Let $\lambda \in \sigma(A)$. The component $P_\lambda x_n(w,b)$ of the solution $x_n(w,b)$ of the equation (3) is given as follows :*

1) *If $\lambda \notin i\omega\mathbb{Z}$, then*

$$\begin{aligned} P_\lambda x_n(w,b) &= e^{n\tau A} \alpha_\lambda(w,b) - X_\lambda(A) P_\lambda b \\ &= e^{n\tau\lambda} \sum_{k=0}^{h(\lambda)-1} n^k A_{k,\lambda} \alpha_\lambda(w,b) - X_\lambda(A) P_\lambda b. \end{aligned}$$

2) *If* $\lambda \in i\omega\mathbb{Z}$, *then*

$$P_\lambda x_n(w,b) = \sum_{k=0}^{h(\lambda)-1} n^{k+1}\frac{1}{k+1}A_{k,\lambda}\beta_\lambda(w,b) + P_\lambda w.$$

Now, we will compare Theorem 2.1 with Theorem 2.2. If $\mu = e^{\tau\lambda} \neq 1$, then

$$B^n P_\lambda \gamma_\mu(w,b) - Z_\mu(B)P_\lambda b = e^{n\tau A}\alpha_\lambda(w,b) - X_\lambda(A)P_\lambda b, \tag{4}$$

that is,

$$\mu^n \sum_{k=0}^{h(\mu)-1}(n)_k B_{k,\mu}P_\lambda\gamma_\mu(w,b) - Z_\mu(B)P_\lambda b = e^{n\tau\lambda}\sum_{k=0}^{h(\lambda)-1} n^k A_{k,\lambda}\alpha_\lambda(w,b) - X_\lambda(A)P_\lambda b.$$

If $\mu = 1$, then

$$\sum_{k=0}^{h(1)-1}(n)_{k+1}\frac{1}{k+1}B_{k,1}P_\lambda\delta(w,b) = \sum_{k=0}^{h(\lambda)-1} n^{k+1}\frac{1}{k+1}A_{k,\lambda}\beta_\lambda(w,b). \tag{5}$$

Notice that the solution $x_n := x_n(w,b)$ of the equation (2) is expressed by

$$x_n = B^n w + S_n(B)b, \quad S_n(B) = \sum_{k=0}^{n-1} B^k.$$

Now, we consider the case $w = 0$ and the case $b = 0$ in the above representation.

A) $B^n = e^{n\tau A}$ $(n \in \mathbb{N}_0)$ if and only if for all $\mu \in \sigma(B), \lambda \in \sigma_\mu(A)$ the relation

$$\sum_{k=0}^{h(\mu)-1}(n)_k B_{k,\mu}P_\lambda = \sum_{k=0}^{h(\mu)-1} n^k A_{k,\lambda}P_\lambda \quad (n \in N_0) \tag{6}$$

holds. From definition of the Stirling number of the second kind, (6) is rewritten as

$$\begin{aligned}\sum_{k=0}^{h(\mu)-1}(n)_k B_{k,\mu}P_\lambda &= \sum_{j=0}^{h(\mu)-1}\sum_{k=0}^{j}\left\{ {j \atop k} \right\}(n)_k A_{j,\lambda}P_\lambda \\ &= \sum_{k=0}^{h(\mu)-1}(n)_k \sum_{j=k}^{h(\mu)-1}\left\{ {j \atop k} \right\} A_{j,\lambda}P_\lambda.\end{aligned}$$

Hence if $0 \le k \le h(\mu) - 1$, then

$$B_{k,\mu}P_\lambda = \sum_{j=k}^{h(\mu)-1}\left\{ {j \atop k} \right\} A_{j,\lambda}P_\lambda.$$

Also, from definition of the Stirling number of the first kind it follows that, for $0 \le j \le h(\mu) - 1$,

$$A_{j,\lambda}P_\lambda = \sum_{k=j}^{h(\mu)-1} \begin{bmatrix} k \\ j \end{bmatrix} B_{k,\mu}P_\lambda.$$

B) $S_n(B) = S_n(e^{\tau A})$ $(n \in \mathbb{N}_0)$ if and only if for all $\mu \in \sigma(B), \lambda \in \sigma_\mu(A)$ the following relations hold :

(1) If $\mu \neq 1$, then

$$Z_\mu(B)P_\lambda = X_\lambda(A)P_\lambda.$$

(2) If $\mu = 1$, then

$$\sum_{k=0}^{h(1)-1} (n)_{k+1}\frac{1}{k+1}B_{k,1}P_\lambda = \sum_{k=0}^{h(1)-1} n^{k+1}\frac{1}{k+1}A_{k,\lambda}Y_\lambda(A)P_\lambda. \tag{7}$$

Indeed, if $\mu \neq 1$, then, taking $w = 0$ in (4), we have that

$$B^n P_\lambda(Z_\mu(B)P_\lambda b - X_\lambda(A)P_\lambda b) = Z_\mu(B)P_\lambda b - X_\lambda(A)P_\lambda b.$$

Put $n = 1$ and $v = Z_\mu(B)P_\lambda b - X_\lambda(A)P_\lambda b$. Since $P_\lambda v = v$, we have $(B - E)v = 0$, that is, $v \in \mathrm{N}(B - E)$. Since $\mu \neq 1$, we get $v = 0$, and hence (7) holds. If $\mu = 1$, then, taking $w = 0$ in (5), we can obtain (7).

The relation (7) is translated as

$$\begin{aligned}
\sum_{k=0}^{h(1)-1} (n)_{k+1}\frac{1}{k+1}B_{k,1}P_\lambda &= \sum_{j=0}^{h(1)-1} \frac{1}{j+1}\sum_{k=0}^{j+1}\begin{Bmatrix} j+1 \\ k \end{Bmatrix}(n)_k A_{j,\lambda}Y_\lambda(A)P_\lambda \\
&= \sum_{j=0}^{h(1)-1} \frac{1}{j+1}\sum_{k=0}^{j}\begin{Bmatrix} j+1 \\ k+1 \end{Bmatrix}(n)_{k+1} A_{j,\lambda}Y_\lambda(A)P_\lambda \\
&= \sum_{k=0}^{h(1)-1} (n)_{k+1} \sum_{j=k}^{h(1)-1}\begin{Bmatrix} j+1 \\ k+1 \end{Bmatrix}\frac{1}{j+1} A_{j,\lambda}Y_\lambda(A)P_\lambda.
\end{aligned}$$

Thus, if $0 \le k \le h(1) - 1$, then

$$\frac{1}{k+1}B_{k,1}P_\lambda = \sum_{j=k}^{h(1)-1}\begin{Bmatrix} j+1 \\ k+1 \end{Bmatrix}\frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda. \tag{8}$$

Also, (8) is equivalent to the following relation for $0 \le j \le h(1) - 1$:

$$\frac{1}{j+1}A_{j,\lambda}Y_\lambda(A)P_\lambda = \sum_{k=j}^{h(1)-1}\begin{bmatrix} k+1 \\ j+1 \end{bmatrix}\frac{1}{k+1}B_{k,1}P_\lambda.$$

Summarizing these, we have translation formulae.

Theorem 2.3 *Let $B = e^{\tau A}, \tau > 0$ and $\lambda \in \sigma_\mu(A)$.*
1) **(Translation formula I)** *If $0 \le k \le h(\mu) - 1$, then*

$$B_{k,\mu} P_\lambda = \sum_{j=k}^{h(\mu)-1} \left\{ \begin{matrix} j \\ k \end{matrix} \right\} A_{j,\lambda} P_\lambda,$$

or equivalently, if $0 \le j \le h(\mu) - 1$, then

$$A_{j,\lambda} P_\lambda = \sum_{k=j}^{h(\mu)-1} \left[\begin{matrix} k \\ j \end{matrix} \right] B_{k,\mu} P_\lambda.$$

2) **(Translation formula II)** *If $\mu \ne 1$, then*

$$Z_\mu(B) P_\lambda = X_\lambda(A) P_\lambda.$$

3) **(Translation formula III)** *Let $\mu = 1$.*
If $0 \le k \le h(1) - 1$, then

$$\frac{1}{k+1} B_{k,1} P_\lambda = \sum_{j=k}^{h(1)-1} \left\{ \begin{matrix} j+1 \\ k+1 \end{matrix} \right\} \frac{1}{j+1} A_{j,\lambda} Y_\lambda(A) P_\lambda,$$

or equivalently, if $0 \le j \le h(1) - 1$, then

$$\frac{1}{j+1} A_{j,\lambda} Y_\lambda(A) P_\lambda = \sum_{k=j}^{h(1)-1} \left[\begin{matrix} k+1 \\ j+1 \end{matrix} \right] \frac{1}{k+1} B_{k,1} P_\lambda.$$

Using Translation formulae, we obtain relationships between $\alpha_\lambda(w,b)$ and $\gamma_\mu(w,b)$ for $\lambda \in \sigma_\mu(A)$ and between $\beta_\lambda(w,b)$ and $\delta(w,b)$ for $\lambda \in \sigma_1(A)$.

Theorem 2.4 *Let $\lambda \in \sigma_\mu(A)$.*
1) *If $\mu \ne 1$, then*

$$P_\lambda \gamma_\mu(w,b) = \alpha_\lambda(w,b).$$

2) *If $\mu = 1$, then*

$$\sum_{k=0}^{h(1)-1} \frac{(-1)^k}{k+1} (B - E)^k P_\lambda \delta(w,b) = \beta_\lambda(w,b).$$

Theorem 2.5 *Let $\lambda \in \sigma_1(A)$. Then the following relation hold true :*

$$\sum_{k=0}^{h(1)-1} (t)_{k+1} \frac{1}{k+1} B_{k,1} P_\lambda \delta(w,b) = \sum_{k=0}^{h(\lambda)-1} t^{k+1} \frac{1}{k+1} A_{k,\lambda} \beta_\lambda(w,b) \quad (t \in \mathbb{R}).$$

3 Representations of Solutions to Equation (1)

Let $U(t,s), (t,s \in \mathbb{R})$ be solution operators to the equation $x'(t) = A(t)x$. Define the periodic map $V(t), t \in \mathbb{R}$ by $V(t) = U(t, t-\tau) = U(t+\tau, t)$, and set $Q_\mu(t) = Q_\mu(V(t))$ $(\mu \in \sigma(V(t)))$. The representation by Floquet is given as $U(t,0) = P(t)e^{tA}$. Clearly, $V(t) = P(t)e^{\tau A}P^{-1}(t)$ and $V(0) = e^{\tau A}$. By the transformation $x = P(t)y$, the equation (1) is reduced to the following equation

$$\frac{d}{dt}y(t) = Ay(t) + h(t), \quad y(0) = w, \tag{9}$$

where $h(t) = P^{-1}(t)f(t)$. It is obvious that $P^{-1}(t)$ and $h(t)$ are τ-periodic.

Put

$$a_h = \int_0^\tau e^{(\tau-s)A}h(s)ds, \quad b_f = \int_0^\tau U(\tau,s)f(s)ds.$$

Then we have $a_h = b_f$. Set

$$\alpha_\lambda(w,a_h) = \alpha_\lambda(w,a_h;A), \quad \beta_\lambda(w,a_h) = \beta_\lambda(w,a_h;A),$$
$$\gamma_\mu(w,b_f) = \gamma_\mu(w,b_f;V(0)), \quad \delta(w,b_f) = \delta(w,b_f;V(0)).$$

First, we give the representation of solutions of the equation (1) which is based on characteristic exponent. By using a solution $y(t)$ of the equation (9), the solution $x(t)$ of the equation (1) is expressed as

$$x(t) = \sum_{\lambda\in\sigma(A)} P(t)P_\lambda y(t) = \sum_{\lambda\in\sigma(A)} P(t)P_\lambda P^{-1}(t)x(t).$$

Then

$$Q_\mu(t) = \sum_{\lambda\in\sigma_\mu(A)} P(t)P_\lambda P^{-1}(t).$$

Set

$$x_\lambda(t) = P(t)P_\lambda P^{-1}(t)x(t), \quad f_\lambda(t) = P(t)P_\lambda P^{-1}(t)f(t).$$

Combining the representation (cf. [1], [2]) of solutions to the equation (9) and Floque's representation, a representation of solution $x_\lambda(t)$ to the equation (1) is easily derived as follows.

Theorem 3.1 *Each component $x_\lambda(t)$ of the solution $x(t)$ of the equation* (1) *is expressed as follows:*

1) *If $\lambda \notin i\omega\mathbb{Z}$, then*

$$\begin{aligned} x_\lambda(t) &= U(t,0)\alpha_\lambda(w,b_f) + u_\lambda(t,f) \\ &= e^{\lambda t}P(t)\sum_{k=0}^{h(\lambda)-1}\frac{t^k}{k!}(A-\lambda E)^k\alpha_\lambda(w,b_f) + u_\lambda(t,f), \end{aligned}$$

where

$$u_\lambda(t,f) = -U(t,0)X_\lambda(A)P_\lambda b_f + \int_0^t U(t,s)f_\lambda(s)ds$$

is a τ-periodic continuous function.

2) *If $\lambda \in i\omega\mathbb{Z}$,*

$$x_\lambda(t) = e^{\lambda t}P(t)\sum_{k=0}^{h(\lambda)-1}\left(\frac{t}{\tau}\right)^{k+1}\frac{1}{k+1}A_{k,\lambda}\beta_\lambda(w,b_f) + e^{\lambda t}P(t)P_\lambda w + v_\lambda(t,f),$$

where

$$v_\lambda(t,f) = -\frac{e^{\lambda t}}{\tau}P(t)\sum_{k=0}^{h(\lambda)-1}\frac{t^{k+1}}{(k+1)!}(A-\lambda E)^k Y_\lambda(A)P_\lambda b_f + \int_0^t U(t,s)f_\lambda(s)ds$$

is a τ-periodic continuous function.

Next, we give a representation of solutions to equation (1), which is based on characteristic multipliers. Our approach is to translate the representation of solutions in Theorem 3.1 into the representation based on characteristic multipliers by using Translation formulae.

Note that

$$Q_\mu(t)x(t) = \sum_{\lambda\in\sigma_\mu(A)} x_\lambda(t), \quad Q_\mu(t)f(t) = \sum_{\lambda\in\sigma_\mu(A)} f_\lambda(t).$$

Lemma 3.1 *Let $\lambda \in \sigma_\mu(A)$. Then*

$$P(t)e^{\lambda t}P_\lambda = U(t,0)e^{-\frac{t}{\tau}W(\mu)}P_\lambda$$

and

$$e^{\frac{t}{\tau}W(\mu)}P_\lambda = \sum_{k=0}^{h(\mu)-1}\left(\frac{t}{\tau}\right)_k V(0)_{k,\mu}P_\lambda,$$

where

$$W(\mu) = \sum_{k=1}^{h(\mu)-1}\begin{bmatrix} k \\ 1 \end{bmatrix} V(0)_{k,\mu} = \sum_{k=1}^{h(\mu)-1}(-1)^{k-1}(k-1)!V(0)_{k,\mu}.$$

Proof Since $P(t) = U(t,0)e^{-tA}$, we have

$$P(t)e^{\lambda t}P_\lambda = U(t,0)e^{-t(A-\lambda E)}P_\lambda.$$

Using Translation formula I, we can easily prove the first relation. Moreover, we have that

$$\begin{aligned} e^{\frac{t}{\tau}W(\mu)}P_\lambda &= e^{t(A-\lambda E)}P_\lambda \\ &= \sum_{k=0}^{h(\mu)-1} \left(\frac{t}{\tau}\right)^k A_{k,\lambda}P_\lambda \\ &= \sum_{k=0}^{h(\mu)-1} \left(\frac{t}{\tau}\right)_k V(0)_{k,\mu}P_\lambda. \end{aligned}$$

This completes the proof. □

Theorem 3.2 *Let $\mu \in \sigma(V(0))$. The component $Q_\mu(t)x(t)$ of solutions $x(t)$ of the equation* (1) *satisfying the initial condition $x(0) = w$ is expressed as follows :*

1) *If $\mu \neq 1$, then*

$$Q_\mu(t)x(t) = U(t,0)\gamma_\mu(w, b_f) + h_\mu(t, f), \quad (t \in \mathbb{R}),$$

where

$$h_\mu(t, f) = -U(t,0)Z_\mu(V(0))Q_\mu(0)b_f + \int_0^t U(t,s)Q_\mu(s)f(s)ds$$

is a τ-periodic continuous function.

2) *If $\mu = 1$, then*

$$\begin{aligned} Q_1(t)x(t) &= U(t,0)e^{-\frac{t}{\tau}W(1)} \sum_{k=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{k+1} V(0)_{k,1}\delta(w, b_f) \\ &\quad + U(t,0)e^{-\frac{t}{\tau}W(1)}Q_1(0)w + h_1(t, f) \quad (t \in \mathbb{R}). \end{aligned}$$

where

$$\begin{aligned} h_1(t, f) &= -U(t,0)e^{-\frac{t}{\tau}W(1)} \sum_{k=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{k+1} V(0)_{k,1}Q_1(0)b_f \\ &\quad + \int_0^t U(t,s)Q_1(s)f(s)ds \end{aligned}$$

is a τ-periodic continuous function.

Outline of Proof Since the proof of 1) is given in [5], we prove 2). The proof follows from the representation of solutions in Theorem 3.1 and Translation formulae.

Since $V(0) = e^{\tau A}$, it follows from Theorem 2.5 that, for any $t \in \mathbb{R}$,

$$\sum_{k=0}^{h(\lambda)-1} \left(\frac{t}{\tau}\right)^{k+1} \frac{1}{k+1} A_{k,\lambda}\beta_\lambda(w, b_f) = \sum_{k=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{k+1} V(0)_{k,1} P_\lambda \delta(w, b_f). \quad (10)$$

Combining the above relation (10) and Lemma 3.1, we obtain

$$\begin{aligned} &e^{\lambda t} P(t) \sum_{k=0}^{h(\lambda)-1} \left(\frac{t}{\tau}\right)^{k+1} \frac{1}{k+1} A_{k,\lambda}\beta_\lambda(w, b_f) \\ &= U(t,0) e^{-\frac{t}{\tau} W(1)} \sum_{k=0}^{h(1)-1} \left(\frac{t}{\tau}\right)_{k+1} \frac{1}{k+1} V(0)_{k,1} P_\lambda \delta(w, b_f). \end{aligned}$$

Since $Q_1(0) = \sum_{\lambda \in \sigma_1(A)} P_\lambda$ and $Q_1(t)x(t) = \sum_{\lambda \in \sigma_1(A)} x_\lambda(t)$, we have the representation of $Q_1(t)x(t)$ in the theorem. □

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