数理解析研究所講究録
第 1586 巻 2008 年 104-108

Hyperbolicity of critically finite maps on complex projective plane

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This is the abstract of my talk in the conference held at RIMS, September 3-6 2007. The results obtained in [M1] and [M2] will be explained.

Our main purpose is to give a necessary and sufficient condition for a critically finite map on complex projective plane to be Axiom A. This is helpful to understand the dynamics of a map f_ϵ which is obtained by a small perturbation of an Axiom A critically finite map f_0.

1 Repellers

We denote by $\mathbb{P}^k$ complex projective space of complex dimension $k(\geq 1)$ and by ω Fubini-Study form such that $\int_{\mathbb{P}^k}\omega^k = 1$. For a holomorphic self-map f of $\mathbb{P}^k$, we define the degree of f by the formula

$$\deg(f) := \int_{\mathbb{P}^k} f^*\omega \wedge \omega^{k-1}.$$

Because the dynamics of degree 1 maps can be understood by linear algebra, in this paper, we will focus on the case when $\deg(f) \geq 2$. Let C denote the critical set of f. We consider the closure of the post-critical set and the critical limit set for f which are respectively defined by

$$D := \overline{\bigcup_{n\geq 1} f^n(C)},\ E := \bigcap_{n\geq 1} \overline{\bigcup_{i\geq n} f^i(C)}.$$

In this section, we will study the dynamics on invariant compact sets outside D. We will describe a 'semi-repelling' structure of such invariant compact sets.

Definition 1.1. Let f be a holomorphic self-map of $\mathbb{P}^k$ of degree ≥ 2. Let T_p denote the holomorphic tangent space at $p \in \mathbb{P}^k$ and let $|\cdot|$ denote Fubini-Study metric.

We say that $p \in \mathbb{P}^k$ is repelling for f if and only if

$$\min_{v\in T_p,\ |v|=1}|Df^j(v)| \to +\infty$$

as $j \to +\infty$, where Df denote the derivative of f.

We say that a compact set K in $\mathbb{P}^k$ is a repeller for f if and only if $f(K) = K$ and there are constants $c > 0,\ \lambda > 1$ such that

$$|Df^n(v)| \geq c\lambda^n |v|$$

for all $v \in \bigcup_{p \in K} T_p$ and all $n \geq 1$.

Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. We say that a holomorphic embedding $\phi : \mathbb{D} \to \mathbb{P}^k$ is a Fatou disk if and only if $\{f^n \circ \phi\}_{n \geq 1}$ is a normal family in $\mathbb{D}$. We say that a Fatou disk $\phi : \mathbb{D} \to \mathbb{P}^k$ is noncontractive if and only if every limit map of $\{f^n \circ \phi\}_{n \geq 1}$ is nonconstant.

By the following theorem, we describe a 'semi-repelling' structure of an invariant compact set outside D, in terms of repeling points and Fatou disks.

Theorem 1.2. *Let f be a holomorphic self-map of $\mathbb{P}^k$ of degree ≥ 2. Let K be a compact set in $\mathbb{P}^k$ such that $f(K) \subset K$ and $K \cap D = \emptyset$. Then, there are subsets $K^u,\ K^c \subset K$ which satisfy the following properties:*

(i) $K^u \cup K^c = K,\ K^u \cap K^c = \emptyset$;

(ii) $f(K^u) \subset K^u,\ f(K^c) \subset K^c$;

(iii) each point in K^u is repelling;

(iv) for each $p \in E^c$, there is a noncontractive Fatou disk through p.

Moreover, if $f(K) = K$ and $K^c = \emptyset$, then K is a repeller.

2 Maps with sparse critical orbits

Let f be a holomorphic self-map of $\mathbb{P}^k$ of degree ≥ 2. As in case when $k = 1$, we will consider the Fatou set and the Julia set for f.

Definition 2.1. We define the Fatou set F for f to be the domain of normality for the sequence of the iterates $\{f^n\}_{n \geq 1}$ and define the Julia set J as $J := \mathbb{P}^k \setminus F$.

The limit $T := \lim_{n \to +\infty} \frac{1}{d^n}(f^*)^n \omega$ exists and we call T the Green (1,1) current for f. The p-fold wedge product $T^p := T \wedge \cdots \wedge T$ is called the Green (p,p) current for f and the support

$$J_p := \mathrm{supp}(T^p)$$

is called the p-th Julia set.

By Fornæss-Sibony and Ueda, it is shown that $J_1 = J$. By Briend-Duval, it is shown that

$$J_k \subset \overline{\{\text{repelling periodic points}\}}.$$

Interestingly, if $k \geq 2$, it is possible that J_k is a proper subset of the one on the right hand side. So, when we study Axiom A maps in higher dimensions, we cannot avoid considering this phenomenon.

Definition 2.2. Let f be a holomorphic self-map of $\mathbb{P}^k$ of degree ≥ 2. We say that f is critically finite if and only if D is algebraic. We say that f is critically sparse if and only if D is pluripolar. (Obviously, critically finite maps are critically sparse.)

When f is critically sparse, we can show that J_k is the 'precise' locus of the distribution of repelling periodic points for f. Actually, we have a stronger therem as follows.

Theorem 2.3. *Suppose that f is critically sparse. Then, all repellers for f are contained in J_k. In particular,*

$$J_k = \overline{\{\text{repelling periodic points}\}}.$$

This theorem seems useful in many cases, not only for critically finite maps. For instance, let us see the following application.

Example 2.4. *Let P be a polynomial self-map of $\mathbb{C}^k$ of degree ≥ 2 which extends holomorphically on $\mathbb{P}^k$. We put*

$$K(P) := \{w \in \mathbb{C}^k \mid \{P^n(w)\}_{n\geq 0} \text{ is bounded}\}.$$

Suppose that $K(P) \cap C = \emptyset$, where C is the critical set of (the extended) P. Since $K(P)$ is a repeller and P is critically sparse in $\mathbb{P}^k$, we can apply Theorem 2.3. Hence, we obtain $K(P) = J_k$.

3 Critically finite maps and hyperbolicity

In this section, we will deal with holomorphic self-maps of $\mathbb{P}^2$. Our philosophy in this section is that a good behavior of critical orbits implies a good structure of global dynamics.

Definition 3.1. Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree ≥ 2. (Then, f is not invertible.)

Let S be a surjectively forward invariant compact set in $\mathbb{P}^2$. We say that S is hyperbolic if and only if the tangent bundle over the space $\widehat{S}$ of histories of points in S has a hyperbolic splitting structure.

We say that f is Axiom A if and only if the nonwandering set Ω for f is hyperbolic and equals to the closure of the set of periodic points of f.

When f is Axiom A, we consider the decomposition of the nonwandering set

$$\Omega = \Omega_0 \sqcup \Omega_1 \sqcup \Omega_2$$

where Ω_i is the part of unstable dimension i.

The following theorem states that a good behavior of critical orbits implies a good structure of Fatou set.

Theorem 3.2. *Let f be a holomorphic self-map of $\mathbb{P}^2$ of degree ≥ 2. Suppose that $J \cap E$ is a hyperbolic set. Then, the Fatou set F consists of the attractive basins for finitely many attracting cycles. Moreover, if the unstable dimension of $J \cap E$ is 1, then*

$$E = \{\text{attracting periodic points}\} \cup \bigcup_{\hat{p} \in \widehat{J \cap E}} W^u(\hat{p})$$

where $W^u(\hat{p})$ is the unstable manifold for $\hat{p}$.

Remark 3.3. Theorem 3.2 is still true if we replace $J \cap E$ with the nonwandering part of $J \cap E$. Note that the hyperbolicity of the nonwandering part of $J \cap E$ is a necessary condition for f to be Axiom A.

Remark 3.4. The first part of Theorem 3.2 can be generalized in any dimension ≥ 2.

By integrating results above, we obtain our main theorems :

Theorem 3.5. *Let f be a critically finite map on $\mathbb{P}^2$. Then, f is Axiom A if and only if $J \cap E$ is a hyperbolic set of unstable dimension 1.*

Theorem 3.6. *Let f be a critically finite map on $\mathbb{P}^2$ which is Axiom A. Then, the following* $(1) - (7)$ *hold :*

(1) all irreducible components of E are rational;

(2) J_2 is connected;

(3) $\Omega_2 = J_2$;

(4) $\Omega_1 = J \cap E$;

(5) $\Omega_0 = \{\text{attracting periodic points}\} \neq \emptyset$;

(6) $E = \{\text{attracting periodic points}\} \cup \bigcup_{\hat{p} \in \widehat{J \cap E}} W^u(\hat{p})$;

(7) $J = J_2 \sqcup \bigcup_{p \in J \cap E} W^s(p)$.

Remark 3.7. The degree of an irreducible component X of E can be any integer ≥ 1. Thus, X is not necessarily smooth.

References

[M1] K.MAEGAWA, On Fatou maps into compact complex manifolds, *Ergod. Th. & Dynam. Sys.*, **25** , 2005, 1551-1560.

[M2] K.MAEGAWA, Holomorphic maps on $\mathbb{P}^k$ with sparse critical orbits, *Submitted.*

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