

## NONCOMMUTATIVE TORI AND MIRROR SYMMETRY

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**ABSTRACT.** This article is a survey on homological mirror symmetry (HMS) of noncommutative tori which includes updated statements obtained by combining some results appeared so far. We include brief reviews of relevant theories in noncommutative geometry, differential geometry, algebraic topology and algebraic geometry so that this article becomes readable for readers in these different fields.

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## 1. INTRODUCTION

Categories of vector bundles and the associated Grothendieck groups provide a way of classifying topological spaces. One can further consider higher  $K$ -groups. These further give a way of generalizing the notion of spaces. The  $C^*$ -algebra  $K_0$ -group  $K_0(C(M))$  of the  $C^*$ -algebra of  $\mathbb{C}$ -valued continuous functions  $C(M)$  on a compact space  $M$  coincides with the topological  $K_0$ -group  $K_0(M)$ , and in general one can consider  $K_0(A)$  and then  $K_i(A)$ ,  $i \geq 1$ , for any *noncommutative*  $C^*$ -algebra  $A$ , where the Bott periodicity holds as in the case of topological  $K$ -theory. The algebra  $A$  is regarded as the space of functions on a noncommutative space, which is the starting point of noncommutative geometry by Connes [6].

A natural category associated to a complex manifold  $M$  is the category  $\text{coh}(M)$  of coherent sheaves on  $M$ . A coherent sheaf is a generalization of a holomorphic vector bundle. This category  $\text{coh}(M)$  forms an abelian category, so one has the derived category  $D^b(\text{coh}(M))$ . A derived category is an example of triangulated categories, where one can define Grothendieck groups [23]. Actually, the derived category  $D^b(\text{coh}(M))$  does depend on the complex structure of  $M$ . Then, the associated Grothendieck group also depends on it.

For a symplectic manifold  $\hat{M}$ , there is an interesting geometric construction of a category called a *Fukaya category* [14]. This should be defined as an  $A_\infty$ -category, a generalization of a differential graded (DG) category, though the complete construction is still under development because of a technical problem (see [18]).

A mirror symmetry is a duality between a complex manifold  $M$  and a symplectic manifold  $\hat{M}$ . In [49], Kontsevich asked a homological algebraic realization of mirror symmetry and proposed a conjecture called *homological mirror symmetry (HMS)*; for a given mirror pair of a complex manifold  $M$  and a symplectic manifold  $\hat{M}$ , the derived category  $D^b(\text{coh}(M))$  of coherent sheaves is equivalent to the derived category  $D^b(\text{Fuk}(\hat{M}))$  of the Fukaya category on  $\hat{M}$ . A definition of the derived category of an  $A_\infty$ -category is also given there [49] so that  $D^b(\text{Fuk}(\hat{M}))$  makes sense (see subsection 5.1).

The Fukaya categories are defined in a geometric way, which means it is not easy to formulate the deformation of Fukaya categories directly. On the other hand, the complex side is more algebraic, which makes it possible to formulate noncommutative analog of the derived category  $D^b(\text{coh}(M))$  of coherent sheaves. Actually, any holomorphic vector bundle on a smooth compact complex manifold  $M$  is given by a vector bundle with a Dolbeault connection (Grothendieck, Malgrange). By Swan [78], a vector bundle  $E$  over  $M$  is equivalent to a finitely generated projective  $C(M)$ -module. The equivalence is given by considering the space of sections  $\Gamma(E)$  of  $E$ , which forms a finitely generated projective module over  $C(M)$ . Now, the noncommutative formulation can be available; one may start from a noncommutative algebra instead of  $C(M)$ !

This article is a survey on HMS of noncommutative tori from the author's viewpoint, as an attempt toward formulating new kinds of geometry in interactions between noncommutative geometry, symplectic geometry and complex geometry via homological algebras and homotopy algebras. Thus, we intend to make this article readable for all readers in these different fields. We include brief review of relevant theory in each field with some of standard references. In section 2, we start from an overview of the theory of noncommutative tori and projective modules over them due to Rieffel, etc. In section 3, we discuss mirror symmetry of tori in a modern setting, generalized geometry. In section 4, we define the (curved) DG-categories of modules over noncommutative complex tori based on projective modules in subsection 2.4. There, we discuss higher dimensional complex tori in general and after that we state what happens in the case of noncommutative two-tori with complex structures. In section 5, we start from a brief

introduction of HMS in subsection 5.1. In subsection 5.2 we discuss HMS of noncommutative two-tori, an example of noncommutative generalization of HMS. The Theorem 5.20 there states the HMS of noncommutative tori in the most updated form in a sense. In section 5.3, we mention partial results toward HMS of higher dimensional noncommutative tori.

**Notation:** Throughout this paper, we treat any (graded) vector space as the one over field  $k = \mathbb{C}$ . A category  $\mathcal{C}$  by definition consists of a class  $\text{Ob}(\mathcal{C})$  of objects, a space  $\text{Hom}_{\mathcal{C}}(X, Y)$  for each  $X, Y \in \text{Ob}(\mathcal{C})$  with the associative composition  $m : \text{Hom}_{\mathcal{C}}(Y, Z) \otimes \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$  for any  $X, Y, Z \in \text{Ob}(\mathcal{C})$  regarded as the composition of morphisms from  $X$  to  $Y$  and those from  $Y$  to  $Z$ . In particular,  $\mathcal{C}$  has the identity morphism  $1_X \in \text{Hom}_{\mathcal{C}}(X, X)$  for any  $X \in \text{Ob}(\mathcal{C})$ . Following the usual category theory, we often denote simply by  $\mathcal{C}$  the class  $\text{Ob}(\mathcal{C})$  of objects in a category  $\mathcal{C}$ , so  $X \in \mathcal{C}$  indicates  $X$  is an object. We also treat categories with additional structures or generalizations of the usual categories such as (curved) DG(=differential graded)-categories and (weak)  $A_{\infty}$ -categories. In those cases, we do not assume that they have the identity morphisms for each objects. If they have the identity morphisms, they are called *unital*.

For a category  $\mathcal{C}$ , one may prefer to express the space  $\text{Hom}_{\mathcal{C}}(X, Y)$  of morphisms from  $X \in \mathcal{C}$  to  $Y \in \mathcal{C}$  as  $\mathcal{C}(Y, X) := \text{Hom}_{\mathcal{C}}(X, Y)$  so that the composition is described as  $m : \mathcal{C}(Z, Y) \otimes \mathcal{C}(Y, X) \rightarrow \mathcal{C}(Z, X)$ . We do not do it for categories in the usual sense, but do so for (weak)  $A_{\infty}$ -categories, then the higher compositions in an  $A_{\infty}$ -category are described as  $m_n : \mathcal{C}(a_1, a_2) \otimes \mathcal{C}(a_2, a_3) \otimes \cdots \otimes \mathcal{C}(a_n, a_{n+1}) \rightarrow \mathcal{C}(a_1, a_n)$ . For a (curved) DG-category  $\mathcal{C}$ , we use both notations since we sometimes need to treat it as a (weak)  $A_{\infty}$ -category, where morphisms are sometimes denoted  $\phi_{ba} \in \mathcal{C}(b, a) = \text{Hom}_{\mathcal{C}}(a, b)$  for  $a, b \in \mathcal{C}$ . Since these are just the problems of notations, we hope the readers are not confused by them. In the case of categories of modules over noncommutative algebras, the naturality of the notations as above is related to whether we consider right modules or left modules.

For any category  $\mathcal{C}$  (in the usual sense, DG,  $A_{\infty}, \dots$ ), by a full subcategory  $\mathcal{C}' \subset \mathcal{C}$  of  $\mathcal{C}$  we mean a category  $\mathcal{C}'$  such that  $\text{Ob}(\mathcal{C}') \subset \text{Ob}(\mathcal{C})$ ,  $\text{Hom}_{\mathcal{C}'}(a, b) = \text{Hom}_{\mathcal{C}}(a, b)$  for  $a, b \in \text{Ob}(\mathcal{C}')$ , with all additional structures in  $\mathcal{C}'$  induced from those in  $\mathcal{C}$  if it has.

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## 2. NONCOMMUTATIVE TORI

In noncommutative geometry in the sense of Connes [6], one usually starts from a  $C^*$ -algebra. A  $\mathbb{C}$ -algebra  $A$  is called a *\*-algebra* if  $A$  is equipped with a  $\mathbb{C}$ -anti-linear involution  $* : A \rightarrow A$  called a *\*-involution*. A *Banach algebra*  $A$  is a normed algebra which is complete with respect to the norm  $\|\cdot\|$ . A *Banach \*-algebra*  $A$  by definition satisfies the compatibility  $\|x^*\| = \|x\|$  for any  $x \in A$ . A Banach \*-algebra  $A$  satisfying  $\|xx^*\| = \|x\|^2$  for any  $x \in A$  is called a  *$C^*$ -algebra*. For a compact space  $M$ , the space  $C(M)$  of continuous functions on  $M$  forms a (commutative)  $C^*$ -algebra. The converse is also true in the sense that any unital commutative  $C^*$ -algebra is isomorphic to  $C(M)$  with an appropriate compact space.

On the other hand, we prefer a noncommutative analog of the space  $C^{\infty}(M)$  of smooth functions on a smooth compact space  $M$ . Thus, in the context of noncommutative *differential*

geometry [6], one often considers an appropriate dense subalgebra of a  $C^*$ -algebra called a pre- $C^*$ -algebra. First, any Banach algebra  $A$  we shall treat in this article is unital, with unit  $1$ , where, to any element  $x \in A$  is associated the *spectrum*  $\sigma(x) := \{\lambda \in \mathbb{C} \mid (\lambda \cdot 1 - x) \text{ is not invertible}\}$ . A subalgebra algebra  $\mathcal{A} \subset A$  is then said to be *stable under holomorphic functional calculus* iff  $f(x) \in \mathcal{A}$  for any  $x \in \mathcal{A} \subset A$  and any analytic function  $f$  on a neighborhood of  $\sigma(x)$ . A *pre- $C^*$ -algebra*  $\mathcal{A}$  is an algebra which is isomorphic to a  $*$ -subalgebra *stable under holomorphic functional calculus* in a  $C^*$ -algebra  $A$  (see [6, p285, Definition 1]). For a pre- $C^*$ -algebra  $\mathcal{A}$ , a  $C^*$ -algebra  $A$  such that  $\mathcal{A} \subset A$  is stable under holomorphic functional calculus is unique, where  $\mathcal{A}$  is dense in  $A$ .<sup>1</sup> Then, the inclusion  $\mathcal{A} \hookrightarrow A$  induces an isomorphism of  $K$ -theory ([6, p298, Proposition 7]).

For a  $\mathbb{C}$ -algebra  $A$ , a *trace* map  $\text{Tr} : A \rightarrow \mathbb{C}$  is a  $\mathbb{C}$ -linear map such that  $\text{Tr}(aa') = \text{Tr}(a'a)$  for any  $a, a' \in A$ . When  $A$  is a  $*$ -algebra, the trace  $\text{Tr}$  is further assumed to satisfy  $\text{Tr}(a^*) = (\text{Tr}(a))^*$ , where  $(\text{Tr}(a))^*$  is the complex conjugation of  $\text{Tr}(a) \in \mathbb{C}$ .

**2.1. Noncommutative tori  $\mathcal{A}_\theta^d$  and  $A_\theta^d$ .** For a fixed  $\theta \in \mathbb{R}$ , consider the  $\mathbb{C}$ -algebra  $\mathbb{C}[U_1, U_2]$  generated by two unitary elements  $U_1, U_2$  with relation

$$U_1 U_2 = e^{-2\pi i \theta} U_2 U_1. \quad (1)$$

Any element  $u \in \mathbb{C}[U_1, U_2]$  is represented as

$$u = \sum_{(n_1, n_2) \in \mathbb{Z}^2} u_{n_1 n_2} (U_1)^{n_1} (U_2)^{n_2}, \quad u_{n_1 n_2} := u(n_1, n_2) \in \mathbb{C}. \quad (2)$$

Thus,  $u$  is regarded as a  $\mathbb{C}$ -valued function on  $\mathbb{Z}^2$ . We call the subalgebra of  $\mathbb{C}[U_1, U_2]$  consisting of elements  $u \in \mathcal{S}(\mathbb{Z}^2)$  a *noncommutative two-torus*  $\mathcal{A}_\theta^2$ . Here,  $\mathcal{S}(\mathbb{Z}^2)$  is the Schwartz space, that is,  $u_{n_1 n_2} := u(n_1, n_2)$ ,  $u \in \mathcal{S}(\mathbb{Z}^2)$ , tends to zero faster than any power of  $|n_1| + |n_2|$ . This is a noncommutative analog of the algebra of smooth functions; one has  $\mathcal{A}_{\theta=0}^2 \simeq C^\infty(T^2)$ . On the other hand, the *universal  $C^*$ -algebra* (see [9]) of the algebra  $\mathbb{C}[U_1, U_2]$  is traditionally called a noncommutative (two-)torus, which we denote by  $A_\theta^2$ . This is the noncommutative analog of the space of continuous functions; one has  $A_{\theta=0}^2 \simeq C(T^2)$ . In fact,  $\mathcal{A}_\theta^2$  is a dense subalgebra of  $A_\theta^2$  stable under holomorphic calculus, i.e.,  $\mathcal{A}_\theta^2$  is a pre  $C^*$ -algebra of  $A_\theta^2$ . Then,  $\mathcal{A}_\theta^2$  is often called the *smooth version* of noncommutative torus to distinguish it from  $A_\theta^2$ . However, the smooth version is our main tool, so we just call  $\mathcal{A}_\theta^2$  a noncommutative two-torus. In any case, those algebras themselves are called noncommutative tori, not algebras over the ones.

Similarly, for a fixed skew-symmetric  $d$  by  $d$  matrix  $\theta := \{\theta^{jk}\} \in \text{Mat}_d(\mathbb{R})$ , consider an algebra  $\mathbb{C}[U_1, \dots, U_d]$  generated by unitary elements  $U_i$ ,  $i = 1, \dots, d$ , with relations

$$U_j U_k = e^{-2\pi i \theta^{jk}} U_k U_j, \quad j, k = 1, \dots, d. \quad (3)$$

We describe elements of  $\mathbb{C}[U_1, \dots, U_d]$  in a slightly different way from those in the two-dimensional case (2). Let

$$U_{\vec{m}} := U_1^{m_1} U_2^{m_2} \dots U_d^{m_d} e^{\pi i \sum_{j < k} m_j \theta^{jk} m_k},$$

where  $\vec{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$ . Then, any element  $u$  of  $\mathbb{C}[U_1, \dots, U_d]$  is a  $\mathbb{C}$ -linear combinations of  $U_{\vec{m}}$ ,  $\vec{m} \in \mathbb{Z}^d$ :

$$u = \sum_{\vec{m} \in \mathbb{Z}^d} u_{\vec{m}} U_{\vec{m}}, \quad u_{\vec{m}} \in \mathbb{C}.$$

In this description, the relation between  $U_{\vec{m}}$  and  $U_{\vec{m}'}$  becomes

$$U_{\vec{m}} U_{\vec{m}'} = e^{\pi i \sum_{j, k} m_j \theta^{jk} m'_k} U_{\vec{m} + \vec{m}'}. \quad (4)$$

<sup>1</sup>The norm  $\|x\|$  in  $\mathcal{A}$  is given by the square root of  $\sup_{\lambda \in \sigma(xx^*)} |\lambda|$ , the spectral radius of  $xx^*$  in  $A$ .

For any element  $u$  represented as above, the  $*$ -involution is defined by

$$u^* := \sum_{\vec{m} \in \mathbb{Z}^d} \overline{u_{\vec{m}}} U_{-\vec{m}},$$

where  $\overline{u_{\vec{m}}}$  is the complex conjugate of  $u_{\vec{m}}$ . Thus,  $\mathbb{C}[U_1, \dots, U_d]$  is a  $*$ -algebra. As in the previous two-dimensional case, we regard an element  $u \in \mathbb{C}[U_1, \dots, U_d]$  as a  $\mathbb{C}$ -valued function on  $\mathbb{Z}^d$  by  $u : \mathbb{Z}^d \ni \vec{m} \mapsto u_{\vec{m}} \in \mathbb{C}$ . We call the subalgebra  $\mathcal{A}_\theta^d$  of  $\mathbb{C}[U_1, \dots, U_d]$  consisting of elements  $u \in \mathcal{S}(\mathbb{Z}^d)$  the (smooth version of) *noncommutative  $d$ -torus*  $\mathcal{A}_\theta^d$ . On the other hand, the universal  $C^*$ -algebra of  $\mathbb{C}[U_1, \dots, U_d]$  is denoted  $A_\theta^d$  (see [69, 47, 13]). Then,  $\mathcal{A}_\theta^d \subset A_\theta^d$  is a pre  $C^*$ -algebra.

The noncommutativity  $\theta$  is called *irrational* if there exists at least one element  $\theta_{jk}$  which is irrational.

There is a canonical normalized trace on  $\mathcal{A}_\theta^d$  specified by the rule

$$\mathrm{Tr}(u) = u_{\vec{m}=0}, \quad u = \sum_{\vec{m}} u_{\vec{m}} U_{\vec{m}}. \quad (5)$$

Let  $\delta_j : \mathcal{A}_\theta^d \rightarrow \mathcal{A}_\theta^d$ ,  $j = 1, \dots, d$ , be derivations defined by

$$\delta_j(U_{\vec{m}}) = 2\pi i m_j U_{\vec{m}}. \quad (6)$$

For the generators  $U_j$  the above relation reads as  $\delta_j U_k = 2\pi i \delta_{jk} U_k$ . These derivations span a  $d$ -dimensional abelian Lie algebra (over  $\mathbb{C}$ ) that we denote  $L$ .

Geometrically, for  $\theta = 0$ , the isomorphism  $\mathcal{A}_0^d \simeq C^\infty(T^d)$  is given by the identification of the generators  $U_i = e^{2\pi i x_i}$ ,  $x_i \in \mathbb{R}$ ,  $i = 1, \dots, d$ . The trace (5) corresponds to the integration  $\int dx_1 \cdots dx_d : C^\infty(T^d) \rightarrow \mathbb{C}$  in Fourier expansion expression. The bases of  $L$  are then regarded as  $\delta_i = d/dx_i$ . Then, for  $\theta \neq 0$ , the relation (4) shows  $\mathcal{A}_\theta^d$  is also described by  $C^\infty(T^d)$  with a Moyal  $\star$ -product (see [47]).

**2.2.  $K$  theory and projective modules.** For an algebra  $A$ , the algebraic  $K_0$ -group  $K_0(A)$  is defined by the formal differences of isomorphism classes of finitely generated projective modules over  $A$ , i.e., the Grothendieck group of the semigroup consisting of isomorphism classes of finitely generated projective modules. A projective  $A$ -module is by definition a direct summand of a free module. When  $A$  is the space  $C(M)$  of continuous functions on a compact space, this  $K_0$ -group  $K_0(C(M))$  corresponds to topological  $K_0$ -group  $K_0(M)$  due to Swan's theorem:

**Theorem 2.1** (Swan [78]). *Let  $M$  be a compact space. For a vector bundle  $E \rightarrow M$  (with finite dimensional fibers), the  $C(M)$ -module of the space  $\Gamma(E)$  of continuous sections of  $E$  is finitely generated and projective. Conversely, every finitely generated projective  $C(M)$ -module arises in this way from a vector bundle over  $M$ . Furthermore, this correspondence induces the equivalence of the category of vector bundles over  $M$  and the category of finitely generated projective  $C(M)$ -modules, where bundle maps correspond to module homomorphisms.*

Note that  $C(M)$  is unital since  $M$  is compact.

In the framework of  $C^*$ -algebras, the  $K_0$ -group  $K_0(A)$  of a unital  $C^*$ -algebra  $A$  is the Grothendieck group of the semigroup consisting of isomorphism classes of projections in  $\mathrm{Mat}_n(A)$  for some  $n \in \mathbb{Z}_{>0}$ . By definition, a projection  $p \in \mathrm{Mat}_n(A)$  satisfies  $p^2 = p = p^*$ , hence defines a finitely generated projective module  $pA^{\oplus n}$  (with an additional 'Hermitian' structure induced from  $p = p^*$ ). Conversely, any finitely generated projective module over the unital  $C^*$ -algebra  $A$  is isomorphic to  $pA^{\oplus n}$  a projection  $p$  with  $n$  large enough (see [80]).

For a given trace  $\mathrm{Tr} : A \rightarrow \mathbb{C}$ , a trace  $\mathrm{Tr} : \mathrm{Mat}_n(A) \rightarrow \mathbb{C}$  is induced in the usual way. Since  $p = p^*$ , one obtains  $\mathrm{Tr}(p) \in \mathbb{R}_{\geq 0}$ . Describe a finitely generated projective module  $E$  as  $E \simeq pA^{\oplus n}$ .

Then, the induced trace on  $\text{End}_A(E)$  is normalized as  $\text{Tr}(p) = \text{Tr}(\mathbf{1}_{\text{End}_A(E)})$ . It is clear that this value  $\text{Tr}(p)$  is the same for isomorphic projective modules and hence this  $\text{Tr}$  induces a map from  $K_0(A)$  to  $\mathbb{R}$ . We denote it by  $\text{tr} : K_0(A) \rightarrow \mathbb{R}$ .

It is shown by the work of Pimsner-Voiculescu [60] that the  $K$ -groups of a noncommutative torus  $A_\theta^d$  are the same as those of a commutative torus  $T^d$ ,  $K_0(A_\theta^d) \simeq \mathbb{Z}^{2^{d-1}} \simeq K_1(A_\theta^d)$ . On the other hand, Rieffel studied the cancellation theorem for these finitely generated projective modules for noncommutative two-tori (irrational rotation algebras) [67, 68] and then for higher dimensional noncommutative two-tori [69]. The answer is positive if  $\theta$  is irrational:

**Theorem 2.2** (Rieffel [69, Theorem 7.1]). *If  $E, F, G$  are finitely generated projective right  $A_\theta^d$ -modules such that  $E \oplus G \simeq F \oplus G$ , then  $E \simeq F$ .*

This implies that  $E \simeq F$  if  $E$  and  $F$  represent the same element in  $K_0(A_\theta^d)$ . This is not true if the cancellation theorem does not hold:  $[E] = [E \oplus G] - [G] = [F \oplus G] - [G] = [F]$  in  $K_0(T_\theta^d)$ . Though the statement is given for  $C^*$ -algebra  $A_\theta^d$ , the result holds true even  $A_\theta^d$  is replaced by the pre- $C^*$ -algebra  $\mathcal{A}_\theta^d$ . Actually, for the proof of Theorem 2.2 and related Theorems, Rieffel [69] employed the differential structure of  $\mathcal{A}_\theta^d$  and modules over it as we explain briefly below.

A *connection* on a right module  $E$  over  $\mathcal{A}_\theta^d$  is a map  $\nabla : L \otimes E \rightarrow E$  which is linear with respect to the vector space  $L$  (6) and satisfies

$$\nabla_X(\xi \cdot u) = \nabla_X(\xi) \cdot u + \xi \cdot X(u)$$

for any  $\xi \in E$  and  $u \in \mathcal{A}_\theta^d$ . In particular, a connection  $\nabla$  is called a *constant curvature connection* if the curvature of the connection is of the following form: for  $\nabla_i := \nabla_{\delta_i}$ ,  $i = 1, \dots, d$ ,

$$[\nabla_i, \nabla_j] = F_{ij} \cdot \mathbf{1}_{\text{End}_{\mathcal{A}_\theta^d}(E)}, \quad \frac{F_{ij}}{2\pi i} = -\frac{F_{ji}}{2\pi i} \in \mathbb{R}. \quad (7)$$

On a noncommutative torus  $\mathcal{A}_\theta^d$ , one can construct a class of finitely generated projective modules called *Heisenberg modules* (see [69, 47]). A Heisenberg module  $E$  over  $\mathcal{A}_\theta^d$  is the Schwartz space  $\mathcal{S}(M)$  on  $M := \mathbb{R}^p \times \mathbb{Z}^q \times F$  for  $p, q \geq 0$ ,  $2p + q = d$ , where  $F$  is a finite abelian group and hence is a product of cyclic group  $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$ . Let  $\hat{M} := \mathbb{R}^p \times T^q \times F$  and call this the dual space of  $M$ . Here, any Heisenberg module is equipped with a constant curvature connection [69] (see [47]). Its Chern character is defined as follows. Recall that  $L$  be the  $d$ -dimensional vector space spanned by  $\delta_1, \dots, \delta_d$ . Here we switch the notation as  $e_i := \delta_i$ ,  $i = 1, \dots, d$ . The basis of the dual vector space  $L^*$  is denoted  $e^1, \dots, e^d$ . The Chern character is defined as

$$\text{ch}(E) = \text{Tr} \exp \left( \frac{F}{2\pi i} \right), \quad F := \frac{1}{2} \sum_{i,j=1}^d F_{ij} e^i \wedge e^j.$$

Let  $D \subset L$  and  $D^* \subset L^*$  be the lattices  $D \simeq \mathbb{Z}^d$ ,  $D^* \simeq \mathbb{Z}^d$ , spanned by linear combinations of basis  $e_1, \dots, e_d$  and  $e^1, \dots, e^d$  with integer coefficients. Denote by  $\wedge^{\text{even}}(L^*) := \bigoplus_{0 \leq 2n \leq d} \wedge^{2n}(L^*)$  (resp.  $\wedge^{\text{odd}}(L^*) := \bigoplus_{0 \leq 2n+1 \leq d} \wedge^{2n+1}(L^*)$ ) the even (resp. odd) part of the exterior algebra  $\bigoplus_{i=1}^d \wedge^i(L^*)$  over  $\mathbb{R}$ . The corresponding integer part is denoted  $\wedge^{\text{even}}(D^*)$  (resp.  $\wedge^{\text{odd}}(D^*)$ ). Then, we have the identifications:

$$K_0(\mathcal{A}_\theta^d) \simeq \mathbb{Z}^{2^{d-1}} \simeq \wedge^{\text{even}}(D^*), \quad K_1(\mathcal{A}_\theta^d) \simeq \mathbb{Z}^{2^{d-1}} \simeq \wedge^{\text{odd}}(D^*), \quad (8)$$

where recall that  $K_i(\mathcal{A}_\theta^d) \simeq K_i(A_\theta^d)$ ,  $i = 0, 1$ , as we mentioned at the beginning of this section. Therefore, we identify an element  $[E] \in K_0(\mathcal{A}_\theta^d)$  with an even form  $\mu(E) \in \wedge^{\text{even}}(D^*)$ . The following is the Elliott's formula [11]:

$$\text{ch}([E]) = \iota_{e^0} \mu(E), \quad (9)$$

where  $\Theta := \frac{1}{2} \sum_{i,j=1}^d \theta^{ij} e_i \wedge e_j$ . This defines the Chern character map  $\text{ch} : K_0(\mathcal{A}_\theta^d) \rightarrow \wedge_{\text{even}}(L^*)$ , which is in particular injective for noncommutative tori [11]. Note that the leading part  $\text{ch}([E])|_{\wedge^0(L^*)}$  coincides with the trace  $\text{tr}(E) \in \mathbb{R}$ . Rieffel showed that, for any image  $\text{ch}(K_0(\mathcal{A}_\theta^d))$  with positive trace, there exists a Heisenberg module. This, together with the cancelation theorem (Theorem 2.2), implies that:

**Theorem 2.3.** *If the matrix  $\theta^{ij}$  is irrational, then any projective module over  $\mathcal{A}_\theta^d$  is isomorphic to a Heisenberg module.*

Originally, in [69, Theorem 7.3], the parallel statement to Theorem 2.3 is given for  $A_\theta^d$  instead of  $\mathcal{A}_\theta^d$ . The relation between the  $\mathcal{A}_\theta^d$  version (Theorem 2.3) to the  $A_\theta^d$  version ([69, Theorem 7.3]) is given by [69, Proposition 3.2]; for any Heisenberg right  $\mathcal{A}_\theta$ -module  $E$ , one can construct a right  $A_\theta^d$ -module by the completion

$$\text{Pmod-}\mathcal{A}_\theta^d \ni E \mapsto E \otimes_{\mathcal{A}_\theta^d} A_\theta^d \in \text{Pmod-}A_\theta^d. \quad (10)$$

**2.3. Morita equivalence of noncommutative tori.** Next, we discuss Morita equivalence of noncommutative tori [70, 71], where Heisenberg modules played a key role.

Let  $\text{Mod-}\mathcal{A}$  be the category of right modules over a (noncommutative)  $\mathbb{C}$ -algebra  $\mathcal{A}$ . For  $E, F \in \text{Mod-}\mathcal{A}$ , elements in the space  $\text{Hom}_{\text{Mod-}\mathcal{A}}(E, F)$  of morphisms from  $E$  to  $F$  are right  $\mathcal{A}$ -module maps. The space  $\text{Hom}_{\mathcal{A}}(E, F)$  has a right  $\text{End}_{\mathcal{A}}(E)$  action and a left  $\text{End}_{\mathcal{A}}(F)$  action; for  $\phi \in \text{Hom}_{\mathcal{A}}(E, F)$ ,  $e \in \text{End}_{\mathcal{A}}(E)$ ,  $f \in \text{End}_{\mathcal{A}}(F)$  and  $\xi \in E$ ,  $(\phi \circ e)(\xi) := \phi(e(\xi))$  and  $(f \circ \phi)(\xi) := f(\phi(\xi))$  in  $F$ . Thus,  $\text{Hom}_{\mathcal{A}}(E, F)$  forms a  $\text{End}_{\mathcal{A}}(F)$ - $\text{End}_{\mathcal{A}}(E)$  bimodule.

A (noncommutative) algebra  $\mathcal{A}$  is called *Morita equivalent* to an algebra  $\mathcal{B}$  iff  $\text{Mod-}\mathcal{A} \simeq \text{Mod-}\mathcal{B}$ . The following conditions are equivalent [58]:

- i) That  $\mathcal{A}$  is Morita equivalent to  $\mathcal{B}$ .
- ii) There exists a  $\mathcal{A}$ - $\mathcal{B}$  bimodule  $P$  which is projective as both a left  $\mathcal{A}$ -module and a right  $\mathcal{B}$ -module such that

$$\text{End}_{\mathcal{A}}(P) \simeq \mathcal{B}, \quad \text{End}_{\mathcal{B}}(P) \simeq \mathcal{A}.$$

- iii) There exists an element  $E \in \text{Pmod-}\mathcal{A}$  such that  $\text{End}_{\mathcal{A}}(E) \simeq \mathcal{B}$ .

In particular, for a  $\mathcal{A}$ - $\mathcal{B}$  bimodule  $P$  as in ii), the functors

$$(\cdot) \otimes_{\mathcal{A}} P : \text{Mod-}\mathcal{A} \rightarrow \text{Mod-}\mathcal{B}, \quad \text{Hom}_{\mathcal{B}}(\cdot, P) : \text{Mod-}\mathcal{B} \rightarrow \text{Mod-}\mathcal{A},$$

give the equivalence  $\text{mod-}\mathcal{A} \simeq \text{mod-}\mathcal{B}$ .

Rieffel introduced the notion of strongly Morita equivalence, which is a (pre-) $C^*$ -analog of Condition ii) above.

**Definition 2.4** ([66, Definition 2.8]). For a unital pre  $C^*$ -algebra  $\mathcal{A}$ , a right  $\mathcal{A}$ -module  $E$  is called a *right  $\mathcal{A}$ -rigged space* if it is equipped with a map  $\langle \cdot, \cdot \rangle_{\mathcal{A}} : E \otimes E \rightarrow \mathcal{A}$  such that

- i)  $\langle x, y_1 + y_2 \rangle_{\mathcal{A}} = \langle x, y_1 \rangle_{\mathcal{A}} + \langle x, y_2 \rangle_{\mathcal{A}}$  for any  $x, y_1, y_2 \in E$ ,
- ii)  $\langle x, y \cdot a \rangle_{\mathcal{A}} = \langle x, y \rangle_{\mathcal{A}} a$  for any  $x, y \in E$  and  $a \in \mathcal{A}$ ,
- iii)  $\langle x, y \rangle_{\mathcal{A}} = (\langle y, x \rangle_{\mathcal{A}})^*$  for any  $x, y \in E$ ,
- iv)  $\langle x, x \rangle_{\mathcal{A}} \geq 0$  for any  $x \in E$ ,

and the linear span of  $\langle E, E \rangle_{\mathcal{A}} \in \mathcal{A}$ , which forms an ideal of  $\mathcal{A}$ , is dense in  $\mathcal{A}$ .<sup>2</sup> A left  $\mathcal{A}$ -rigged space is also defined in a similar way.

Here, the inequality in iv) is defined for self adjoint elements  $a \in \mathcal{A}$ ,  $a = a^*$ ;  $a \leq a'$  for two self-adjoint elements  $a, a' \in \mathcal{A}$  iff  $a' - a$  belongs to the positive cone, i.e., the spectrum of  $a' - a$  is contained in  $[0, \infty)$  (see [9, p2 and p9],[80]). Note that the spectrum is real for any self-adjoint elements in a Banach  $*$ -algebra  $A$  ([9, Corollary I.3.4 (ii)].)

**Definition 2.5** ([66, Definition 6.10]). For pre- $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , an  $\mathcal{A}$ - $\mathcal{B}$  bimodule  $P$  is called a *strongly Morita equivalence bimodule*<sup>3</sup> if it is a left  $\mathcal{A}$ -rigged and right  $\mathcal{B}$ -rigged space satisfying

- $\langle x, y \rangle_{\mathcal{A}} z = x \langle y, z \rangle_{\mathcal{B}}$  for any  $x, y, z \in P$ ,
- $\langle a \cdot x, a \cdot x \rangle_{\mathcal{A}} \leq \|a\|^2 \langle x, x \rangle_{\mathcal{A}}$  for any  $x \in E$  and  $a \in \mathcal{A}$ ,
- $\langle x \cdot b, x \cdot b \rangle_{\mathcal{B}} \leq \|b\|^2 \langle x, x \rangle_{\mathcal{B}}$  for any  $x \in E$  and  $b \in \mathcal{B}$ .

Two pre- $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  are called *strongly Morita equivalent* iff there exists a strongly Morita equivalence  $\mathcal{A}$ - $\mathcal{B}$  bimodule.

If  $P$  is a strongly Morita equivalence  $A$ - $B$  bimodule of two unital  $C^*$ -algebras  $A$  and  $B$ , then  $P$  is finitely generated projective both as a left  $A$ -module and a right  $B$ -module with  $\text{End}_A(P) = B$  and  $\text{End}_B(P) = A$  [67, Proposition 2.1]. Thus, two strongly Morita equivalent unital  $C^*$ -algebras  $A$  and  $B$  are always Morita equivalent as  $\mathbb{C}$ -algebras. The converse is also true in the sense that any Morita equivalence bimodule is equipped with a strongly Morita equivalence bimodule (see [80, Theorem 15.4.2]). In this reason, hereafter we drop the term 'strongly'.

Let  $O(d, d; \mathbb{Z})$  be the group defined by

$$O(d, d; \mathbb{Z}) := \{g \in \text{Mat}_{2n}(\mathbb{Z}) \mid g^t J g = J\}, \quad J := \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix}. \quad (11)$$

The group  $SO(d, d; \mathbb{Z})$  consists of elements  $g \in O(d, d; \mathbb{Z})$  such that  $\det(g) = 1$ . An  $SO(d, d; \mathbb{Z})$  action on a generic skew-symmetric matrix  $\theta \in \text{Mat}_d(\mathbb{R})$  is defined by

$$g(\theta) := (\mathcal{R}\theta + \mathcal{S})(\mathcal{P}\theta + \mathcal{Q})^{-1}, \quad g := \begin{pmatrix} \mathcal{R} & \mathcal{S} \\ \mathcal{P} & \mathcal{Q} \end{pmatrix} \in SO(d, d, \mathbb{Z}).$$

In fact,  $g(\theta)$  is again a skew-symmetric matrix in  $\text{Mat}_d(\mathbb{R})$  due to the condition  $g \in SO(d, d; \mathbb{Z})$ , and is well-defined iff  $\mathcal{P}\theta + \mathcal{R}$  is invertible. One can define a dense subspace of the space of  $d$  skew-symmetric matrices on which the  $SO(d, d; \mathbb{Z})$  action is well-defined, where it is shown that a noncommutative torus  $\mathcal{A}_\theta^d$  is Morita equivalent to  $\mathcal{A}_{g(\theta)}^d$  if [70] and only if [71, 13] they are related by  $\theta' = g(\theta)$ ,  $g \in SO(d, d, \mathbb{Z})$ .

In order to show that  $\mathcal{A}_\theta^d$  and  $\mathcal{A}_{g(\theta)}^d$  is equivalent for  $g \in SO(d, d; \mathbb{Z})$ , it is enough to show it for each generator  $g \in SO(d, d; \mathbb{Z})$  [70]. The following elements generate the group  $SO(d, d; \mathbb{Z})$

<sup>2</sup>For a  $C^*$ -algebra  $A$ , a right  $A$ -module  $E$  satisfying the conditions i)-iv) with the condition iv'),  $\langle x, x \rangle_A = 0$  iff  $x = 0$ , is called a *pre-Hilbert right  $A$ -module*. A pre-Hilbert right  $A$ -module  $E$  is called a *Hilbert right  $A$ -module* if the norm  $\|\cdot\| : E \rightarrow A$  defined by  $\|x\| := \sqrt{\|\langle x, x \rangle_A\|}$ ,  $x \in E$ , is complete. A Hilbert right  $A$ -module  $E$  is called *full* if it forms a right  $A$ -rigged space  $E$  (see [80]).

<sup>3</sup>The term 'Morita' is omitted in the literatures [69], etc. Also, in [66, Definition 6.10], it was called an imprimitivity bimodule.



[25, 70]:

$$\rho(\mathcal{R}) = \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{R}^{t,-1} \end{pmatrix}, \quad \mathcal{R} \in SL(d; \mathbb{Z}) \quad (12)$$

$$\nu(S) = \begin{pmatrix} 1 & S^{ij} \\ 0 & 1 \end{pmatrix}, \quad S^{ji} = -S^{ij} \in \mathbb{Z}, \quad i, j = 1, \dots, d \quad (13)$$

$$\sigma_k = \begin{pmatrix} 0_k & & 1_k & \\ & 1_q & & 0_q \\ 1_k & & 0_k & \\ & 0_q & & 1_q \end{pmatrix}, \quad k + q = d, \quad 0 < k \leq d, \quad k : \text{even}. \quad (14)$$

To see the Morita equivalence for elements  $\rho(\mathcal{R})$  and  $\nu(S)$  is easy. To see it for  $\sigma_k$  (it is enough to consider only the case  $k = 2$ ), the corresponding Morita equivalence bimodule is constructed explicitly in [70].

The converse, that  $\mathcal{A}_\theta^d$  and  $\mathcal{A}_{\theta'}^d$  is Morita equivalent only if  $\theta' = g\theta$ , is first discussed in [71] by introducing a stronger notion, gauge Morita equivalence, which employs constant curvature connections on Morita equivalence bimodules (see also [47, 13]). This notion turns out to be equivalent to the ordinary Morita equivalence for noncommutative tori [13] essentially since there exists a constant curvature connection for any Morita equivalence bimodule due to Rieffel (Theorem 2.3). If  $\mathcal{A}_\theta^d$  and  $\mathcal{A}_{\theta'}^d$  are Morita equivalent to each other, then so are the  $C^*$ -algebras  $A_\theta^d$  and  $A_{\theta'}^d$ . The strongly Morita equivalence  $A_\theta^d$ - $A_{\theta'}^d$  bimodule is obtained by the completion of the strongly Morita equivalence  $\mathcal{A}_\theta^d$ - $\mathcal{A}_{\theta'}^d$  bimodule via eq.(10). However, the converse is not true for some special cases. See, [81, 54, 13, 12] and references therein, where more precise statements on Morita equivalence of noncommutative tori are developed both for  $\mathcal{A}_\theta^d$  and  $A_\theta^d$  carefully.

**2.4. Categories of projective modules over noncommutative tori.** For a  $C$ -algebra  $\mathcal{A}$ , the full subcategory of  $\text{Mod-}\mathcal{A}$  consisting of finitely generated projective right modules is denoted  $\text{Pmod-}\mathcal{A}$ .

**Definition 2.6.** Let  $\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d$  be the category of finitely generated projective right modules with connections. For two objects  $(E_a, \nabla_a), (E_b, \nabla_b) \in \text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d$ , the space of morphisms is the same as in  $\text{Mod-}\mathcal{A}_\theta^d$ :  $\text{Hom}_{\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d}((E_a, \nabla_a), (E_b, \nabla_b)) := \text{Hom}_{\text{Mod-}\mathcal{A}_\theta^d}(E_a, E_b)$ . The composition in this category is the composition of  $\mathcal{A}_\theta^d$ -bimodule maps.

The category  $\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d$  is equipped with the following structure: for any  $X \in L$ ,  $\xi \in E$  and  $\phi \in \text{Hom}_{\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d}((E_a, \nabla_a), (E_b, \nabla_b))$ ,

$$\nabla_{ba, X}(\phi) := \nabla_{b, X}(\phi(\xi)) - \phi(\nabla_{a, X}(\xi)). \quad (15)$$

Also, it is clear that

**Lemma 2.7.** For  $(E_a, \nabla_a), (E_b, \nabla_b), (E_c, \nabla_c) \in \text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d$  and  $\phi_{ba} \in \text{Hom}_{\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d}((E_a, \nabla_a), (E_b, \nabla_b))$ ,  $\phi_{cb} \in \text{Hom}_{\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d}((E_b, \nabla_b), (E_c, \nabla_c))$ , one has

$$\nabla_{ca}(\phi_{cb} \circ \phi_{ba}) = (\nabla_{cb}(\phi_{cb})) \circ \phi_{ba} + \phi_{cb} \circ (\nabla_{ba}(\phi_{ba})).$$

□

For an element  $(E, \nabla) \in \text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d$  and an isomorphism  $\phi : E' \rightarrow E$  in  $\text{Mod-}\mathcal{A}_\theta^d$ , a connection  $\nabla'$  on  $E'$  is induced as

$$\nabla' := \phi^{-1} \circ \nabla \circ \phi. \quad (16)$$

On the other hand, as we saw in Theorem 2.3, any finitely generated projective module is isomorphic to a Heisenberg module, which is equipped with constant curvature connections. By eq.(16), if  $\nabla$  is a constant curvature connection, the induced connection  $\nabla'$  on  $E'$  is also a constant curvature connection. Thus, any finitely projective module  $E$  over  $\mathcal{A}_\theta^d$  is equipped with a constant curvature connection. Let us conclude this fact in terms of categories.

**Definition 2.8.** Denote by  $\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^d$  the full subcategory of  $\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^d$  consisting of finitely generated projective right modules with connections. The full subcategory of  $\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^d$  consisting of modules with constant curvature connections is denoted  $\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^d$ .

The upper script  $^{st}$  stands for 'standard'; the Heisenberg modules with constant curvature connections are often called *standard* modules, see [63]. This full subcategory  $\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^d$  plays a key role in discussing homological mirror symmetry in section 5.

**Corollary 2.9.** *There exists a surjective map  $\text{Ob}(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^d) \rightarrow \text{Ob}(\text{Pmod}\text{-}\mathcal{A}_\theta^d)$  by forgetting the structure of connections.*  $\square$

For a given (constant curvature) connection  $\nabla$  on  $E$ , any connection  $\nabla'$  on  $E$  is described of the form

$$\nabla'_i = \nabla_i + \phi_i, \quad \phi_i \in \text{End}_{\mathcal{A}_\theta^d}(E).$$

We shall employ the following lemmas later for the case  $\mathcal{A} = \mathcal{A}_\theta^d$ , but the fact itself holds true for any (noncommutative) algebra  $\mathcal{A}$ .

**Lemma 2.10** (See [67, Proposition 2.2]). *For any  $E_a, E_b \in \text{Pmod}\text{-}\mathcal{A}$  and  $\phi_{ba} \in \text{Hom}_{\text{Pmod}\text{-}\mathcal{A}}(E_a, E_b)$ ,  $\phi_{ab} \in \text{Hom}_{\text{Pmod}\text{-}\mathcal{A}}(E_b, E_a)$ , one has*

$$\text{Tr } m(\phi_{ab}, \phi_{ba}) = \text{Tr } m(\phi_{ba}, \phi_{ab}).$$

**Proof.** Any finitely generated projective  $\mathcal{A}$ -module is by definition isomorphic to a module of the form  $p(\mathcal{A}^{\oplus n})$  for sufficiently large  $n$ , where  $p$  is a projection in  $\text{Mat}_n(\mathcal{A})$ . For  $E_a \simeq p_a(\mathcal{A}^{\oplus n_a})$  and  $E_b \simeq p_b(\mathcal{A}^{\oplus n_b})$ , any element in  $\text{Hom}_{\text{Pmod}\text{-}\mathcal{A}}(E_a, E_b)$  is described as

$$p_b \phi p_a, \quad \phi \in \text{Mat}_{n_b \times n_a}(\mathcal{A}).$$

Similarly, any element in  $\text{Hom}_{\text{Pmod}\text{-}\mathcal{A}}(E_b, E_a)$  is described as  $p_a \psi p_b$ ,  $\psi \in \text{Mat}_{n_a \times n_b}(\mathcal{A})$ . As the trace  $\text{Tr}$  on  $\text{Mat}(\mathcal{A})$ , one has

$$\text{Tr } m(p_a \phi p_b, p_b \psi p_a) = \text{Tr } m(p_b \psi p_a, p_a \phi p_b)$$

from which the lemma follows.  $\square$

**Lemma 2.11.** *For  $E_a, E_b \in \text{Pmod}\text{-}\mathcal{A}_\theta^d$ , the map*

$$\text{Tr } m : \text{Hom}_{\text{Pmod}\text{-}\mathcal{A}_\theta^d}(E_b, E_a) \otimes \text{Hom}_{\text{Pmod}\text{-}\mathcal{A}_\theta^d}(E_a, E_b) \rightarrow \mathcal{A}_{\theta_a}^d$$

*is nondegenerate.*

**Proof.** This follows from the Morita equivalence of  $\mathcal{A}_{\theta_a}^d$  with  $\mathcal{A}_\theta^d$  and the construction of Heisenberg modules in [69]. We shall see this explicitly in the case of noncommutative two-tori  $\mathcal{A}_\theta$  in the next subsection.  $\square$

### 2.5. Explicit construction of Heisenberg modules over noncommutative (two-)tori.

In order to discuss homological mirror symmetry, we prefer an explicit description of the space  $\text{Hom}_{\text{Pmod-}\mathcal{A}}(E, F)$  of morphisms in the category  $\text{Pmod-}\mathcal{A}$  for  $\mathcal{A} = \mathcal{A}_\theta^d$ . In this subsection, we first explain that the space  $\text{Hom}_{\text{Pmod-}\mathcal{A}}(E, F)$  is again described by a Heisenberg module over a noncommutative torus  $\mathcal{A}_\theta^d$ , which is Morita equivalent to  $\mathcal{A}_\theta^d$  (Corollary 2.14). After that, we shall concentrate on the case of two-dimensional tori, where the Heisenberg modules are presented explicitly and the composition of morphisms, described again by Heisenberg modules, are constructed explicitly.

Again, let us start from a general noncommutative algebra  $A$ . First, the following is a standard fact (for instance, see [52, p489, 18.25]).

**Lemma 2.12.** *For any finitely generated projective right  $A$ -module  $P \in \text{Pmod-}A$  and  $E \in \text{Mod-}A$ , one has*

$$\text{Hom}_{\text{Mod-}A}(P, E) \simeq E \otimes_A (P^*).$$

□

**Lemma 2.13.** *For a Morita equivalence  $\mathcal{B}$ - $A$  bimodule  $P$  and a finitely generated projective right  $A$ -module  $E$ , the tensor product  $E \otimes_A (P^*)$  is finitely generated and projective as a right  $\mathcal{B}$ -module.*

**Proof.** The tensor product of finitely generated modules is finitely generated. On the other hand,  $(\cdot) \otimes_A (P^*) : \text{Mod-}A \rightarrow \text{Mod-}\mathcal{B}$  induces an equivalence of categories, so a projective module  $E \in \text{Mod-}A$  is sent to be a projective module in  $\text{Mod-}\mathcal{B}$ , where recall that  $E \in \text{Mod-}A$  is projective iff any map  $f : E \rightarrow F$  can have a lift  $f' : E \rightarrow F'$ ,  $f = s \circ f'$  for any surjection  $s : F' \rightarrow F$ . □

These lemmas together with Theorem 2.3 lead:

**Corollary 2.14.** *For a Morita equivalence  $\mathcal{A}_\theta^d$ - $\mathcal{A}_{\theta'}^d$  bimodule  $E$  and  $F \in \text{Pmod-}\mathcal{A}_\theta^d$ , the space  $\text{Hom}_{\text{Pmod-}\mathcal{A}_\theta^d}(E, F)$  is isomorphic to a Heisenberg module over  $\mathcal{A}_{\theta'}^d$ . □*

Here, recall that for any element  $g \in SO(d, d; \mathbb{Z})$  such that  $g(\theta)$  is well-defined, there exists a Morita equivalence  $\mathcal{A}_\theta^d$ - $\mathcal{A}_{g(\theta)}^d$  bimodule and a Morita equivalence  $\mathcal{A}_{g(\theta)}^d$ - $\mathcal{A}_\theta^d$  bimodule. Let us label the Morita equivalence Heisenberg  $\mathcal{A}_{g(\theta)}^d$ - $\mathcal{A}_\theta^d$  bimodule by  $E_{g, \theta}$ . (cf. By the discussion in [71] we see that the constant curvature of  $E_{g, \theta}$  is given by  $\mathcal{P}(\mathcal{Q} + \mathcal{P}\theta)^{-1}$ .) This labeling is useful though it has some overcounting in the sense that  $E_{g, \theta} \simeq E_{g', \theta}$  can happen even if  $g \neq g'$ . We shall see this fact in two-tori case later below. Since  $\text{End}_{\mathcal{A}_\theta^d}(E_{g, \theta}) \simeq \mathcal{A}_\theta^d$  is not decomposed into a direct sum of smaller algebras,  $E_{g, \theta}$  is not decomposed into a direct sum of smaller right  $\mathcal{A}_\theta^d$ -modules. Such a Heisenberg module is called *basic* in [47]. The Morita equivalence theorems in the previous subsection guarantee that any basic Heisenberg  $\mathcal{A}_\theta^d$  module is of the form. On the other hand, since any Heisenberg module becomes a Morita equivalence bimodule, any finitely projective right  $\mathcal{A}_\theta^d$  module is isomorphic to a direct sum of basic Heisenberg right  $\mathcal{A}_\theta^d$  modules  $E_{g, \theta}$ . Thus, to understand the structure of  $\text{Pmod-}\mathcal{A}_\theta^d$ , it is enough to discuss it for these basic modules  $E_{g, \theta}$ .

**Lemma 2.15.** *For two Heisenberg modules  $E_{g_a, \theta}, E_{g_b, \theta} \in \text{Pmod-}\mathcal{A}_\theta^d$ , one has*

$$\text{Hom}(E_{g_a}, E_{g_b}) \simeq E_{g_b} \otimes_{\mathcal{A}_\theta^d} (E_{g_a})^* \simeq E_{g_b g_a^{-1}, g_a \theta}.$$

**Proof.** For a given  $g \in SO(d, d; \mathbb{Z})$ , the Chern character of  $E_{g, \theta}$  determines the isomorphism class, which leads this lemma. □

Next, we would like to construct a bilinear map

$$m : E_{g_c g_b^{-1}, g_b \theta} \otimes E_{g_b g_a^{-1}, g_a \theta} \rightarrow E_{g_c g_a^{-1}, g_a \theta} \quad (17)$$

so that the following diagram commutes:

$$\begin{array}{ccc} m : & \text{Hom}_{\text{Pmod-}\mathcal{A}_\theta^d}(E_{g_b}, E_{g_c}) \otimes \text{Hom}_{\text{Pmod-}\mathcal{A}_\theta^d}(E_{g_a}, E_{g_b}) & \longrightarrow \text{Hom}_{\text{Pmod-}\mathcal{A}_\theta^d}(E_{g_a}, E_{g_c}) \\ & \downarrow & \downarrow \\ m : & E_{g_c g_b^{-1}, g_b \theta} \otimes E_{g_b g_a^{-1}, g_a \theta} & \longrightarrow E_{g_c g_a^{-1}, g_a \theta}. \end{array}$$

Here, the vertical arrows are isomorphisms. The map (17) is given by constructing the tensor product  $E_{g_c g_b^{-1}, g_b \theta} \otimes_{\mathcal{A}_\theta^d} E_{g_b g_a^{-1}, g_a \theta} \simeq E_{g_c g_a^{-1}, g_a \theta}$  explicitly so that the above diagram commutes.

When it is defined, an isomorphism  $E_{g_b g_a^{-1}, g_a \theta} \rightarrow \text{Hom}_{\text{Pmod-}\mathcal{A}_\theta^d}(E_{g_a}, E_{g_b})$  is given by

$$E_{g_a} \rightarrow \phi_{ba} \otimes_{\mathcal{A}_\theta^d} E_{g_a} \subset E_{g_b} \quad (18)$$

for  $\phi_{ba} \in E_{g_b g_a^{-1}, g_a \theta}$ . Then, the linear map

$$\nabla_{ab} : E_{g_b g_a^{-1}, g_a \theta} \rightarrow E_{g_b g_a^{-1}, g_a \theta}$$

is induced from  $\nabla_{ab}$  on  $\text{Hom}_{\text{Pmod-}\mathcal{A}_\theta^d}(E_{g_a, \theta}, E_{g_b, \theta})$  in eq.(15).

For higher dimensional noncommutative tori, a class of Heisenberg modules (corresponding to line bundles) and the product (17) are constructed explicitly in [34].

Now, let us concentrate on two-dimensional noncommutative tori [63, 32]. We follow the arguments and notation in [32] (but see Remark 2.18). For  $d = 2$ , the group  $SO(d, d; \mathbb{Z})$  reduces to  $SO(2, 2; \mathbb{Z}) \simeq SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ . Clearly, the generators (12) form one side of  $SL(2, \mathbb{Z})$ . The other generators (13) and (14) then commute with the generator (12) and form another  $SL(2, \mathbb{Z})$ . More explicitly, the embedding  $SL(2, \mathbb{Z}) \rightarrow SO(2, 2; \mathbb{Z})$  is given by

$$\begin{pmatrix} r & s \\ p & q \end{pmatrix} \mapsto \begin{pmatrix} r \cdot \mathbf{1}_2 & s \cdot \mathbf{J} \\ -p \cdot \mathbf{J} & q \cdot \mathbf{1}_2 \end{pmatrix}, \quad \mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This  $SL(2, \mathbb{Z})$  acts on  $\theta := \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$  by

$$g\theta = \frac{r\theta + s}{p\theta + q}, \quad g = \begin{pmatrix} r & s \\ p & q \end{pmatrix} \in SL(2, \mathbb{Z}) \quad (19)$$

and the  $SL(2, \mathbb{Z})$  consisting of (12) acts freely on  $\theta$ . Thus, we concentrate on this  $SL(2, \mathbb{Z})$  of this side and denote by  $g$  an element in  $SL(2, \mathbb{Z})$ .

The Heisenberg modules over a noncommutative two-torus  $\mathcal{A}_\theta^2$  are given as follows. For each  $g = \begin{pmatrix} r & s \\ p & q \end{pmatrix} \in SL(2, \mathbb{Z})$ , if  $p = 0$  we just set  $E_{g, \theta} := \mathcal{A}_\theta^2$ . If  $p \neq 0$ , the Heisenberg module  $E_{g, \theta}$  over  $\mathcal{A}_\theta^2$  is given by the Schwartz space  $\mathcal{S}(\mathbb{R} \times (\mathbb{Z}/p\mathbb{Z}))$ . The right action of  $\mathcal{A}_\theta^2$  is defined by

$$(fU_1)(x, j) = f(x, j)e^{2\pi i(x - j\frac{s}{p})}, \quad (fU_2)(x, j) = f\left(x - \frac{q}{p} - \theta, j - 1\right) \quad (20)$$

for  $f \in E_{g, \theta}$ , where  $x \in \mathbb{R}$  and  $j \in \mathbb{Z}/p\mathbb{Z}$ . One can check directly that  $U_1$  and  $U_2$  in fact satisfy the noncommutativity relation (1). Note that the Heisenberg module  $E_{g, \theta}$  depend only on  $p$  and  $q$  and is independent of  $r$  and  $s$ . If  $(p, q) = (p', q')$  for two elements  $g = \begin{pmatrix} r & s \\ p & q \end{pmatrix}, g' = \begin{pmatrix} r' & s' \\ p' & q' \end{pmatrix} \in SL(2, \mathbb{Z})$ , then there exists an integer  $m \in \mathbb{Z}$  such that  $(r', s') = m(p, q)$ . The endomorphism algebra

$\text{End}_{\mathcal{A}_\theta^2} E_{g,\theta}$  is isomorphic to the noncommutative two-torus  $\mathcal{A}_{g\theta}^2$ . One can find generators  $Z_1, Z_2$  of its right action as

$$(Z_1 f)(x, j) = e^{2\pi i(\frac{x}{q+p\theta} - \frac{j}{p})} f(x, j), \quad (Z_2 f)(x, j) = f\left(x - \frac{1}{p}, j - r\right). \quad (21)$$

These generators satisfy the following relation

$$Z_1 Z_2 = e^{-2\pi i(g\theta)} Z_2 Z_1, \quad (22)$$

where one sees that the replacement of  $(r, s)$  by  $(r', s') = (r, s) + m(p, q)$  leads to  $g'\theta = g\theta + m$  and gives isomorphic algebras  $\mathcal{A}_{g'\theta}^2 \simeq \mathcal{A}_{g\theta}^2$ . These  $E_{g,\theta}$  complete the list of all basic Heisenberg  $\mathcal{A}_\theta^2$  modules.

The Heisenberg module  $E_{g,\theta}$  is equipped with the following constant curvature connection:

$$\nabla_1 = \frac{\partial}{\partial x} - \frac{2\pi i \beta}{q + p\theta}, \quad \nabla_2 = \frac{2\pi i p}{q + p\theta} x - \frac{2\pi i \alpha}{q + p\theta}, \quad \alpha, \beta \in \mathbb{R}, \quad (23)$$

where the curvature is  $[\nabla_1, \nabla_2] = 2\pi i p / (q + p\theta)$ . In eq.(23),  $\alpha$  and  $\beta$  parameterize the moduli of constant curvature connections [8, 47]. By gauge transformation  $\nabla_i \rightarrow (Z_j)^{-1} \nabla_i Z_j$ , we have  $\alpha \sim \alpha + 1$  and  $\beta \sim \beta + 1$ . The Chern character turns out to be

$$\text{ch}(E_{g,\theta}) = \text{Tr} \exp\left(\frac{[\nabla_1, \nabla_2] dx_1 \wedge dx_2}{2\pi i}\right) = |q + p\theta| + \frac{|q + p\theta|}{q + p\theta} p,$$

that is,  $\text{rank}(E_{g,\theta}) = \text{Tr}(\mathbf{1}_{\text{End}_{T_2}(E_{g,\theta})}) = |q + p\theta|$  and first Chern class is  $(|q + p\theta| / (q + p\theta))p$ . This implies that  $\mu(E_{g,\theta}) = \pm(q + p dx_1 \wedge dx_2) \in K_0(\mathcal{A}_\theta^2)$  by the Elliott's formula (9), where the sign is determined as  $\pm = |q + p\theta| / (q + p\theta)$ .

Next, we would like to define the space of morphisms  $\text{Hom}(E_{g_a,\theta}, E_{g_b,\theta})$  between two Heisenberg modules  $E_{g_a,\theta}$  and  $E_{g_b,\theta}$ , where  $g_a, g_b \in SL(2, \mathbb{Z})$ . Denote

$$g_{ba} = g_b g_a^{-1} = \begin{pmatrix} r_b & s_b \\ p_b & q_b \end{pmatrix} \begin{pmatrix} q_a & -s_a \\ -p_a & r_a \end{pmatrix} = \begin{pmatrix} r_{ba} & s_{ba} \\ p_{ba} & q_{ba} \end{pmatrix}. \quad (24)$$

If  $p_{ba} = 0$ , we define  $\text{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) := \mathcal{A}_{\theta_a}^2$ , where  $\theta_a := g_a \theta$ . If  $p_{ab} \neq 0$ , we define  $\text{Hom}(E_{g_a,\theta}, E_{g_b,\theta}) \simeq E_{g_{ba},\theta_a}$  as  $\mathcal{A}_{\theta_b}^2$ - $\mathcal{A}_{\theta_a}^2$  bimodules. Here,  $E_{g_{ba},\theta_a} := \mathcal{S}(\mathbb{R} \times (\mathbb{Z}/p_{ba}\mathbb{Z}))$ , where the left  $\mathcal{A}_{\theta_b}^2$  action and the right  $\mathcal{A}_{\theta_a}^2$  action are defined by eq.(20) and eq.(21) with the replacement of  $g$  and  $\theta$  by  $g_{ba}$  and  $\theta_a$ , respectively. However, we prefer to rescale elements  $\phi_{ba} \in E_{g_{ba},\theta_a}$  such as

$$\phi'_{ba}(x, j) := \phi_{ba}\left(\frac{x}{q_a + p_a \theta}, j\right). \quad (25)$$

We denote by  $E_\theta(g_b, g_a)$  the  $\mathcal{A}_{\theta_b}^2$ - $\mathcal{A}_{\theta_a}^2$  bimodule obtained by the rescaling of  $E_{g_{ba},\theta_a}$ . Elements in  $\phi'_{ba} \in E_\theta(g_b, g_a)$  are again denoted  $\phi_{ba}$ , etc.

The bilinear map (17) is then constructed as follows. If  $p_{ba} = p_{cb} = 0$ , this tensor product is just the usual product in  $\mathcal{A}_{\theta_a}^2$ . If  $p_{ba} = 0$  and  $p_{cb} \neq 0$ , it is given by the right action of  $\mathcal{A}_{\theta_b}^2 \simeq \mathcal{A}_{\theta_a}^2$ . In the case  $p_{ba} \neq 0$  and  $p_{cb} = 0$ , it is given by the left action of  $\mathcal{A}_{\theta_c}^2 \simeq \mathcal{A}_{\theta_b}^2$ . In the case  $p_{cb} p_{ba} \neq 0$ , if  $p_{ca} = 0$ , the product  $m_2 : E_\theta(g_c, g_b) \otimes E_\theta(g_b, g_a) \rightarrow E_\theta(g_c, g_a)$  is given by

$$m_2(\phi_{cb}, \phi_{ba}) = \frac{1}{q_a + p_a \theta} \sum_{(n_1, n_2) \in \mathbb{Z}^2} (U_1)^{n_1} (U_2)^{n_2} \sum_{j \in \mathbb{Z}/|p_{cb}|} \int dx \phi_{cb}(x, -q_{cb} j) (\phi_{ba}(x, j) (U_2)^{-n_2} (U_1)^{-n_1}), \quad (26)$$

where  $E_\theta(g_c, g_a) \simeq \mathcal{A}_{\theta_c}^2 \simeq \mathcal{A}_{\theta_a}^2$ . Then, for the remaining generic case  $p_{cb}p_{ba}p_{ca} \neq 0$ , it is given by

$$m_2(\phi_{cb}, \phi_{ba})(x, j) = \sum_{u \in \mathbb{Z}} \phi_{cb} \left( x + \frac{q_c + p_c \theta}{p_{bc}} \left( u - \frac{p_{cb}}{p_{ca}} j \right), u \right) \cdot \phi_{ba} \left( x - \frac{q_a + p_a \theta}{p_{ba}} \left( u - \frac{p_{cb}}{p_{ca}} j \right), -r_{ba} u + j \right). \quad (27)$$

One sees that this is essentially the pointwise product in  $\mathcal{S}(\mathbb{R})$ , with a summation which runs over  $u \in \mathbb{Z}$  corresponding to translations on  $\mathbb{R}$ .

The linear map  $\nabla_{ba} : E_\theta(g_b, g_a) \rightarrow E_\theta(g_b, g_a)$  corresponding to eq.(15) can also be given explicitly [32].

**Remark 2.16.** For any  $g_a, g_b, \dots$  with fixed  $(p_a, q_a), (p_b, q_b), \dots$ , the bimodule structure of  $E_\theta(g_b, g_a)$  and their composites do not depend on the choices of  $(r_a, s_a), (r_b, s_b), \dots$ . First, for  $g_a, g_b$  with fixed  $(p_a, q_a), (p_b, q_b)$ ,  $p_{ba}$  is unique, and  $q_{ba}$  and  $r_{ba}$  are unique up to  $\mathbb{Z}/p_{ba}\mathbb{Z}$  (see eq.(24)). This, together with the formula  $q_{ba} + p_{ba}\theta_a = (q_b + p_b\theta)/(q_a + p_a\theta)$ , shows that the bimodule structure is independent of the choices. One can check similar facts for the formula of products (26) (27).

**Remark 2.17.** The Heisenberg module  $E_\theta(g_b, g_a)$  in fact defines a strongly Morita equivalence bimodule in the sense of Definition 2.5. First, for any  $g_a, g_b \in SL(2, \mathbb{Z})$ , there exists a canonical isomorphism  $\dagger : E_\theta(g_b, g_a) \rightarrow E_\theta(g_b, g_a)$ . If  $p_{ba} = 0$ , then  $E_\theta(g_b, g_a) \simeq \mathcal{A}_{\theta_a}^2 \simeq \mathcal{A}_{\theta_b}^2$  and  $u^\dagger := u^*$ ,  $u \in \mathcal{A}_{\theta_a}^2$ , the star conjugation. If  $p_{ba} \neq 0$ , it is given by

$$(\phi_{ba}^\dagger)(x, j) := \overline{(\phi_{ba})(x, -r_{ab}j)} \quad (28)$$

for any  $\phi_{ba} \in E_\theta(g_b, g_a)$ . Using this operation, for  $\phi, \phi' \in E_\theta(g_b, g_a)$ , define the inner products by

$$\langle \phi', \phi \rangle_{\mathcal{A}_{\theta_a}^2} := m_2((\phi')^\dagger, \phi), \quad \langle \phi', \phi \rangle_{\mathcal{A}_{\theta_b}^2} := m_2(\phi', \phi^\dagger).$$

One can check that these inner products satisfy the conditions in Definitions 2.4 and 2.5. Lemma 2.11 is also checked directly, which implies that  $\langle \phi, \phi \rangle_{\mathcal{A}_{\theta_a}^2} = 0$  iff  $\phi = 0$ , etc. See also [68, section 1.3].

**Remark 2.18.** In [32], the structure of the categories of Heisenberg *left* modules is discussed instead of right modules here in order to compare it to the corresponding Fukaya category as in subsection 5.2. The relation between the notations here and those in [32] is obtained by  $\theta \leftrightarrow -\theta$  and  $p \leftrightarrow -p$ .

### 3. MIRROR SYMMETRY OF TORI

Mirror symmetry is now interpreted in various ways. We define mirror symmetry of flat tori in a modern framework called generalized geometry [24].

**3.1. Generalized geometry.** Let  $M$  be a real  $2d$ -dimensional manifold. If  $M$  is equipped with a linear map  $I : \Gamma(TM) \rightarrow \Gamma(TM)$  on the space of smooth sections  $\Gamma(TM)$  of the tangent bundle  $TM$  such that  $I^2 = -1$ ,  $(M, I)$  is called an almost complex manifold, where  $I$  is the almost complex structure. If the almost complex structure  $I$  is integrable, i.e., the  $+i$  eigenspace of  $I$  in  $\Gamma(TM) \otimes \mathbb{C}$  is closed with respect to the Lie bracket  $[\cdot, \cdot]$  in  $\Gamma(TM) \otimes \mathbb{C}$ , then  $(M, I)$  forms a complex manifold.

On the other hand, given a two form  $\omega \in \Omega^2(M) := \Gamma(\wedge^2 T^*M)$ ,  $(M, \omega)$  is called a symplectic manifold iff the two-form  $\omega$  is *nondegenerate*, i.e.,  $(\omega)^d \in \Omega^{2d}(M)$  is a nowhere vanishing  $2d$ -form,

and is closed. Note that  $\omega$  can be regarded as a linear map  $\omega : \Gamma(TM) \rightarrow \Gamma(T^*M)$  by  $X \mapsto \iota_X(\omega)$ , where  $\iota_X$  is the inner derivation of  $X \in \Gamma(TM)$ .

Now, in order to treat complex manifolds and symplectic manifolds in a uniform way, extend the canonical pairing between  $\Gamma(TM)$  and  $\Gamma(T^*M)$  to a quadratic form

$$\langle \cdot, \cdot \rangle : \Gamma(TM \oplus TM^*) \otimes \Gamma(TM \oplus TM^*) \rightarrow C^\infty(M) \quad (29)$$

defined by  $\langle X + \alpha, Y + \beta \rangle := \alpha(Y) + \beta(X)$  for any  $X, Y \in \Gamma(TM)$  and  $\alpha, \beta \in \Gamma(T^*M)$ .

For an almost complex manifold  $(M, I)$ , if we consider the adjoint maps  $I^* : \Gamma(T^*M) \rightarrow \Gamma(T^*M)$ ,

$$(I^*\alpha)(X) := \alpha(I(X)),$$

of the almost complex structure  $I : \Gamma(TM) \rightarrow \Gamma(TM)$ , then one sees that  $\mathcal{I}_I := I \oplus (-I^*) : \Gamma(TM \oplus TM^*) \rightarrow \Gamma(TM \oplus TM^*)$  preserves the quadratic form (29):

$$\langle \mathcal{I}(X + \alpha), \mathcal{I}(Y + \beta) \rangle = \langle X + \alpha, Y + \beta \rangle.$$

(cf.  $-(I^*\alpha)(I(X)) = -\alpha(I^2(X)) = \alpha(X)$ .) On the other hand, for a nondegenerate two-form  $\omega \in \Omega^2(M)$  and the associated linear map  $\omega : \Gamma(TM) \rightarrow \Gamma(T^*M)$ , we can define  $\omega^* : \Gamma(T^*M) \rightarrow \Gamma(TM)$  by

$$(\omega(Y))(\omega^*(\alpha)) = \alpha(Y)$$

for any  $\alpha \in \Gamma(T^*M)$  and  $Y \in \Gamma(TM)$ . Then,  $\mathcal{I}_\omega := -\omega - \omega^* : \Gamma(TM \oplus TM^*) \rightarrow \Gamma(TM \oplus TM^*)$  again preserves the quadratic form (29), where the natural lifts of  $\omega : \Gamma(TM) \rightarrow \Gamma(T^*M)$  and  $\omega^* : \Gamma(T^*M) \rightarrow \Gamma(TM)$  to those on  $\Gamma(TM) \oplus \Gamma(T^*M)$  are denoted by the same letters  $\omega$  and  $\omega^*$ .<sup>4</sup> One sees that the condition that  $(\mathcal{I}_I)^2 = -1$  is equivalent to that  $I^2 = -1$ . Similarly, the condition that  $(\mathcal{I}_\omega)^2 = -1$  is equivalent to that  $\omega : \Gamma(TM \otimes TM) \rightarrow C^\infty(M)$  is skew-symmetric. Thus, we arrive at the following definition:

**Definition 3.1** (Generalized almost complex manifold [24]). A *generalized almost complex structure*  $\mathcal{I}$  on a smooth manifold  $M$  is a linear map  $\mathcal{I} : \Gamma(TM \oplus T^*M) \rightarrow \Gamma(TM \oplus T^*M)$  which preserves the quadratic form (29) and satisfies  $(\mathcal{I})^2 = -1$ .

We prepare terminologies of Lie algebroids (see [56]) to define integrability conditions.

**Definition 3.2** (Courant bracket). A *Courant bracket* on  $\Gamma(TM \oplus TM^*)$  is a skew symmetric bilinear map  $[\cdot, \cdot] : \Gamma(TM \oplus TM^*) \otimes \Gamma(TM \oplus TM^*) \rightarrow \Gamma(TM \oplus TM^*)$  given by

$$[X + \alpha, Y + \beta] = [X, Y] + \iota_X d\beta - \iota_Y d\alpha + \frac{1}{2}d(\iota_Y \alpha - \iota_X \beta)$$

for any  $X + \alpha, Y + \beta \in \Gamma(TM \oplus TM^*)$ .

Note that this bracket does not satisfy the Jacobi identity, so  $\Gamma(TM \oplus TM^*)$  does not form a Lie algebra with respect to the Courant bracket. A systemization of  $\Gamma(TM \oplus TM^*)$  with the Courant bracket leads to the axiom of Courant algebroids [55].

**Definition 3.3** (Lie algebroid [65]). A vector bundle  $\mathcal{L} \rightarrow M$  on a smooth manifold  $M$  is called a *Lie algebroid* if  $\mathcal{L}$  is equipped with a Lie bracket  $[\cdot, \cdot] : \Gamma(\mathcal{L}) \otimes \Gamma(\mathcal{L}) \rightarrow \Gamma(\mathcal{L})$  and a bundle map  $\alpha : \Gamma(\mathcal{L}) \rightarrow \Gamma(TM)$ , called an *anchor map*, satisfying the following conditions:

- $\alpha$  is a Lie algebra homomorphism, i.e.,  $\alpha([X, Y]) = [\alpha(X), \alpha(Y)]$  for any  $X, Y \in \Gamma(\mathcal{L})$ ,
- $[X, fY] = f[X, Y] + (\alpha(X)f)Y$  for any  $X, Y \in \Gamma(\mathcal{L})$  and  $f \in C^\infty(M)$ .

<sup>4</sup>The minus sign for  $\omega$  and  $\omega^*$  is just for conventional reason. See the matrix expression in Example 3.6.

Now, suppose we are given a generalized almost complex manifold  $(M, \mathcal{I})$ . Then, since  $(\mathcal{I})^2 = -1$ , we can consider the  $\pm i$  eigenspace  $\mathcal{L}_\pm$  of  $(TM \oplus T^*M) \otimes \mathbb{C}$ , i.e., the direct sum decomposition  $\mathcal{L}_+ \oplus \mathcal{L}_- = (TM \oplus T^*M) \otimes \mathbb{C}$  as vector bundles over  $M$  such that  $\mathcal{I}(\mathcal{L}_\pm) = \pm i \cdot \mathcal{L}_\pm$ . The Courant bracket in  $\Gamma(TM \oplus T^*M)$  is extended to that in  $\Gamma(TM \oplus T^*M) \otimes \mathbb{C}$ . If  $\Gamma(\mathcal{L}_+)$  is closed with respect to the Courant bracket, then so is  $\Gamma(\mathcal{L}_-)$ , and vice versa, since  $\Gamma(\mathcal{L}_\pm)$  are complex conjugate to each other. In this case,  $\mathcal{I}$  is called *integrable*. The vector bundle  $\mathcal{L}_+ \rightarrow M$  (or  $\mathcal{L}_- \rightarrow M$ ) then forms a Lie algebroid iff  $\mathcal{I}$  is integrable.

**Definition 3.4** (Generalized complex manifold [24]). A *generalized complex manifold*  $(M, \mathcal{I})$  is a generalized almost complex manifold  $(M, \mathcal{I})$  such that  $\mathcal{I}$  is integrable.

One can see that, for an almost complex manifold  $(M, I)$ , the condition that  $I$  is integrable is equivalent to that  $\mathcal{I}_I$  is integrable. Similarly, a nondegenerate two form  $\omega \in \Omega^2(M)$  is a closed two form iff  $\mathcal{I}_\omega$  is integrable. Hence, complex manifolds and symplectic manifolds actually give typical examples of generalized complex manifolds.

We discuss tori with flat background, so the integrability of any almost generalized complex structure is automatically satisfied.

For later convenience, let us discuss a local expression of these structures in terms of basis. We choose a basis  $(e_1, \dots, e_d; e^1, \dots, e^d)$  of  $\Gamma(TM \oplus T^*M)$ , where  $e_a, a = 1, \dots, d$ , are bases of  $\Gamma(TM)$  and  $e^a, a = 1, \dots, d$ , are bases of  $\Gamma(TM^*)$  such that  $e^a(e_b) = \delta_b^a$ . The condition that  $\mathcal{I} : \Gamma(TM \oplus T^*M) \rightarrow \Gamma(TM \oplus T^*M)$  is an almost generalized complex manifold structure is expressed as

$$\mathcal{I}^2 = -1, \quad \mathcal{I}^t q \mathcal{I} = q, \quad q := \begin{pmatrix} \mathbf{0}_d & \mathbf{1}_d \\ \mathbf{1}_d & \mathbf{0}_d \end{pmatrix}.$$

We give some examples of generalized complex manifolds.

**Example 3.5** (A complex manifold  $(M, I)$ ). Express  $I(e_a) = e_b I_a^b$ , where  $I_a^b$  is locally a function in  $C^\infty(M)$ . The corresponding matrix is also denoted  $I := \{I_a^b\}_{ab}$ . By definition, it satisfies  $I^2 = -1$  as a matrix. On the other hand, express  $I^*(e^a) = e^b I_b^a$  and  $I^* := \{I_a^b\}_{ab}$ . By definition,  $I_b^a = e^a(I(e_b)) = I^*(e^a)(e_b) = I_b^a$  and hence

$$I^* = I^t$$

as matrices, where  $t$  indicates the transpose. The corresponding generalized complex structure  $\mathcal{I}_I$  is expressed as

$$\mathcal{I}_I := \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^t \end{pmatrix}.$$

**Example 3.6** (Symplectic manifold  $(M, \omega)$ ). Similarly, for  $\omega(e_a, e_b) =: \omega_{ab}$ , by  $\omega(e_a, \omega^*(e^b)) = e^b(e_a) = \delta_a^b$  one obtains

$$\mathcal{I}_\omega := \begin{pmatrix} \mathbf{0} & -\omega^{-1} \\ -\omega^t & \mathbf{0} \end{pmatrix}.$$

One sees that the condition  $(\mathcal{I}_\omega)^2 = -1$  is equivalent to  $\omega^t = -\omega$ .

**Example 3.7** (B-field transformation). In local matrix expression, let us consider the following transformation on  $(TM \oplus T^*M)$ :

$$\begin{pmatrix} \mathbf{1}_d & \mathbf{0}_d \\ B & \mathbf{1}_d \end{pmatrix}.$$



This means that the matrix  $B$  defines a two-form  $B \in \Gamma(\wedge^2 T^*M)$ . One sees that this transformation is invertible, where the inverse transformation is  $\begin{pmatrix} 1_d & 0_d \\ -B & 1_d \end{pmatrix}$ , and preserves the quadratic form  $q$ . Thus, for a given generalized almost complex manifold  $(M, \mathcal{I})$ ,

$$\mathcal{I}(B) := \begin{pmatrix} 1_d & 0_d \\ B & 1_d \end{pmatrix} \mathcal{I} \begin{pmatrix} 1_d & 0_d \\ -B & 1_d \end{pmatrix}$$

defines a new generalized almost complex manifold structure. In particular,  $\mathcal{I}(B)$  is integrable and  $(M, \mathcal{I}(B))$  forms a generalized complex manifold iff  $B$  is a closed two-form [24]. In this case,  $\mathcal{I}(B)$  is called a *B-field transform* of  $\mathcal{I}$ . Given a complex manifold  $(M, I)$ , the *B-field transform* of  $\mathcal{I}_I$  is of the form

$$\begin{pmatrix} 1_d & 0 \\ B & 1_d \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix} \begin{pmatrix} 1_d & 0 \\ -B & 1_d \end{pmatrix} = \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix}. \quad (30)$$

On the other hand, given a symplectic manifold  $(M, \omega)$ , the *B-field transform* of  $\mathcal{I}_\omega$  is of the form

$$\begin{pmatrix} 1_d & 0 \\ B & 1_d \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 1_d & 0 \\ -B & 1_d \end{pmatrix} = \begin{pmatrix} \omega^{-1} B & -\omega^{-1} \\ \omega + B\omega^{-1} B & -B\omega^{-1} \end{pmatrix}. \quad (31)$$

**Remark 3.8.** The equation (30) implies that the *B-transformation* preserves the complex structure  $I$ ,  $\mathcal{I}_I(B) = \mathcal{I}$ , iff  $B$  is a  $(1, 1)$ -form, see Lemma 3.11 iii) in the next subsection. If  $B$  is not a  $(1, 1)$ -form, then the  $\mathcal{I}_I(B)$  no more defines a complex structure. This  $\mathcal{I}_I(B)$  is believed to correspond to gerby deformation of the complex structure  $I$  [1].

Next, we discuss a generalization of Kähler manifolds in this framework. Recall that  $(M, I, \omega)$  is called a *Kähler manifold* iff  $(M, I)$  is a complex manifold,  $(M, \omega)$  is a symplectic manifold, and  $G := \omega(I \otimes 1) : \Gamma(TM) \otimes \Gamma(TM) \rightarrow C^\infty(M)$  defines a Riemannian metric, i.e.,  $G$  is symmetric and positive definite. In the expression where we regard the symplectic two form as a linear map  $\omega : \Gamma(TM) \rightarrow \Gamma(T^*M)$ , the metric is given by

$$G(\xi, \eta) = ((\omega \circ I)(\xi))(\eta).$$

In local matrix expression  $I(e_a) = e_b I^b_a$ ,  $\omega = \{\omega_{ab} = \omega(e_a, e_b)\}_{ab}$  and  $G := \{G_{ab} = G(e_a, e_b)\}_{ab}$ , the condition that the metric  $G$  is symmetric is

$$G = I^t \omega = -\omega I. \quad (32)$$

This implies that the Kähler form  $\omega$  is a  $(1, 1)$ -form with respect to the complex structure  $I$ , see Lemma 3.11 iii).

**Definition 3.9** (Generalized Kähler manifold [24]). For a smooth manifold  $M$  with two given generalized complex manifold structures  $(M, \mathcal{I}_+)$  and  $(M, \mathcal{I}_-)$ ,  $(M, \mathcal{I}_+, \mathcal{I}_-)$  is called a *generalized Kähler manifold* if  $\mathcal{I}_+$  and  $\mathcal{I}_-$  commute with each other,  $\mathcal{I}_+ \mathcal{I}_- = \mathcal{I}_- \mathcal{I}_+$ , and

$$\mathcal{G} := -\langle \cdot, \mathcal{I}_+ \mathcal{I}_- (\cdot) \rangle : \Gamma(TM \oplus T^*M) \otimes \Gamma(TM \oplus T^*M) \rightarrow C^\infty(M) \quad (33)$$

defines a positive definite bilinear map on  $\Gamma(TM \oplus T^*M)$ .

For a Kähler manifold  $(M, I, \omega)$ ,  $\mathcal{I}_I$  and  $\mathcal{I}_\omega$  commute with each other since  $\omega$  is a  $(1, 1)$ -form (32). Thus, a Kähler manifold  $(M, I, \omega)$  is an example of a generalized Kähler manifold  $(M, \mathcal{I}_I, \mathcal{I}_\omega)$ .

A general expression of a generalized Kähler manifold structure  $(M, \mathcal{I}_+, \mathcal{I}_-)$  is known [39, 40, 24]. For a given smooth manifold  $M$ , any generalized Kähler manifold structure  $\mathcal{I}_\pm$  is described in the local matrix expression as

$$\mathcal{I}_\pm := \frac{1}{2} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ B & \mathbf{1} \end{pmatrix} \begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+ \pm I_-)^t \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -B & \mathbf{1} \end{pmatrix}, \quad (34)$$

where  $(M, I_+, \omega_+)$  and  $(M, I_-, \omega_-)$  are two Kähler manifold structures and  $B \in \Omega^2(M)$  is a closed two form called the  $B$ -field.

The case  $I = I_+ = I_-$  and  $B = 0$  corresponds to that  $(M, I, \omega)$  is a Kähler manifold, where  $\omega = \omega_+ = \omega_-$ . The condition that  $\mathcal{I}_+$  commutes with  $\mathcal{I}_-$  correspond to that the bilinear form  $G = \omega(I \otimes \mathbf{1})$  is symmetric, i.e.,  $\omega$  is a  $(1, 1)$ -form. For the case  $I = I_+ = I_-$ , let us turn on  $B \neq 0$ . If  $B$  satisfies  $BI + I^t B = 0$ , i.e.,  $B$  is a  $(1, 1)$ -form (see Lemma 3.11), then  $I$  is preserved as we saw in eq.(30), though  $\omega$  is changed as in eq.(31). Then, we may think of the  $(1, 1)$ -form  $\omega - iB$  as a complexified Kähler form (see Definition 3.13). If  $B$  is not a  $(1, 1)$ -form, the deformation  $\mathcal{I}_I(B)$  of the complex structure  $I$  is expected to describe a gerby deformation in the sense, for instance, of Barannikov-Kontsevich [1]. On the other hand, the  $I_+ \neq I_-$  is expected to describe noncommutative deformation of a complex manifold [39]. An attempt to understand these deformation should be to consider some category associated to a generalized complex manifold.

**3.2. Local calculation for complex and Kähler manifolds.** In this subsection, we discuss some details on local structures of complex and Kähler manifolds.

For a given complex structure  $I$ , one can consider the  $\pm i$ -eigenspace  $L_\pm$  of  $I$  in  $TM \otimes \mathbb{C}$  which is described locally by  $L_\pm \in Mat_{2d \times d}(C^\infty(M))$  such that  $L = (L_+, L_-) \in Mat_{2d \times 2d}(C^\infty(M))$  satisfies

$$IL = LJ_0, \quad J_0 := \begin{pmatrix} i \cdot \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -i \cdot \mathbf{1} \end{pmatrix}.$$

As above, we denote by the same notation  $L_\pm$  the matrices and the corresponding vector spaces. We prefer another convention; since  $I^2 = -1$ , the transpose  $I^t$  also satisfies  $(I^t)^2 = -1$  and hence has its  $\pm i$ -eigenspace  $L_\pm^*$ . Thus,

$$I^t L^* = L^* J_0.$$

Namely,  $(e^1, \dots, e^{2d})L_\pm^*$  defines the  $\pm i$  eigenvector space with respect to  $I^t$ , which implies  $L_-^* = \bar{L}_+^*$ , the complex conjugate of  $L_-^*$ . Then  $I^t = L^* J_0 (L^*)^{-1}$ , and  $(I^t)^2 = -1$ , which implies that  $L^*$  is at least nondegenerate. These facts lead that, by an appropriate choice of basis  $(e^1, \dots, e^d)$ , one can express  $L_\pm^*$  locally as

$$L_+^* = \begin{pmatrix} \mathbf{1} \\ \tau \end{pmatrix}, \quad L_-^* = \begin{pmatrix} \mathbf{1} \\ \bar{\tau} \end{pmatrix},$$

where  $\tau \in Mat_{n \times n}(C^\infty(M))$  and  $\bar{\tau}$  is the complex conjugate. Denote  $\text{Im}(\tau) := \tau_I$ ,  $\text{Re}(\tau) := \tau_R$ , then

$$(L^*)^{-1} = \begin{pmatrix} -(2i\tau_I)^{-1}\bar{\tau} & (2i\tau_I)^{-1} \\ \mathbf{1} + (2i\tau_I)^{-1}\bar{\tau} & -(2i\tau_I)^{-1} \end{pmatrix} = \begin{pmatrix} (2i\tau_I)^{-1} & \mathbf{0} \\ \mathbf{0} & (2i\tau_I)^{-1} \end{pmatrix} \begin{pmatrix} -\bar{\tau} & \mathbf{1} \\ \tau & -\mathbf{1} \end{pmatrix}$$

and

$$I^t = \begin{pmatrix} -\tau_I^{-1}\tau_R & \tau_I^{-1} \\ -\tau_R\tau_I^{-1}\tau_R - \tau_I & \tau_R\tau_I^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \tau_R & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \tau_I^{-1} \\ -\tau_I & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ -\tau_R & \mathbf{1} \end{pmatrix}. \quad (35)$$

The transpose of  $I^t$  above then gives the local expression of  $I$ . Now, the space  $\Omega^1(M) \otimes \mathbb{C}$  of smooth sections of  $T^*M \otimes \mathbb{C}$ , spanned locally by  $e^1, \dots, e^d$  over  $C^\infty(M) \otimes \mathbb{C}$ , has a decomposition

$\Omega^1(M) \otimes \mathbb{C} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ , where  $\Omega^{1,0}(M)$  and  $\Omega^{0,1}(M)$  are the space of smooth sections of  $L_+^*$  and that of  $L_-^*$ , respectively. More generally, for a given almost complex manifold  $(M, I)$ , one has the decomposition

$$\Omega^r(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Omega^{p,q}(M),$$

where  $\Omega^{p,q}(M) := \Gamma((\wedge^p(L_+^*)) \wedge (\wedge^q(L_-^*)))$  is the space of smooth sections of  $(\wedge^p(L_+^*)) \wedge (\wedge^q(L_-^*))$ .

**Definition 3.10.** An element in  $\Omega^{p,q}(M)$  is called a  $(p, q)$ -form.

The following fact is used frequently in this article.

**Lemma 3.11.** For an almost complex manifold  $(M, I)$ , consider the local expression as in eq.(35). For any two form  $\hat{f} \in \Omega^2(M)$  and its local expression

$$\hat{f} := \frac{1}{2} \sum_{i,j=1}^d F_{ij} e^i \wedge e^j, \quad F := \{F_{ij}\}_{i,j=1,\dots,2n} \in \text{Mat}_n(C^\infty(M)),$$

the following statements are equivalent:

- i)  $\hat{f}$  is a  $(1, 1)$ -form.
- ii) The matrix  $F$  satisfies  $(\tau \quad -\mathbf{1}_n) F_a \begin{pmatrix} \tau^t \\ -\mathbf{1}_n \end{pmatrix} = 0$ .
- iii) The matrix  $F$  satisfies  $I^t F + F I = 0$ .
- iv) The matrix  $F$  is expressed as

$$F = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \tau_R & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ -f_2 & f_1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I^t \end{pmatrix} \begin{pmatrix} \mathbf{1} & \tau_R^t \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

for a skew-symmetric matrix  $f_1 \in \text{Mat}_n(C^\infty(M))$  and a symmetric matrix  $f_2 \in \text{Mat}_n(C^\infty(M))$ .

**Proof.** These equivalences are obvious when we describe (with loss of generality) the matrix  $F$  corresponding to the two-form  $\hat{f} \in \Omega^2(M)$  locally as

$$F = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \tau_R & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ -f_2^t & f_3 \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I^t \end{pmatrix} \begin{pmatrix} \mathbf{1} & \tau_R^t \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

for skew-symmetric matrices  $f_1, f_3 \in \text{Mat}_n(C^\infty(M))$  and  $f_2 \in \text{Mat}_n(C^\infty(M))$ . One sees that the condition that  $F$  is of  $(1, 1)$ -form is equivalent to that  $f_1 = f_3$  and  $f_2 = f_2^t$ .  $\square$

Recall that, for a given Kähler manifold  $(M, I, \omega)$ , as  $2n$  by  $2n$  matrices, one has the relation  $G = I^t \omega = -\omega I$  and hence the constant two-form  $\omega \in \Omega^2(M)$  is a Kähler form only if it is a  $(1, 1)$ -form by Lemma 3.11.

**Corollary 3.12.** Given a Kähler manifold  $(M, I, \omega)$ , the Kähler structure is expressed locally as  $\tau \in \text{Mat}_n(C^\infty(M))$  such that  $\text{Im}(\tau)$  is positive definite and

$$\omega = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \tau_R & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_2 \\ -\omega_2 & \omega_1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I^t \end{pmatrix} \begin{pmatrix} \mathbf{1} & \tau_R^t \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \in \text{Mat}_{2n}(C^\infty(M))$$

with  $\omega_1, \omega_2 \in \text{Mat}_n(C^\infty(M))$  skew-symmetric and symmetric, respectively, such that  $\begin{pmatrix} \omega_2 & -\omega_1 \\ \omega_1 & \omega_2 \end{pmatrix}$  is positive definite.

**Proof.** We see that the corresponding metric  $G$  is written as

$$G = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \tau_R & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I \end{pmatrix} \begin{pmatrix} \omega_2 & -\omega_1 \\ \omega_1 & \omega_2 \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I^t \end{pmatrix} \begin{pmatrix} \mathbf{1} & \tau_R^t \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Thus, these data define a Kähler structure iff the matrix  $\begin{pmatrix} \omega_2 & -\omega_1 \\ \omega_1 & \omega_2 \end{pmatrix}$  is positive definite.  $\square$

**Definition 3.13.** A symplectic manifold  $(M, \omega)$  with a two-form  $B$  is called a *complexified symplectic manifold* if there exists an almost complex structure  $I$  on  $M$  such that  $\omega$  and  $B$  are  $(1, 1)$ -form with respect to  $I$ . Furthermore, when  $(M, I, \omega)$  is a Kähler manifold,  $(M, I, \omega, B)$  is called a *complexified Kähler manifold*.

A statement similar to Corollary 3.12 applies to complexified Kähler manifolds.

**3.3. T-duality and mirror symmetry for tori.** For a generalized Kähler manifold  $(M, \mathcal{I}_\pm)$ , where  $\mathcal{I}_\pm$  are given by eq.(33), let us describe the quadratic form  $\mathcal{G}$  in terms of  $G$  and  $B$ . One has

$$-q\mathcal{I}_+\mathcal{I}_- = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \quad (36)$$

This matrix is the one which is familiar to people in string theory. In string theory, there is one of the most important duality called T-duality. There are various generalizations, but mainly flat tori are discussed for the T-duality. A flat torus is by definition a torus with a metric described by a constant matrix  $G$  and a constant two form  $B$  called the  $B$ -field globally with respect to a basis  $(e_1, \dots, e_d)$ . So, let us consider this situation, where the above matrix (36) is just a real valued matrix in  $Mat_d(\mathbb{R})$ . This matrix (36) in  $Mat_d(\mathbb{R})$  defines a quadratic form  $H_{zero} : (\mathbb{Z}^d \oplus \mathbb{Z}^d) \rightarrow \mathbb{R}$  as

$$H_{zero}(w, m) = (w^t \ m^t) \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \begin{pmatrix} w \\ m \end{pmatrix} \quad (37)$$

for  $(w, m) \in (\mathbb{Z}^d \oplus \mathbb{Z}^d)$ . This  $H_{zero}$  is just the zero mode part of the Hamiltonian of a closed string on the flat torus  $(T^d, G, B)$  (up to a constant). Here  $w^t = (w_1, \dots, w_d)$  is called the winding mode;  $w_i \in \mathbb{Z}$  corresponds to the degree of the map from  $S^1$  (closed string state) to the cycle in  $i$ -th direction of the torus  $T^d$ . The number  $m^t = (m_1, \dots, m_d)$  then corresponds to the momentum of the closed string; they take the value in  $\mathbb{Z}^d$  up to constant since the target space  $T^d$  is compact (in the sense in Physics). The lift of these data to  $N = 2$  superstring setting by Kapustin-Orlov [40] became an origin of generalized geometry. However, we first give definitions of T-duality group and mirror symmetry in our restricted case of flat tori. Some relevant background in physics will be mentioned in order at the end of this subsection. Providing full details on the background of mirror symmetry needs 1000 pages and is out of our purpose, see [28].

The group  $g : \mathbb{Z}^d \oplus \mathbb{Z}^d \rightarrow \mathbb{Z}^d \oplus \mathbb{Z}^d$  preserving the lattice  $(\mathbb{Z}^d \oplus \mathbb{Z}^d, q)$  is  $O(d, d; \mathbb{Z})$  (eq.(11)). For any  $g \in O(d, d; \mathbb{Z})$ , there exists a transformation  $(G, B) \mapsto (g(G), g(B))$  which preserves the quadratic form (see [22, 40])

$$H_{zero}(g(G), g(B))(w, m) = H_{zero}(G, B)(g(w, m)). \quad (38)$$

Thus, the group  $O(d, d; \mathbb{Z})$  is called the *T-duality group*, where  $(T^d, G, B)$  and  $(T^d, g(G), g(B))$  are said *T-dual* to each other.

Next, we discuss a lift of this symmetry on flat tori to generalized Kähler flat tori. Here, we say a generalized flat Kähler manifold  $M$  is *flat* if  $TM \rightarrow M$  and then  $T^*M \rightarrow M$  are trivial vector bundles and the matrices (34) describing a generalized Kähler structure are constant globally with a suitable basis of  $TM \oplus T^*M$ . So, now,  $M = T^{2n}$ . First, for a given generalized complex flat torus  $(T^{2n}, \mathcal{I})$ , so  $\mathcal{I} \in Mat_{2n}(\mathbb{R})$ , a constant matrix, consider the transformation

$$g(\mathcal{I}) := g^{-1}\mathcal{I}g, \quad g \in O(2n, 2n|\mathbb{Z}).$$

Since  $O(2n, 2n; \mathbb{Z})$  preserves the inner product  $q$ ,  $g(\mathcal{I})$  again defines a generalized flat torus. Similarly, for a given generalized Kähler flat torus  $(T^{2n}, \mathcal{I}_\pm)$ ,  $(T^{2n}, g(\mathcal{I}_\pm))$  again defines a generalized Kähler flat torus. Since a generalized Kähler flat torus  $(T^{2n}, \mathcal{I}_\pm)$  is determined by the constant

matrices  $(G, B, I_+, I_-)$ ,  $g(\mathcal{I}_\pm)$  gives a transformation  $g(G, B, I_+, I_-)$ . Thus, we obtained the lift of the T-duality to  $N = 2$  superstring setting.

Now, we are interested in the special transformation given by  $\sigma_n \in O(2n, 2n; \mathbb{Z})$ :

**Definition 3.14.** For a  $2n$ -dimensional flat generalized Kähler torus  $(T^{2n}, \mathcal{I}_\pm)$ , its *mirror transform*  $(T^{2n}, \hat{\mathcal{I}}_\pm)$  is defined by

$$\hat{\mathcal{I}}_\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \mathcal{I}_\pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (39)$$

where  $\mathbf{1} := \mathbf{1}_n$ ,  $\mathbf{0} := \mathbf{0}_n$ , the  $n$  by  $n$  matrices.

We define the *mirror transform*  $\hat{\mathcal{I}}$  of a flat generalized complex torus  $\mathcal{I}$  by the same formula as in eq.(39).

It is clear that this gives a  $\mathbb{Z}_2$  symmetry (involution) on the set of flat generalized complex or Kähler tori. The mirror-transformation (39) means we fixed the base torus  $T^n$  corresponding to the first  $n$  entries and take T-duality for the remaining  $T^n$  regarded as a fiber over the base  $T^n$ , see at the end of this subsection.

Let us observe this mirror symmetry explicitly in an example. In order to do that, we first study the moduli of flat Kähler tori. Recall that a flat Kähler torus is described by constant matrices  $I$  and  $\omega$  and then the metric  $G := \omega(I \otimes \mathbf{1})$  is also described by a constant matrix. All the arguments in the previous subsection apply here by regarding the matrix elements as constants. In particular, by Corollary 3.12, one immediately obtains the followings.

**Proposition 3.15.** *The space of flat Kähler structures on a torus of real dimension  $2n$  is a manifold of dimension  $3n^2$ . A complex structure  $I$  is described by  $\tau \in \text{Mat}_n(\mathbb{C})$  such that  $\text{Im}(\tau)$  is positive definite as in eq.(35). Then, the Kähler metric  $\omega \in \text{Mat}_n(\mathbb{R})$  is then of the form*

$$\omega = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \tau_R & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I \end{pmatrix} \begin{pmatrix} \omega_1 & \omega_2 \\ -\omega_2 & \omega_1 \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \tau_I^\dagger \end{pmatrix} \begin{pmatrix} \mathbf{1} & \tau_R^\dagger \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

with  $\omega_1 \in \text{Mat}_n(\mathbb{R})$  and  $\omega_2 \in \text{Mat}_n(\mathbb{R})$  a skew-symmetric matrix and a symmetric matrix, respectively, such that  $\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2^\dagger & \omega_1 \end{pmatrix}$  is nondegenerate.

**Proof.** The dimension of the space of constant complex structures  $I = \{\tau_I, \tau_R\}$  is  $2 \cdot n^2$  and the dimension of the space  $\{\omega_1, \omega_2\}$  is  $n^2$ .  $\square$

**Corollary 3.16.** *The space of flat complexified Kähler tori of real dimension  $2n$  is a manifold of dimension  $4n^2$ .*

**Proof.** The space of constant B-fields is also of dimension  $n^2$  since they are  $(1, 1)$ -forms.  $\square$

Now, we observe the mirror dual for a complex torus  $(T^{2n}, I)$  (as is done for instance in [46]). Using eq.(35), the corresponding generalized complex structure  $\mathcal{I}_I$  is

$$\mathcal{I}_I = \begin{pmatrix} -\tau_R^\dagger \tau_I^{t,-1} & -\tau_R^\dagger \tau_I^{t,-1} \tau_R^\dagger - \tau_I^\dagger & \mathbf{0} & \mathbf{0} \\ \tau_I^{t,-1} & \tau_I^{t,-1} \tau_R^\dagger & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tau_I^{-1} \tau_R & -\tau_I^{-1} \\ \mathbf{0} & \mathbf{0} & \tau_R \tau_I^{-1} \tau_R + \tau_I & -\tau_R \tau_I^{-1} \end{pmatrix}$$

for any  $\tau_R$  and any positive definite  $\tau_I$ .

On the other hand, consider a symplectic torus  $(T^{2n}, \omega, B)$  of the following form:

$$\omega = \begin{pmatrix} 0 & a^t \\ -a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b^t \\ -b & 0 \end{pmatrix}.$$

The corresponding generalized complex structure  $\mathcal{I}_\omega(B)$  is (see eq.(31))

$$\mathcal{I}_\omega(B) = \begin{pmatrix} a^{-1}b & 0 & 0 & a^{-1} \\ 0 & a^{t,-1}b^t & -a^{t,-1} & 0 \\ 0 & a^t + b^t a^{t,-1}b^t & -b^t a^{t,-1} & 0 \\ -a - ba^{-1}b & 0 & 0 & -ba^{-1} \end{pmatrix}$$

and its mirror dual  $\hat{\mathcal{I}}_\omega(B)$  is

$$\begin{pmatrix} a^{-1}b & a^{-1} & 0 & 0 \\ -a - ba^{-1}b & -ba^{-1} & 0 & 0 \\ 0 & 0 & -b^t a^{t,-1} & a^t + b^t a^{t,-1}b^t \\ 0 & 0 & -a^{t,-1} & a^{t,-1}b^t \end{pmatrix}.$$

Thus,  $\hat{\mathcal{I}}_\omega(B)$  defines a complex structure. In fact, one sees that there is a bijection between the space  $\{(\tau_I, \tau_R)\}$  defining  $\mathcal{I}_I$  and the space  $\{(a, b)\}$  defining  $\hat{\mathcal{I}}_\omega(B)$ . The bijection relating them is given by the T-duality. Consider  $g(\mathcal{I}_I)$  in the case  $g = \sigma_{2n}$ . The generalized complex structure  $\sigma_{2n}(\mathcal{I}_I)$  again defines a complex structure:

$$\sigma_{2n}(\mathcal{I}_I) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -I^t & 0 \\ 0 & I \end{pmatrix}, \quad I := \begin{pmatrix} -\tau_R^t \tau_I^{t,-1} & -\tau_R^t \tau_I^{t,-1} \tau_R^t - \tau_I^t \\ \tau_I^{t,-1} & \tau_I^{t,-1} \tau_R^t \end{pmatrix},$$

that is,  $\sigma_{2n}(I) = -I^t$ . This  $\sigma_{2n}$  corresponds to T-dualizing  $T^{2n}$  for all directions. Then, one sees that the correspondence between  $(\tau_I, \tau_R)$  and  $(a, b)$  is given by

$$(a, b) = \sigma_{2n}(\tau_I, \tau_R). \quad (40)$$

To summarize, for a flat complex torus  $(T^{2n}, \mathcal{I}_I)$ , where  $I$  is determined by  $\tau$ , the mirror dual symplectic torus  $(T^{2n}, \hat{\mathcal{I}}_I)$  is given by  $\mathcal{I}_\omega(B) = \hat{\mathcal{I}}_I$  with  $(\omega, B)$  determined from  $(a, b)$  by eq.(40). Thus, the mirror transformation (Definition 3.14) exchange a complex structure with a symplectic structure and vice versa, as expected, see remarks below. (However, one sees in this example that the mirror transformation does not preserve the subset consisting of flat (complexified) Kähler tori.)

For two tori, this mirror symmetry between a complex one-torus  $(T^2, \tau)$  and a symplectic torus  $(T^2, \rho)$ ,  $\rho \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} := i\omega + B$ , has the following expression also:

$$\rho = -\frac{1}{\tau}. \quad (41)$$

We use this mirror relation in subsection 5.2.

We end with some remarks on T-duality, mirror symmetry, and generalized geometry from the viewpoints of string theory. When we consider  $N = 1$  *superstring theory* on a manifold  $M$ ,<sup>5</sup> the supersymmetry is enhanced to  $N = 2$  supersymmetry if  $M$  is a Kähler manifold. More precisely, the condition the symmetry is enhanced to  $N = 2$  supersymmetry was given in [19]. The condition is that  $M$  is equipped with a generalized Kähler structure twisted by a three form  $H = dB$ , which is now called a twisted generalized Kähler structure [24].

<sup>5</sup>More precisely,  $N = 1$  supersymmetric nonlinear sigma model [28].

Mirror symmetry is discovered as a duality of some good class of these  $N = 2$  superstring theories [28]. For instance, two target Calabi-Yau manifolds <sup>6</sup>  $M$  and  $\hat{M}$  are called mirror dual to each other if they define equivalent  $N = 2$  superstring theories. In fact, they form  $N = 2$  superconformal field theory since these target spaces are Ricci flat. This mirror symmetry in general transforms the symplectic structure defined by the Kähler form  $\omega$  of  $M$  to the complex structure  $I$  of  $\hat{M}$  and vice versa. For instance, for a  $N = 2$  superconformal field theory on  $M$ , one can define two topological field theories called A-model and B-model via A-twist and B-twist. Here, the A-model (resp. B-model) depends on the symplectic structure  $\omega$  (resp. the complex structure  $I$ ) only. For mirror dual Calabi-Yau manifolds  $M$  and  $\hat{M}$ , the A-model on  $M$  is equivalent to the B-model on  $\hat{M}$  and vice versa. This turns out to become various mathematical statements. For tree closed strings, the mirror symmetry between A-model on  $M$  and the B-model on  $\hat{M}$  is formulated in terms of the equivalence of Frobenius manifolds. For tree open strings, it is formulated as HMS, see subsection 5.1. String theory suggests that such a duality should exist for the full quantum open-closed string setting.

On the other hand, T-duality has been discussed as the duality of bosonic (closed) strings mainly for flat tori; two flat tori are T-dual to each other iff the Hamiltonians of *bosonic* closed strings on the flat tori are equivalent (see [22]). This leads the definition of the T-duality around eq.(38). In particular, the duality given by  $\sigma_d \in O(d, d; \mathbb{Z})$  is called the T-duality in a narrow sense (see eq.(40)). For instance, for  $d = 1$ ,  $B = 0$  and  $G$  is a positive definite one by one matrix, i.e., a positive real number. Then, one obtain  $\sigma_1(G) = G^{-1}$ . Namely, the  $S^1$  with radius  $\sqrt{G}$  is T-dual to the  $S^1$  with radius  $(\sqrt{G})^{-1}$ . This happens because as follows. Now, the zero mode Hamiltonian (37) reduces to

$$H_{zero} = Gw^2 + G^{-1}m^2, \quad (w, m) \in \mathbb{Z}^2,$$

where the first term corresponds to the mass of closed string winding  $w$  times in  $S^1$ , and the second term corresponds to the mass (= energy!) of closed strings of momentum  $m$ . In the case of a point particle, instead of a closed string, the first term is absent. However, one sees, in the case of (closed) string, the role of the winding number  $w$  and the momentum  $m$  via the T-duality. Thus, *the T-duality is a duality coming from nonlocality of strings.*

Strominger-Yau-Zaslow [77] proposed a way of understanding mirror symmetry via T-duality. They proposed regarding a Calabi-Yau  $n$ -fold as a torus fibration of fiber  $T^n$  which is in general singular at some points in the base space. For the case of flat Kähler tori  $T^{2n}$ , it is clear that they are trivial torus fibration with fiber  $T^n$  and the base  $T^n$ . It is discussed that the mirror symmetry follows from the T-duality of the fiber  $T^n$ . Thus, the mirror of  $T^{2n}$  is  $T^{2n}$  topologically. The torus  $T^{2n}$  has larger symmetry as the T-duality group  $O(d, d; \mathbb{Z})$  (bosonic string setting) is lifted to generalized Kähler tori (superstring theory setting) by Kapustin-Orlov [40]. For the mirror duality of semi-flat torus fibrations in the framework of generalized geometry, see [2].

One may notice the similarity of the T-duality group  $O(d, d; \mathbb{Z})$  for flat tori with the Morita equivalence of noncommutative tori in subsection 2.3. Actually, the similarity is first focused by Connes-Douglas-Schwarz [7] in noncommutative two-tori case (in the context of Matrix theory), which is then extended by Rieffel and Schwarz [70, 71] for higher dimensional case. However, noncommutative tori are interpreted in terms of open strings (modules and bimodules correspond

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<sup>6</sup>A Calabi-Yau manifold in general indicates a Ricci flat Kähler manifold, but it is often assumed that the fundamental group is trivial,  $\pi_1 = 0$ , in particular in discussing mirror symmetry of Calabi-Yau three manifolds. By the former definition, flat Kähler tori are Calabi-Yau, but the latter definition excludes flat Kähler tori.

to D-branes and open strings on them, respectively, see [74]), and the T-duality has been discussed in closed string setting. Thus, the relation should be realized by studying T-duality for open strings [30]. The corresponding superstring setting (with topological twist) is to consider appropriate categories (D-brane category; see [53]) associated to them, which is the main subject of this article.

#### 4. CURVED DG CATEGORY OF VECTOR BUNDLES OVER $\mathcal{A}_\theta^{2n}$

In this section, we introduce complex structures on noncommutative tori  $\mathcal{A}_\theta^d$  following A. Schwarz [72] and lift the categories of modules with connections over noncommutative tori  $\mathcal{A}_\theta^d$  to those over complex noncommutative tori  $(\mathcal{A}_\theta^d, \tau)$  following [34]. See also [3, 46].

**4.1. Curved DG categories.** A differential graded (DG) algebra  $(V, d, m)$  is a  $\mathbb{Z}$ -graded vector space  $V := \bigoplus_{r \in \mathbb{Z}} V^r$  equipped with a degree one differential  $d : V^r \rightarrow V^{r+1}$ ,  $d^2 = 0$ , and a degree preserving associative product  $m : V^r \otimes V^s \rightarrow V^{r+s}$  satisfying the Leibniz rule

$$dm(v, v') = m(d(v), v') + (-1)^{|v|} m(v, d(v')) \quad (42)$$

for any degree homogeneous elements  $v, v' \in V$ , where  $|v|$  is the degree of  $v \in V$ . A familiar example is the DG algebra  $(\Omega^*(M), d, \wedge)$  of the space of smooth differential forms  $\Omega^*(M)$  on a smooth manifold  $M$  with the exterior differential  $d$  and the wedge product  $\wedge$ . Another example which is more relevant to us is the DG algebra  $(\Omega^{0,\bullet}(M, I), \bar{d}, \wedge)$  of smooth  $(0, \bullet)$ -forms on a complex manifold  $(M, I)$  (cf. Definition 3.10). Note that its cohomology is isomorphic to  $H^*(\mathcal{O}_M)$ .

The notions of curved DG algebras [64] (or  $Q$ -algebras [73]) and  $A_\infty$ -algebras (J. Stasheff [75, 76]) are generalizations of DG algebras in different ways, which can be uniformly described as special cases of weak  $A_\infty$ -algebras.

**Definition 4.1** (Weak  $A_\infty$ -algebra). A weak  $A_\infty$ -algebra  $(V, m)$  consists of a  $\mathbb{Z}$ -graded vector space  $V := \bigoplus_{r \in \mathbb{Z}} V^r$  with a collection of multilinear maps  $m := \{m_n : V^{\otimes n} \rightarrow V\}_{n \geq 0}$  of degree  $(2 - n)$  satisfying

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^\sigma m_k(w_1, \dots, w_j, m_l(w_{j+1}, \dots, w_{j+l}), w_{j+l+1}, \dots, w_n) \quad (43)$$

for  $n \geq 0$  with homogeneous elements  $w_i \in V^{|w_i|}$ ,  $i = 1, \dots, n$ , where  $\sigma = (j+1)(l+1) + l(|w_1| + \dots + |w_j|)$ . That the multilinear map  $m_k$  has degree  $(2 - k)$  indicates the degree of  $m_k(w_1, \dots, w_k)$  is  $|w_1| + \dots + |w_k| + (2 - k)$ .

**Definition 4.2** ([75, 76]). A weak  $A_\infty$ -algebra  $(V, m)$  with  $m_0 = 0$  is called a (*strongly*) *homotopy associative algebra* or an  $A_\infty$ -algebra.

**Definition 4.3.** A weak  $A_\infty$ -algebra  $(V, m)$  with higher products all zero,  $m_3 = m_4 = \dots = 0$  is a *curved DG algebra* (Positsel'skii [64]).

One sees that a curved DG algebra with  $m_0 = 0$  is a DG algebra. Note that a (curved) DG algebra  $(V, m)$  forms an associative algebra  $(V, m_2)$ . However, in general, a (weak)  $A_\infty$ -algebra does not form an associative algebra. Let us see the relations (43) in the case  $m_0 = 0$ , that is,  $(V, m)$  is an  $A_\infty$ -algebra. Then, the relations (43) starts from  $n = 1$ . Let us write  $m_1 = d$ ,  $m_2 = \cdot$ .



For  $x, y, z \in V$ , the first three relations are:

- i)  $d^2 = 0$ ,
- ii)  $d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y)$ ,
- iii)  $(x \cdot y) \cdot z - x \cdot (y \cdot z) = d(m_3)(x, y, z)$ ,

where  $d(m_3) := m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$ . Thus, the  $A_\infty$ -relation for  $n = 1$  implies that  $(V, m_1)$  is a complex. That for  $n = 2$  is the the Leibniz rule (42) of the differential  $m_1$  with respect to the product  $m_2$ . That for  $m = 3$  implies the associativity of  $m_2$  if  $m_3 = 0$ . Thus, in general  $m_2$  is not strictly associative and it is said *homotopy associative*, where  $m_3$  is a homotopy between  $(x \cdot y) \cdot z$  and  $x \cdot (y \cdot z)$ . The remaining higher products  $m_4, m_5, \dots$  then define higher homotopy.

There are the notions of weak  $A_\infty$ -modules over a weak  $A_\infty$ -algebra (see [42]), which include curved DG modules and  $A_\infty$ -modules as special cases.

**Definition 4.4.** A *right curved DG module*  $(\mathcal{E}_a, d_a, m)$  over a curved DG algebra  $(V, -f, d, m)$  is a  $\mathbb{Z}$ -graded vector space  $\mathcal{E}_a$  equipped with a degree one linear map  $d_a : \mathcal{E}_a \rightarrow \mathcal{E}_a$  and a right action  $m : \mathcal{E}_a \otimes V \rightarrow \mathcal{E}_a$  satisfying the following condition: for any  $v, v' \in V$  and  $v_a \in \mathcal{E}_a$ ,

$$\begin{aligned} (d_a)^2(v_a) &= m(v_a, f) , \\ d_a m(v_a, v) &= m(d_a(v_a), v) + (-1)^{|v_a|} m(v_a, d(v)) , \\ m(v_a, m(v, v')) &= m(m(v_a, v), v') . \end{aligned}$$

In particular, if  $f = 0$ , then  $(\mathcal{E}_a, d_a, m)$  is called a *DG-module* over the DG algebra  $(V, d, m)$ .

However, it is more natural for us to consider more general modules for a DG algebra.

**Definition 4.5** (Module over a DG algebra). A *right module*  $(\mathcal{E}_a, d_a, m)$  over a DG algebra  $(V, d, m)$  is a  $\mathbb{Z}$ -graded right module  $\mathcal{E}_a$  over  $(V, m)$  equipped with a degree one linear map  $d_a : \mathcal{E}_a \rightarrow \mathcal{E}_a$  satisfying the Leibniz rule

$$d_a m(v_a, v) = m(d(v_a), v) + (-1)^{|v_a|} m(v_a, d(v))$$

for any homogeneous elements  $v_a \in \mathcal{E}_a$  and  $v \in V$ .

**Remark 4.6.** If  $f$  of a curved DG algebra  $(V, -f, d, m)$  is a center in  $V$ , then  $(V, d, m)$  forms a DG algebra. In this situation, let us compare Definition 4.5 with Definition 4.4. The third condition in Definition 4.4 implies that  $\mathcal{E}_a$  is a right module over  $V$ , so is a module over the DG algebra  $(V, d, m)$ . The second condition in Definition 4.4 is the Leibniz rule in Definition 4.5. Thus, dropping the first condition in Definition 4.4, one obtains Definition 4.5.

**Remark 4.7.** A module over a Q-algebra is introduced by Schwarz [73]. Definition 4.5 is obtained as the special case of it where the Q-algebra is a DG algebra.

A curved DG category is a generalization of a curved DG algebra, where morphisms in a curved DG-category correspond to elements of a curved DG algebra. It is defined as a special case of weak  $A_\infty$ -categories. We need the categorical version of these terminologies.

**Definition 4.8** ((Weak)  $A_\infty$ -category [14]). A weak  $A_\infty$ -category  $\mathcal{C}$  consists of a class of objects  $\text{Ob}(\mathcal{C}) = \{a, b, \dots\}$ , a  $\mathbb{Z}$ -graded vector space  $\mathcal{C}(a, b) =: V_{ab}$  for each two objects  $a, b \in \mathcal{C}$  and a collection of multilinear maps

$$m := \{m_n : V_{a_1 a_2} \otimes \dots \otimes V_{a_n a_{n+1}} \rightarrow V_{a_1 a_{n+1}}\}_{n \geq 0}$$

of degree  $(2-n)$  satisfying the  $A_\infty$ -relations (43). In particular, a weak  $A_\infty$ -category with  $m_0 = 0$  is an  $A_\infty$ -category.

**Definition 4.9.** A weak  $A_\infty$ -category  $\mathcal{C}$  with higher products all zero,  $m_3 = m_4 = \dots = 0$ , is called a *curved DG category*.

**Remark 4.10.** A weak  $A_\infty$ -category with only one object is a weak  $A_\infty$ -algebra. Similar facts apply to its special cases such as an  $A_\infty$ -category and a curved DG category.

For a curved DG category  $\mathcal{C}$ , denote  $m_0 = f_a : \mathbb{C} \rightarrow V_{aa}^2$ ,  $m_1 = d_{ba} : V_{ba}^r \rightarrow V_{ba}^{r+1}$ , and  $m_2 = m$ . The defining relations for a curved DG-category turn out to be

$$d(f_a) = 0, \quad (44)$$

$$(d)^2(v_{ba}) = m(f_b, v_{ba}) - m(v_{ba}, f_a), \quad (45)$$

$$dm(v_{cb}, v_{ba}) = m(d(v_{cb}), v_{ba}) + (-1)^{|v_{cb}|} m(v_{cb}, d(v_{ba})), \quad (46)$$

$$m(m(v_{dc}, v_{cb}), v_{ba}) = m(v_{dc}, m(v_{cb}, v_{ba})). \quad (47)$$

**Lemma 4.11.** *The category  $\text{Mod-}V$  of right modules over a DG algebra  $(V, d, m)$  forms a curved DG category with the space  $\mathcal{C}(b, a) = V_{ba} = \text{Hom}_{\text{Mod-}V}(\mathcal{E}_a, \mathcal{E}_b)$  of morphisms the space  $\text{Hom}_V(\mathcal{E}, \mathcal{E}')$  of right  $V$ -module maps.*

**Proof.** By the Leibniz rule of  $d_a$ , one has

$$(d_a)^2(m_a(v_a, v)) = m_a((d_a)^2(v_a), v)$$

which implies there exists an element  $f_a \in \text{Hom}_{\text{Mod-}V}^2(\mathcal{E}_a, \mathcal{E}_a)$  such that  $(d_a)^2(v_a) = f_a(v_a)$ . Next, for any  $\mathcal{E}_a, \mathcal{E}_b \in \text{Mod-}V$ , the degree one linear map  $d_{ba} : V_{ba}^r \rightarrow V_{ba}^{r+1}$  is given by

$$d_{ba}(v_{ba}) = d_b(v_{ba}) - (-1)^{|v_{ba}|} v_{ba} d_a.$$

Then, the square of  $d_{ba}$  yields

$$(d_{ba})^2(v_{ba}) = m(f_b, v_{ba}) - m(v_{ba}, f_a), \quad (48)$$

which is the condition (45). The remaining conditions are clear.  $\square$

We also consider an additional structure, cyclicity, for curved DG categories and  $A_\infty$ -categories.

**Definition 4.12 (Cyclicity).** A weak  $A_\infty$ -category  $\mathcal{C}$  is called a *cyclic weak  $A_\infty$ -category* iff  $\mathcal{C}$  is equipped with a graded symmetric nondegenerate bilinear map

$$\eta : \mathcal{C}^k(a, b) \otimes \mathcal{C}^l(b, a) \rightarrow \mathbb{C} \quad (49)$$

of a fixed degree  $|\eta| \in \mathbb{Z}$  for any  $a, b \in \mathcal{C}$  and it satisfies

$$\eta(m_n(v_{12}, \dots, v_{n(n+1)}), v_{(n+1)1}) = (-1)^{n+(|v_{23}|+\dots+|v_{(n+1)1}|)|v_{12}|} \eta(m_n(v_{23}, \dots, v_{(n+1)1}), v_{12}), \quad (50)$$

for each  $n \geq 0$ .

Here,  $\eta$  is of degree  $|\eta|$  means the inner product  $\eta$  in eq.(49) is nonzero only if  $k+l+|\eta| = 0$ . That  $\eta$  is graded symmetric means it satisfies  $\eta(V_{ab}^k, V_{ba}^l) = (-1)^{kl} \eta(V_{ba}^l, V_{ab}^k)$  for  $V_{ab}^k := \mathcal{C}^k(a, b)$ , etc.

**Remark 4.13.** The inner product defining cyclicity in an  $A_\infty$ -category is related to the Serre duality, see [44].

In the case when  $\mathcal{C}$  is a cyclic curved DG category, the conditions (50) turns out to be

$$\eta(f_a, v_{aa}) = \eta(v_{aa}, f_a), \quad (51)$$

$$\eta(d(v_{ab}), v_{ba}) + (-1)^{|v_{ab}|} \eta(v_{ab}, d(v_{ba})) = 0, \quad (52)$$

$$\eta(m(v_{cb} \otimes v_{ba}), v_{ac}) = (-1)^{|v_{cb}|(|v_{ba}|+|v_{ac}|)} \eta(m(v_{ba} \otimes v_{ac}), v_{cb}), \quad (53)$$

where we wrote  $m_0 = f_a \in V_{aa}^2$  and  $m_1 = d$ . In the case  $\mathcal{C}$  with  $\eta$  is a cyclic DG category, since  $f_a = 0$  for any  $a \in \mathcal{C}$ , we simply do not have the first condition above.

**4.2. Curved DG category of modules over a DG algebra: a general construction.** Let us start from a unital (noncommutative) algebra  $A$ . We shall consider the case  $\mathcal{A} = \mathcal{A}_\theta^d$  in the next subsection. Define a DG algebra which is as a  $\mathbb{Z}$ -graded vector space  $\Omega^{0,\bullet}(\mathcal{A}) := \bigoplus_{r=0}^n \Omega^{0,r}(\mathcal{A})$  for some positive integer  $n$  with  $\Omega^{0,0}(\mathcal{A}) = \mathcal{A}$ . Clearly,  $\Omega^{0,\bullet}(\mathcal{A})$  forms left and right  $\mathcal{A}$ -modules. Thus, one can consider the curved DG category  $\text{Mod-}\Omega^{0,\bullet}(\mathcal{A})$  of modules over the DG algebra  $\Omega^{0,\bullet}(\mathcal{A})$  by Lemma 4.11.

We are interested in a full curved DG subcategory of  $\text{Mod-}\Omega^{0,\bullet}(\mathcal{A})$  consisting of modules over  $\Omega^{0,\bullet}(\mathcal{A})$  of the form

$$\mathcal{E} := E \otimes_{\mathcal{A}} \Omega^{0,\bullet}(\mathcal{A})$$

for any right  $\mathcal{A}$ -module  $E \in \text{Mod-}\mathcal{A}$ . Here, the tensor product  $\otimes_{\mathcal{A}}$  is taken as that of a right  $\mathcal{A}$ -module  $E$  with a left  $\mathcal{A}$ -module  $\Omega^{0,\bullet}(\mathcal{A})$ . We denote this curved DG category by

$$\Omega^{0,\bullet}(\text{Mod-}\mathcal{A}) \subset \text{Mod-}\Omega^{0,\bullet}(\mathcal{A}). \quad (54)$$

**4.3. Curved DG categories  $\Omega^\tau(\text{Pmod-}\mathcal{A}_\theta^{2n})$  of modules over  $\Omega^\tau(\mathcal{A}_\theta^{2n})$ .** Let us consider a complex structure on the noncommutative torus  $\mathcal{A}_\theta^{2n}$  as introduced by A. Schwarz [72] and define the DG-algebra  $\Omega^{0,\bullet}(\mathcal{A})$  for  $\mathcal{A} = \mathcal{A}_\theta^{2n}$ . We take a different notation which fits our arguments, though it is equivalent to the one in [72]. When we define a complex structure on a commutative torus  $\mathcal{A}^{2n}$ , we may take a  $\mathbb{C}$ -valued  $n$  by  $n$  matrix  $\tau = \{\tau_j^i\}$ ,  $i, j = 1, \dots, n$ , whose imaginary part  $\tau_I := \text{Im}(\tau)$  is positive definite. A commutative complex torus is then described by  $\mathbb{C}^n / (\mathbb{Z}^n + \tau^t \mathbb{Z}^n)$ , where  $\tau^t$  is the transpose of  $\tau$ . The complex coordinates of  $\mathbb{C}^n$  are given by  $(z_1, \dots, z_n)$ ,  $z^i = x^i + \sum_j y^j \tau_j^i$ ,  $i = 1, \dots, n$ . The corresponding Dolbeault operator  $\bar{\partial}$  is given by

$$\bar{\partial} = \sum_{i=1}^n d\bar{z}^i \frac{\partial}{\partial \bar{z}^i}, \quad \frac{\partial}{\partial \bar{z}^i} := \frac{1}{2i} \sum_{j=1}^n \left( ((\tau_I)^{-1} \tau)_i^j \frac{\partial}{\partial x^j} - ((\tau_I)^{-1})_i^j \frac{\partial}{\partial y^j} \right),$$

where we denote  $\text{Im}(\tau) =: \tau_I$  which is by definition positive definite.

Based on these formula, for a noncommutative torus  $\mathcal{A}_\theta^{2n}$  and a fixed complex structure  $\tau$ , let us define  $\bar{\delta}_i \in L$ ,  $i = 1, \dots, n$ , by

$$\bar{\delta}_i := \frac{1}{2i} \sum_{j=1}^n \left( ((\tau_I)^{-1} \tau)_i^j \delta_j - ((\tau_I)^{-1})_i^j \delta_{n+j} \right).$$

Also, for  $E_a := (E_{g_a, \theta}, \nabla_a) \in \text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^{2n}$ , a Heisenberg module  $E_{g_a, \theta}$  over  $\mathcal{A}_\theta^{2n}$  with a constant curvature connection  $\nabla_a$ , define a holomorphic structure  $\bar{\nabla}_{a,i} : E_{g_a, \theta} \rightarrow E_{g_a, \theta}$ ,  $i = 1, \dots, n$ , by

$$\bar{\nabla}_{a,i} := \frac{1}{2i} \sum_{j=1}^n \left( ((\tau_I)^{-1} \tau)_i^j \nabla_{a,j} - ((\tau_I)^{-1})_i^j \nabla_{a,n+j} \right). \quad (55)$$

Let  $\Lambda$  be the Grassmann algebra (with unit) generated by  $d\bar{z}^1, \dots, d\bar{z}^n$  of degree one. Namely, they satisfy  $d\bar{z}^i d\bar{z}^j = -d\bar{z}^j d\bar{z}^i$  for any  $i, j = 1, \dots, n$ , so in particular  $(d\bar{z}^i)^2 = 0$ . These generators are thought of as a formal basis of the anti-holomorphic one forms on the complex noncommutative

torus  $\mathcal{A}_\theta^{2n}$ . By  $\Lambda^k$  we denote the degree  $k$  graded piece of  $\Lambda$ . The graded vector space  $\Omega^{0,\bullet}(\mathcal{A}_\theta^{2n}) := \mathcal{A}_\theta^{2n} \otimes \Lambda$  is then thought of as the space of smooth anti-holomorphic forms on the complex noncommutative torus  $\mathcal{A}_\theta^{2n}$ . It has the graded decomposition:

$$\Omega^{0,\bullet}(\mathcal{A}_\theta^{2n}) = \bigoplus_{r=0}^n \Omega^{0,r}(\mathcal{A}_\theta^{2n}), \quad \Omega^{0,r}(\mathcal{A}_\theta^{2n}) = \mathcal{A}_\theta^{2n} \otimes \Lambda^r.$$

Any element in  $V^k := \Omega^{0,k}(\mathcal{A}_\theta^{2n})$  can be written as

$$v = \sum_{\vec{m} \in \mathbb{Z}^n} \sum_{i_1, \dots, i_k} v_{\vec{m}; i_1 \dots i_k} U_{\vec{m}} \cdot (d\bar{z}^{i_1} \dots d\bar{z}^{i_k}),$$

where  $v_{\vec{m}; i_1 \dots i_k} \in \mathbb{C}$  is skew-symmetric with respect to the indices  $i_1 \dots i_k$ . A product  $m : V^k \otimes V^l \rightarrow V^{k+l}$  is defined naturally by combining the product on  $\mathcal{A}_\theta^{2n}$  with the one on the Grassmann algebra  $\Lambda$ , and then  $(V, m)$  forms a graded algebra. One can define a differential  $d : V^k \rightarrow V^{k+1}$ ,

$$d := \sum_{i=1}^n d\bar{z}^i \cdot \bar{\delta}_i,$$

which satisfies the Leibniz rule with respect to the product  $m$ . Thus,  $(\Omega^{0,\bullet}(\mathcal{A}_\theta^{2n}), d, m)$  is a DG algebra. Note that this is isomorphic to the DG algebra  $(\Omega^{0,\bullet}(\mathcal{A}_\theta^{2n}), \bar{\delta}, \wedge)$  in the commutative case  $\theta = 0$ .

Consider the curved DG category  $\Omega^{0,\bullet}(\text{Mod-}\mathcal{A}_\theta^{2n})$  (defined in eq.(54)). In order to express the  $\tau$  dependence explicitly; denote

$$\Omega^\tau(\mathcal{A}_\theta^{2n}) := (\Omega^{0,\bullet}(\mathcal{A}_\theta^{2n}), d, m), \quad \Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n}) := \Omega^{0,\bullet}(\text{Mod-}\mathcal{A}_\theta^{2n}). \quad (56)$$

Any object  $\mathcal{E}_a \in \Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})$  is of the form

$$\mathcal{E}_a := E_a \otimes_{\mathcal{A}_\theta^{2n}} \Omega^{0,\bullet}(\mathcal{A}_\theta^{2n}) = E \otimes \Lambda$$

for some  $E_a \in \text{Mod-}\mathcal{A}_\theta^{2n}$ . Thus, it is clear that

**Lemma 4.14.** *Any module  $\mathcal{E}_a \in \Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})$  over the DG algebra  $\Omega^\tau(\mathcal{A}_\theta^{2n})$  is the lift of an element  $(E_a, \nabla_a) \in \text{Mod}^\nabla\text{-}\mathcal{A}_\theta^{2n}$ , where  $d_a : \mathcal{E}_a \rightarrow \mathcal{E}_a$  is given by*

$$d_a := \sum_{i=1}^n d\bar{z}^i \cdot \bar{\nabla}_{a,i}$$

with  $\bar{\nabla}_{a,i}$  the holomorphic structure (55), and one has

$$\text{Hom}_{\Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})}(\mathcal{E}_a, \mathcal{E}_b) \simeq \text{Hom}_{\text{Mod}^\nabla\text{-}\mathcal{A}_\theta^{2n}}(E_a, E_b) \otimes \Lambda$$

for any  $\mathcal{E}_a = E_a \otimes \Lambda$ ,  $\mathcal{E}_b = E_b \otimes \Lambda \in \Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})$ .  $\square$

In general, a module  $\mathcal{E}_a \in \Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})$  has its curvature:

$$(d_a)^2(v_a) = f_a \cdot v_a, \quad f_a := -\pi i (d\bar{z}^t \tau_I^{-1}) (\tau \quad -1_n) F_a \begin{pmatrix} \tau^t \\ -1_n \end{pmatrix} (\tau_I^{t,-1} d\bar{z}) \in \Lambda^2$$

for any  $v_a \in \mathcal{E}_a$ , where  $d\bar{z}^t := (d\bar{z}^1 \dots d\bar{z}^n)$ . This  $d_a$  defines a differential on  $\mathcal{E}_a$ , that is,  $f_a = 0$  if and only if the curvature  $-2\pi i F_a$  of  $E_a$  is a  $(1, 1)$ -form with respect to the complex structure defined by  $\tau$  by Lemma 3.11. In this case,  $(\mathcal{E}_a, d_a, m_a)$  forms a DG module over  $V$ . In the commutative case ( $\theta = 0$ ), this implies that  $\mathcal{E}_a$  forms a holomorphic vector bundle. However, for general  $\theta$ ,  $f_a$  may not be zero even if it is zero when  $\theta$  is set to be zero [34] (see also [46]).

Now, we would like to discuss additional structures in full subcategories of  $\Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})$  which are necessary to discuss homological mirror symmetry in the next section.

**Definition 4.15.** The curved DG full subcategory of  $\Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})$  consisting of objects  $E \otimes \Lambda$ ,  $E \in \text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^{2n}$ , is denoted  $\Omega^\tau(\text{Pmod-}\mathcal{A}_\theta^{2n})$ . The curved DG full subcategory of  $\Omega^\tau(\text{Pmod-}\mathcal{A}_\theta^{2n}) \subset \Omega^\tau(\text{Mod-}\mathcal{A}_\theta^{2n})$  consisting of objects  $E \otimes \Lambda$ ,  $E \in \text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n}$  (Definition 4.15), is denoted  $\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$ .

**Proposition 4.16.** *The curved DG category  $\Omega^\tau(\text{Pmod-}\mathcal{A}_\theta^{2n})$  forms a cyclic curved DG category. The inner product  $\eta : \text{Hom}(a, b) \otimes \text{Hom}(b, a) \rightarrow \mathbb{C}$  of degree  $-n$  is given by*

$$\eta = \int_{\mathcal{A}_\theta^{2n}} m, \quad \int_{\mathcal{A}_\theta^{2n}} (u \otimes \lambda) = \text{Tr}(u) \otimes \int_{\Lambda} \lambda$$

for  $u \in \mathcal{A}_\theta^{2n}$  and  $\lambda \in \Lambda$ . Here  $\int_{\Lambda} : \Lambda \rightarrow \mathbb{C}$  is the linear map defined by

$$\begin{cases} \int_{\Lambda} (d\bar{z}^{i_1} \cdots d\bar{z}^{i_k}) = 0 & (k \neq n) \\ \int_{\Lambda} (d\bar{z}^1 \cdots d\bar{z}^n) = 1 \end{cases}.$$

**Proof.** The cyclicity of the inner product  $\eta$  follows from Lemma 2.10. The nondegeneracy follows from Lemma 2.11.  $\square$

For  $\mathcal{E}_a \in \Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$ , the two form  $f_a \in \text{End}_{\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})}^2(\mathcal{E}_a)$  can be written of the form

$$f_a = \hat{f}_a \cdot \mathbf{1}_{\text{End}_{\mathcal{A}_\theta^{2n}}(E_a)}, \quad \hat{f}_a \in \Lambda^2.$$

We call  $\hat{f}_a$  the potential two-form of  $\mathcal{E}_a \in \Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$  and denote by  $\Omega^\tau(\text{Pmod}^{\hat{f}}\text{-}\mathcal{A}_\theta^{2n})$  the cyclic curved DG full subcategory of  $\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$  consisting of objects with  $\hat{f} \in \Lambda^2$  as their potential two-form.

**Proposition 4.17** ([34, Proposition 3.5]). *For any  $\hat{f} \in \Lambda^2$ , the cyclic curved DG full subcategory  $\Omega^\tau(\text{Pmod}^{\hat{f}}\text{-}\mathcal{A}_\theta^{2n})$  forms a cyclic DG category.*

**Proof.** By looking at eq.(48), one sees  $(d_{ab})^2 = 0$  if  $\mathcal{E}_a, \mathcal{E}_b \in \Omega^\tau(\text{Pmod}^{\hat{f}}\text{-}\mathcal{A}_\theta^{2n})$ .  $\square$

**Remark 4.18.** Since  $\hat{f} \in \Lambda^2$  is a center in  $V$ , i.e.,  $m(\hat{f}, v) - m(v, \hat{f}) = 0$  for any  $v \in V := \Omega^{0,\bullet}(\mathcal{A}_\theta^{2n})$ . Thus, for any  $\hat{f} \in \Lambda^2$ , one can regard the DG algebra  $\Omega^\tau(\mathcal{A}_\theta^{2n}) = (V, d, m)$  as a curved DG algebra  $(V, \hat{f}, d, m)$  by  $(d)^2(v) = 0 = m(\hat{f}, v) - m(v, \hat{f})$ ,  $v \in V$ . The curved DG category  $\Omega^\tau(\text{Pmod}^{\hat{f}}\text{-}\mathcal{A}_\theta^{2n})$  is then the category of curved DG modules over  $(V, \hat{f}, d, m)$  [34, Proposition 3.5], which by definition forms a DG-category.

In the case of a noncommutative two-torus, a complex structure is defined by  $\tau \in H_+$ . For any  $\mathcal{E}_a \in \Omega^\tau(\text{Mod-}\mathcal{A}_\theta^2)$ , the degree one linear map  $d_a : \mathcal{E}_a \rightarrow \mathcal{E}_a$  is given by

$$d_a := \left( \nabla_{a,1} - \frac{1}{\tau} \nabla_{a,2} \right) d\bar{z},$$

where  $(d_a)^2 = 0$  holds automatically since  $(d\bar{z})^2 = 0$ . Thus,  $\Omega^\tau(\text{Mod-}\mathcal{A}_\theta^2)$  and then  $\Omega^\tau(\text{Pmod-}\mathcal{A}_\theta^2)$  form DG-categories, before being restricted to  $\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^2)$ .

**4.4. (Weak)  $A_\infty$ -categories and Functors between them.** When we compare different DG-categories and/or  $A_\infty$ -categories, the fundamental tools are homotopy equivalence in the framework of  $A_\infty$ -categories, defined by functors between them satisfying appropriate conditions. In this subsection, we briefly recall those notions. The reader can skip to the next section, where we shall refer terminologies in this subsection and so then return to this subsection later if needed.

We start from the case of  $A_\infty$ -algebras. See [57] for extensive background, [35] for direct proofs of fundamental properties, and [42] for a review from the viewpoint of representation

theory and homological algebras. The category extension is straightforward, which comes after that.

First, for a given  $A_\infty$ -algebra  $(V, m)$ , consider the degree shift

$$s : V^r \rightarrow (V[1])^{r-1}$$

which is called the *suspension*. For the  $A_\infty$ -structure  $m_n$ ,  $n \geq 1$ , the induced multilinear map  $sm_n(s^{-1} \otimes \cdots \otimes s^{-1})$  on  $V[1]$ , which we again denote by  $m_n$ , turns out to be of degree one for any  $n \geq 1$ . This simplifies the formulas we shall discuss below. In this suspended notation, the defining equation (43) for an  $A_\infty$ -algebra  $(V[1], m)$  turns out to be

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{|o_1|+\cdots+|o_j|} m_k(o_1, \dots, o_j, m_l(o_{j+1}, \dots, o_{j+l}), o_{j+l+1}, \dots, o_n) \quad (57)$$

with  $o_i := s(w_i)$ ,  $i = 1, \dots, n$ . One sees that the sign has been simplified (see [21]).

**Definition 4.19** ( $A_\infty$ -morphism). Given two  $A_\infty$ -algebras  $(V, m)$  and  $(V', m')$ , a collection of degree preserving (= degree zero) multilinear maps  $\mathcal{F} := \{f_k : (V[1])^{\otimes k} \rightarrow V'[1]\}_{k \geq 1}$ , is called an  $A_\infty$ -morphism  $\mathcal{F} : (V, m) \rightarrow (V', m')$  iff it satisfies the following conditions

$$\sum_{k_1+\cdots+k_i=n} m'_i(f_{k_1} \otimes \cdots \otimes f_{k_i}) = \sum_{k+l=n+1} \sum_{j=0}^{k-1} f_k(\mathbf{1}^{\otimes j} \otimes m_l \otimes \mathbf{1}^{\otimes(n-j-l)}) \quad (58)$$

for  $n \geq 1$ .

Note that the condition (58) for  $n = 1$  implies that  $m'_1 f_1 = f_1 m_1$ , i.e.,  $f_1 : (V[1], m_1) \rightarrow (V'[1], m'_1)$  forms a cochain map.

**Definition 4.20.** An  $A_\infty$ -morphism  $\mathcal{F} : (V, m) \rightarrow (V', m')$  between two  $A_\infty$ -algebras  $(V, m)$  and  $(V', m')$  is called an  $A_\infty$ -quasi-isomorphism iff the cochain map  $f_1 : (V[1], m_1) \rightarrow (V'[1], m'_1)$  is a quasi-isomorphism of cochain complexes, i.e.,  $f_1$  induces an isomorphism on the cohomologies of the cochain complexes. In particular,  $\mathcal{F}$  is called an  $A_\infty$ -isomorphism iff  $f_1 : V[1] \rightarrow V'[1]$  is an isomorphism.

**Remark 4.21.** The suspension further enables us to deal with these tools define in terms of coalgebras. Let  $T^c(V[1]) := \bigoplus_{k \geq 1} (V[1])^{\otimes k}$  be the tensor coalgebra of  $V[1]$ . The degree one multilinear map  $\sum_k m_k \in \text{Hom}(T^c(V[1]), V[1])$  is lifted to be a coderivation  $m$  satisfying  $(m)^2 = 0$ . Thus, an  $A_\infty$ -algebra  $(V, m)$  is equivalent to a DG coalgebra  $(T^c(V[1]), m)$ . For two  $A_\infty$ -algebras  $(V, m)$ ,  $(V', m')$ , an  $A_\infty$ -morphism  $\mathcal{F} : (V, m) \rightarrow (V', m')$  is a degree zero coalgebra homomorphism  $\mathcal{F} : T^c(V[1]) \rightarrow T^c(V'[1])$  such that  $m' \circ \mathcal{F} = \mathcal{F} \circ m$ . Though after the preparation of these terminologies this coalgebra description can give the definitions clearer, in this article we do not use it. See [35] and reference therein.

**Definition 4.22.** An  $A_\infty$ -algebra  $(V, m)$  is called *minimal* if  $m_1 = 0$ .

The following is a key theorem in homotopy algebra:

**Theorem 4.23** (Minimal model theorem (Kadeishvili [29])). *For any  $A_\infty$ -algebra  $(V, m)$ , there exists a minimal  $A_\infty$ -algebra  $(H(V), m')$  and an  $A_\infty$ -quasi-isomorphism  $\mathcal{F} : (H(V), m') \rightarrow (V, m)$ .*

Such an  $A_\infty$ -algebra  $(H(V), m')$  is called a *minimal model* of  $(V, m)$ . The minimal model of  $(V, m)$  is unique up to  $A_\infty$ -isomorphisms on  $H(V)$ .

**Remark 4.24.** Even the original  $A_\infty$ -algebra  $(V, m)$  is a DG-algebra, its minimal model are equipped with higher  $A_\infty$ -products  $m_3, m_4, \dots$  in general. When there exists a minimal model with higher  $A_\infty$ -products all zero, the original DG algebra  $(V, m)$  is often called *formal* in algebraic topology (see [10, 48]).

There exists a canonical way to construct a minimal model of  $(V, m)$  when a Hodge decomposition of  $(V, d := m_1)$  is given:

$$dh + hd = \text{Id}_V - P, \quad h : V^r \rightarrow V^{r-1},$$

where  $P$  is an idempotent,  $P^2 = P$ , and  $h$  is a homotopy operator, a linear map of degree minus one (see [35, subsection 5.4] and reference therein.)

Now, we turn to the category version. The suspension  $s(\mathcal{C})$  of an  $A_\infty$ -category  $\mathcal{C}$  is defined by the shift

$$s : \mathcal{C}(a, b) \rightarrow s(\mathcal{C}(a, b)) =: s(\mathcal{C})(a, b) \quad (59)$$

for any  $a, b \in \text{Ob}(\mathcal{C}) = \text{Ob}(s(\mathcal{C}))$ , where the degree  $|m_n|$  of the  $A_\infty$ -products becomes one for all  $n \geq 1$  as in the case of  $A_\infty$ -algebras.

**Definition 4.25** ( $A_\infty$ -functor). Given two  $A_\infty$ -categories  $\mathcal{C}, \mathcal{C}'$ , an  $A_\infty$ -functor  $\mathcal{F} := \{f; f_1, f_2, \dots\} : s(\mathcal{C}) \rightarrow s(\mathcal{C}')$  is a map  $f : \text{Ob}(s(\mathcal{C})) \rightarrow \text{Ob}(s(\mathcal{C}'))$  of objects with degree preserving multilinear maps

$$f_k : s(\mathcal{C})(a_1, a_2) \otimes \dots \otimes s(\mathcal{C})(a_k, a_{k+1}) \rightarrow s(\mathcal{C}')(f(a_1), f(a_{k+1}))$$

for  $k \geq 1$  satisfying the defining relations of an  $A_\infty$ -morphism (58).

In particular, if  $f : \text{Ob}(s(\mathcal{C})) \rightarrow \text{Ob}(s(\mathcal{C}'))$  is a bijection and  $f_1 : s(\mathcal{C})(a, b) \rightarrow s(\mathcal{C}')(f(a), f(b))$  induces an isomorphism between the cohomologies for any  $a, b \in \text{Ob}(s(\mathcal{C}))$ , we call the  $A_\infty$ -functor a *homotopy equivalence*.

**Definition 4.26** (Minimal  $A_\infty$ -category). An  $A_\infty$ -category  $\mathcal{C}$  is called *minimal* if  $m_1 = 0$ .

One can see that, for a minimal  $A_\infty$ -category  $\mathcal{C}$ , the  $A_\infty$  relations (43) for  $n = 3$  reduces to the associativity condition of the composition of morphisms  $m_2$ . Thus,  $(\mathcal{C}, m_2)$  forms a category in a usual sense.

The minimal model theorem holds true for an  $A_\infty$ -category as a straightforward generalization of the for an  $A_\infty$ -algebra.

The  $A_\infty$ -categories we shall deal with are equipped with the additional structure, the cyclicity (Definition 4.12). For a cyclic (weak)  $A_\infty$ -category  $\mathcal{C}$  with an inner product  $\eta$  (49), after the suspension  $s : \mathcal{C} \rightarrow s(\mathcal{C})$ , the inner product  $s(\eta) =: \omega$  in  $s(\mathcal{C})$  is given by

$$\omega = \eta(s^{-1}, s^{-1}),$$

where the cyclicity condition (50) turns out to be [32]

$$\omega(m_n(o_{12}, \dots, o_{n(n+1)}), o_{(n+1)1}) = (-1)^{(|o_{23}| + \dots + |o_{(n+1)1}|)|o_{12}|} \omega(m_n(o_{23}, \dots, o_{(n+1)1}), o_{12})$$

for homogeneous elements  $o_{i(i+1)} \in s(\mathcal{C})(a_i, a_{i+1})$ ,  $i = 1, \dots, n+1$  (with the identification  $i + (n+1) = i$ ).

**Definition 4.27** (Cyclic  $A_\infty$ -functor). For two cyclic  $A_\infty$ -categories  $\mathcal{C}$  and  $\mathcal{C}'$  with the inner products  $\eta$  and  $\eta'$ , respectively, we call an  $A_\infty$ -functor  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}'$  *cyclic* when

$$\omega'(f_1(o_{ab}), f_1(o_{ba})) = \omega(o_{ab}, o_{ba}), \quad (60)$$

and, for fixed  $n \geq 3$ ,

$$\sum_{k,l \geq 1, k+l=n} \omega'(f_k(o_{12}, \dots, o_{k(k+1)}), f_l(o_{(k+1)(k+2)}, \dots, o_{n(n+1)})) = 0 \quad (61)$$

holds, where  $\omega = s(\eta)$  and  $\omega' = s(\eta')$ .

**Definition 4.28** (Homotopy equivalence of cyclic  $A_\infty$ -categories). For two given cyclic  $A_\infty$ -categories and a cyclic  $A_\infty$ -functor  $\mathcal{C} \rightarrow \mathcal{C}'$ , we call the cyclic  $A_\infty$ -functor  $\mathcal{C} \rightarrow \mathcal{C}'$  a *homotopy equivalence* if it is a homotopy equivalence of  $A_\infty$ -categories. Then, the two (cyclic)  $A_\infty$ -categories  $\mathcal{C}$  and  $\mathcal{C}'$  are called *homotopy equivalent* to each other.

The minimal model theorem holds for any cyclic  $A_\infty$ -category  $\mathcal{C}$ ; there exists a minimal cyclic  $A_\infty$ -category  $\mathcal{C}'$  which is homotopy equivalent to  $\mathcal{C}$ . This is shown in [35, subsection 5.2, 5.5] for cyclic  $A_\infty$ -algebras. As discussed there, an explicit construction of minimal cyclic  $A_\infty$ -algebra also exists ([35, subsection 5.5]).<sup>7</sup>

The  $A_\infty$ -categories we shall deal with have the unit. We end this subsection with giving the definition.

**Definition 4.29** (Unital  $A_\infty$ -category). An  $A_\infty$ -category  $\mathcal{C}$  is called *unital* if there exists an element  $1 \in V_{aa}^0$ , called the *unit*, for any  $a \in \mathcal{C}$  such that  $m_2(1_a, w) = m_2(w, 1_b) = w$  for any  $w \in \mathcal{C}(a, b)$  and  $m_k(\dots, 1_a, \dots) = 0$  for any  $k \geq 3$ .

## 5. HOMOLOGICAL MIRROR SYMMETRY FOR NONCOMMUTATIVE TORI

**5.1. Homological mirror symmetry (HMS).** For a given complex manifold  $(M, I)$  and the mirror dual symplectic manifold  $(\hat{M}, \omega)$ , the homological mirror symmetry proposed by Kontsevich [49] states the following equivalence of triangulated categories:

$$D^b(Fuk(\hat{M}, \omega)) \simeq D^b(coh(M, I)). \quad (62)$$

Here, we need to explain what are the Fukaya category  $Fuk(\hat{M}, \omega)$ . It should be defined as an  $A_\infty$ -category. For an  $A_\infty$ -category, there is a canonical way to construct a triangulated category due to Bondal-Kapranov (the case of DG-categories [4]) and Kontsevich (the case of  $A_\infty$ -categories [49]). Then, the derived category  $D^b(Fuk(\hat{M}, \omega))$  of the Fukaya category  $Fuk(\hat{M}, \omega)$  is the triangulated category obtained in this way. On the other hand,  $D^b(coh(M, I))$  is the derived category of the abelian category  $coh(M, I)$  of coherent sheaves on  $(M, I)$ . Then, the claim of HMS is that the equivalence above holds as triangulated categories. Thus, the homological mirror symmetry, if it exists, gives geometric interpretation of the DG-category on  $(M, I)$  and also help constructing the Fukaya category  $Fuk(\hat{M}, \omega)$  fully. In particular, the Fukaya category  $Fuk(\hat{M}, \omega)$  may be obtained as a minimal model (or a 'smaller' model) of the DG-category on  $(M, I)$ .

In this subsection, we briefly recall these terminologies in order. We first define Fukaya categories for symplectic manifolds, and give a brief introduction of homological algebra such as derived categories, triangulated categories, and the Bondal-Kapranov-Kontsevich construction together with references. The construction of DG-categories associated to complex manifolds is algebraic and comparably easy. On the other hand, to define Fukaya categories on symplectic manifolds is still under construction because of the difficulty on transversality. Finally, we would

<sup>7</sup>In [35], the arguments are concentrated on the case the inner product  $\omega$  defining cyclicity is of degree minus one for an application to string field theory. However, it is clear that the arguments are valid for the case of inner products with any degree (as mentioned in [32, 36, 37], etc.)



like to stress the problem to set-up the homological mirror symmetry conjecture as above, and propose a resolution for it.

**A rough definition of  $\mathcal{C} := Fuk(M, \omega)$ :** There are some variation for Fukaya categories. We first present a simpler Fukaya category which expresses the main idea, and then discuss its possible generalization.

We fix a symplectic manifold  $(M, \omega)$  of dimension  $2n$ . Let  $\text{Ob}(\mathcal{C})$ ,  $\mathcal{C} := Fuk(M, \omega)$ , be the set of Lagrangian submanifolds  $L$  in  $M$ . A Lagrangian submanifold is by definition a  $n$ -dimensional submanifold in  $M$  on which  $\omega$  vanishes. For  $L_a, L_b \in \mathcal{C}$ , when they intersect to each other transversally, the intersection  $L_a \cap L_b$  is a set of points since  $L_a$  and  $L_b$  are  $n$ -dimensional submanifolds of the  $2n$ -dimensional manifold  $M$ . In this case, the space  $\mathcal{C}(L_a, L_b)$  of morphisms is defined by

$$\mathcal{C}(L_a, L_b) := \bigoplus_{v \in L_a \cap L_b} \mathbb{C}[v], \quad (63)$$

where  $[v]$  is the base of  $V_{ab}$  associated to an intersection point  $v \in L_a \cap L_b$  with its degree  $|[v]| \in \mathbb{Z}$  being determined by the Maslov index of the intersection point  $v$  [14]. For  $L_{a_1}, \dots, L_{a_{n+1}} \in Fuk(M)$  such that  $L_{a_i}$  and  $L_{a_{i+1}}$  are transversal for each  $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ , with the identification  $L_{a_i} \sim L_{a_{i+(n+1)}}$ , a multilinear map  $m_n$  of degree  $(2-n)$  is defined by

$$m_n([v_{a_1 a_1}], \dots, [v_{a_n a_{n+1}}]) = \sum_{v \in L_{a_1} \cap L_{a_{n+1}}} \sum_{\phi} \pm e^{-\int_D \phi^* \omega} [v], \quad (64)$$

where  $D$  is a disk with cyclic ordered points  $(z_{12}, \dots, z_{n(n+1)}, z_{(n+1)1})$  on  $\partial(D)$ ,  $\phi: D \rightarrow M$  is a pseudo holomorphic map s.t.  $\phi(\partial_i(D)) \subset L_{a_i}$ ,  $\phi(z_{i(i+1)}) = v_{a_i a_{i+1}}$ ,  $\phi(z_{(n+1)1}) = v$ , and  $\int_D \phi^* \omega$  is the symplectic area of the disk  $D$  (see Figure 1). There is a generalization of this Fukaya

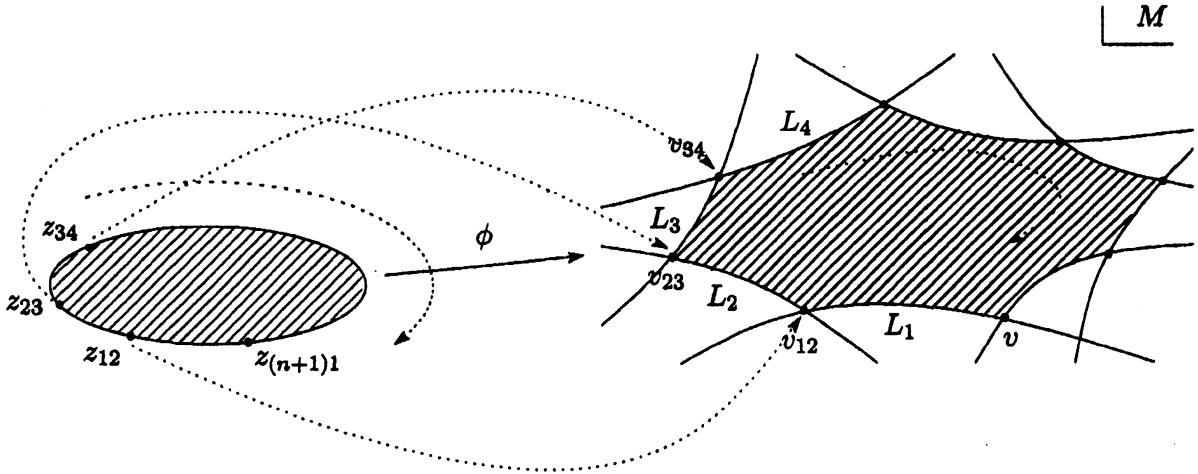


FIGURE 1. A pseudo holomorphic map defining a higher products  $m_n$ .

category due to Kontsevich [49]. Instead of Lagrangian submanifolds for objects, we consider pairs of Lagrangian submanifolds  $L$  with unitary local systems (vector bundles)  $U$  equipped with flat connections. For two objects  $(L_1, U_1), (L_2, U_2)$  such that  $L_1$  and  $L_2$  intersects transversally, the space  $\mathcal{C}((L_1, U_1), (L_2, U_2))$  of morphisms is defined by

$$\mathcal{C}((L_1, U_1), (L_2, U_2)) := \bigoplus_{v \in L_1 \cap L_2} \text{Hom}_{\mathbb{C}}(U_2|_v, U_1|_v).$$

The formula of the multilinear map (64) is modified by adding a new factor to each term equal to the trace of the composition of holonomy maps along the boundary of  $D^2$ . Later we consider line bundles only for the unitary local systems, so we skip giving the precise formula about it.

In any case, one sees that the space  $\mathcal{C}(L, L')$  of morphisms is defined only when  $L$  and  $L'$  intersect transversally. However, even if we consider some full subcategory consisting of Lagrangians intersecting with each other transversally, a Lagrangian  $L$  can not intersect transversally with  $L$  itself. Hence,  $\mathcal{C}(L, L)$  is not defined and so  $A_\infty$ -products  $m_n$  including  $\mathcal{C}(L, L)$  are not defined yet. In this sense, at present, Fukaya category is not defined completely as an  $A_\infty$ -category.<sup>8</sup> In an original paper it is called a *topological  $A_\infty$ -category* [17], in [51] it is called a *pre  $A_\infty$ -category*. We shall return to this problem later in discussing HMS in Problem 5.17.

We let eqs.(63) and (64) or their appropriate generalizations be the axiom of an Fukaya category; *when, on a subset  $\text{Ob}(\mathcal{C}') \subset \text{Ob}(\mathcal{C})$ , there exists an  $A_\infty$ -structure satisfying the axiom, we just say  $\mathcal{C}'$  is a Fukaya  $A_\infty$ -category*, though  $\mathcal{C}'$  should be a full subcategory of the Fukaya  $A_\infty$ -category  $\mathcal{C}$  when it will be defined.

**The derived category  $D^b(\text{coh}(M, I))$ :** For a complex manifold  $(M, I)$ , the derived category  $D^b(\text{coh}(M, I))$  of coherent sheaves on  $(M, I)$  is already a well-established notion, so we do not intend to give the complete definition. The standard references are [20, 41, 27], etc. Let  $\mathcal{O}_M$  is the sheaf of holomorphic functions on the complex manifold  $(M, I)$ . A coherent sheaf  $\mathcal{E}$  on  $(M, I)$  is a sheaf of  $\mathcal{O}_M$ -modules which is obtained locally by an exact sequence

$$(\mathcal{O}_M)^{\oplus m} \rightarrow (\mathcal{O}_M)^{\oplus n} \rightarrow \mathcal{E} \rightarrow 0$$

with some  $n, m \in \mathbb{Z}_{\geq 0}$ , i.e.,  $\mathcal{E}$  is obtained as a quotient of a finitely generated locally free sheaf by a finitely generated locally free sheaf. The case  $m = 0$  corresponds to that  $\mathcal{E}$  is a finitely generated locally free sheaf, i.e., a holomorphic vector bundle. A coherent sheaf is a generalization of a holomorphic vector bundle so that its 'rank' is *not* necessarily constant on  $M$ .

Recall that a category  $\mathcal{C}$  is called an *additive* category iff  $\mathcal{A}$  has a zero object, the space  $\text{Hom}_{\mathcal{C}}(X, Y)$  of morphisms from  $X \in \mathcal{C}$  to  $Y \in \mathcal{C}$  is an abelian group, the composition is bilinear, and it has a structure of direct sums of objects. An additive category  $\mathcal{C}$  is an *abelian* category iff, for any morphism  $f : X \rightarrow Y$ ,  $X, Y \in \mathcal{C}$ , objects, usually denoted,  $\text{Ker}(f)$ ,  $\text{Im}(f)$ ,  $\text{Coim}(f)$ ,  $\text{Coker}(f) \in \mathcal{C}$  are defined and  $\text{Im}(f) \simeq \text{Coim}(f)$  holds for any morphism  $f$  (see [20, 41]). It is known that the category  $\text{coh}(M, I)$  of coherent sheaves on  $(M, I)$  forms an abelian category (see [41, p443]). For an additive category  $\mathcal{C}$ , a *complex* in  $\mathcal{C}$  is a pair

$$X^\bullet := \{X^i, d^i\}_{i \in \mathbb{Z}}$$

such that  $X_i \in \mathcal{C}$ ,  $d_i \in \text{Hom}_{\mathcal{C}}(X^i, X^{i+1})$  and  $d^{i+1} \circ d^i = 0$ . Complexes in  $\mathcal{C}$  form an additive category  $\text{Comp}(\mathcal{C})$ , where the space  $\text{Hom}_{\text{Comp}(\mathcal{C})}(X^\bullet, Y^\bullet)$  is the space of cochain maps from  $X^\bullet \in \text{Comp}(\mathcal{C})$  to  $Y^\bullet \in \text{Comp}(\mathcal{C})$ . In particular, if  $\mathcal{C}$  is abelian, so is  $\text{Comp}(\mathcal{C})$ . The full subcategory consisting of *bounded complexes* (resp. *bounded below*, *bounded above*) is denoted  $\text{Comp}^b(\mathcal{C})$  (resp.  $\text{Comp}^+(\mathcal{C})$ ,  $\text{Comp}^-(\mathcal{C})$ ). The *homotopy category*  $\text{Ho}(\text{Comp}(\mathcal{C}))$  of  $\text{Comp}(\mathcal{C})$  consists of the same objects as in  $\text{Comp}(\mathcal{C})$  but the space  $\text{Hom}_{\text{Ho}(\text{Comp}(\mathcal{C}))}(X^\bullet, Y^\bullet)$  of morphisms is  $\text{Hom}_{\text{Comp}(\mathcal{C})}(X^\bullet, Y^\bullet)$  modulo null-homotopic morphisms. The *derived category*  $D(\mathcal{C})$  of an abelian category  $\mathcal{C}$  is defined by the localization of the homotopy category  $\text{Ho}(\text{Comp}(\mathcal{C}))$  by quasi-isomorphisms in  $\text{Comp}(\mathcal{C})$ . The derived categories  $D^b(\mathcal{C})$ ,  $D^+(\mathcal{C})$  and  $D^-(\mathcal{C})$  are defined

<sup>8</sup>In order to clear the essential difficulty of this problem, the complete construction of  $A_\infty$ -structure is under development in [18] for the case of only one Lagrangian.

in a similar way. These derived categories form triangulated categories (see [20, 41]).<sup>9</sup> Thus, sometimes the notation  $D$  (or  $D^b$ , etc.) is used for a triangulated category. A triangulated category  $\mathcal{T}$  is by definition an additive category equipped with an automorphism  $T : \mathcal{T} \rightarrow \mathcal{T}$  and the structure of triangles defined by a certain axiom (due to Verdier, see [20, 41]). A triangle is a sequence

$$\dots T^{-1}(X) \xrightarrow{T^{-1}(u)} T^{-1}(Y) \xrightarrow{T^{-1}(v)} T^{-1}(Z) \xrightarrow{T^{-1}(w)} X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{T(u)} T(Y) \xrightarrow{T(v)} T(Z) \xrightarrow{T(w)} \dots \quad (65)$$

for  $X, Y, Z \in \mathcal{T}$  and  $u \in \text{Hom}_{\mathcal{T}}(X, Y)$ ,  $v \in \text{Hom}_{\mathcal{T}}(Y, Z)$ ,  $w \in \text{Hom}_{\mathcal{T}}(Z, T(X))$ . As is clear in the description above, the axiom of triangles includes that  $(Y, Z, T(X), v, w, T(u))$  forms a triangle iff  $(X, Y, Z, u, v, w)$  forms a triangle as above (65). Another fundamental statement is that any morphism  $u : X \rightarrow Y$  is embedded in a triangle as above, i.e., the triangulated category  $\mathcal{T}$  includes  $Z$ .

Instead of  $\text{Comp}(\mathcal{C})$ , one can consider a DG-category  $DG(\mathcal{C})$  of complexes in  $\mathcal{C}$ . The objects in  $DG(\mathcal{C})$  are the same as that in  $\text{Comp}(\mathcal{C})$ . For two objects  $X^\bullet, Y^\bullet \in DG(\mathcal{C})$ , the space  $\text{Hom}_{DG(\mathcal{C})}^r(X^\bullet, Y^\bullet)$  consists of any collections  $\phi^\bullet := \{\phi^i : X^i \rightarrow Y^{i+r}\}_{i \in \mathbb{Z}}$ , where the differential  $d : \text{Hom}_{DG(\mathcal{C})}^r(X^\bullet, Y^\bullet) \rightarrow \text{Hom}_{DG(\mathcal{C})}^{r+1}(X^\bullet, Y^\bullet)$  is defined by

$$d(\phi^i) = d_Y^{i+r} \circ \phi^i - \phi^i \circ d_X^i.$$

**Definition 5.1** (The zero-th cohomology category  $H^0(\mathcal{C})$ ). For a DG-category  $\mathcal{C}$ , a category (in a usual sense)  $H^0(\mathcal{C})$  is defined by  $\text{Ob}(H^0(\mathcal{C})) := \text{Ob}(\mathcal{C})$  and, for any  $X, Y \in H^0(\mathcal{C})$ , the space of morphisms is

$$\text{Hom}_{H^0(\mathcal{C})}(X, Y) := H^0(\text{Hom}_{\mathcal{C}}^\bullet(X, Y)).$$

It is clear that  $\text{Ho}(\mathcal{C}) = H^0(DG(\mathcal{C}))$ . Once we obtain a triangulated category  $\mathcal{T}$ , the *Grothendieck group*  $K_0(\mathcal{T})$  of the triangulated category  $\mathcal{T}$  is defined as the free abelian group of isomorphism classes in  $\mathcal{T}$  modulo relation  $[X] - [Y] + [Z] = 0$  for any triangle (65) in  $\mathcal{T}$  [23].

**The Bondal-Kapranov-Kontsevich construction** On the other hand, for any DG-category  $\mathcal{C}$ , there is a canonical way to construct a triangulated category due to Bondal-Kapranov [4]. It is done in the following three steps.

- (a) Construct a DG-category  $(\tilde{\mathcal{C}})^\oplus$  as an additive DG-category which has  $\mathcal{C}$  as a full subcategory and is equipped with an automorphism  $T : (\tilde{\mathcal{C}})^\oplus \rightarrow (\tilde{\mathcal{C}})^\oplus$ .
- (b) Construct a DG-category  $\text{Tw}(\tilde{\mathcal{C}})$  of *one-sided twisted complexes* in  $(\tilde{\mathcal{C}})^\oplus$ .
- (c) Define the category  $\text{Tr}(\mathcal{C})$  as the zero-th cohomology category  $H^0(\text{Tw}(\tilde{\mathcal{C}}))$ .

The precise definitions are in order.

**Definition 5.2** (Additive completion  $\mathcal{C}^\oplus$ ). For a given DG-category  $\mathcal{C}$ , let us add each finite direct sum of objects in  $\mathcal{C}$  and the zero object if they are not in  $\mathcal{C}$ . The DG-category structure on them is induced from that of  $\mathcal{C}$ . We call the resulting DG-category the *additive completion* of  $\mathcal{C}$  and denote it by  $\mathcal{C}^\oplus$ .

**Definition 5.3** (Shift functor completion  $\tilde{\mathcal{C}}$ ). For a given DG-category  $\mathcal{C}$ , the DG-category  $\tilde{\mathcal{C}}$  as follows. The objects are of the form  $X[n]$ , where  $X \in \mathcal{C}$  and  $n \in \mathbb{Z}$ . For  $X[n], Y[m] \in \tilde{\mathcal{C}}$ , the

<sup>9</sup>Before localizations, the homotopy categories already form triangulated categories. A localization is then defined as a quotient of a triangulated category by a triangulated full subcategory, see [41].

space  $\text{Hom}_{\tilde{\mathcal{C}}}^r(X[n], Y[m])$  of morphisms is defined by

$$\text{Hom}_{\tilde{\mathcal{C}}}^r(X[n], Y[m]) := \text{Hom}_{\mathcal{C}}^{r+(m-n)}(X, Y),$$

where the differential and the composition in  $\tilde{\mathcal{C}}$  is induced from  $\mathcal{C}$  by the relation above. Then, there exists an automorphism  $T : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$  such that  $T(X[n]) = X[n+1]$ .

**Definition 5.4** (Twisted complex in a DG-category  $\mathcal{C}^\oplus$ ). For a given DG-category  $\mathcal{C}$ , a *twisted complex*  $(X, \Phi)$  in  $\mathcal{C}^\oplus$  is a pair of an object  $X \in \mathcal{C}^\oplus$  and an element  $\Phi \in \text{Hom}_{\mathcal{C}^\oplus}^1(X, X)$  satisfying

$$d(\Phi) + m(\Phi, \Phi) = 0.$$

A twisted complex  $(X, \Phi)$  is called *one-sided* if  $(X, \Phi)$  is of the form:  $X = \bigoplus_{i=1}^l X_i$ ,  $X_i \in \mathcal{C}$ , and  $\Phi = \{\phi_{ji}\}_{i,j=1,\dots,l}$ ,  $\phi_{ji} \in \text{Hom}_{\mathcal{C}}^1(X_i, X_j)$ , such that  $\phi_{ji} = 0$  for  $i \geq j$ .

**Definition 5.5** (DG category  $Tw(\mathcal{C})$ ). For a given DG-category  $\mathcal{C}$ , the DG-category  $Tw(\mathcal{C})$  of one-side twisted complexes is defined as follows. The objects are one-sided complexes  $(X, \Phi)$  in  $\mathcal{C}^\oplus$ . For any two one-sided complexes  $(X, \Phi), (Y, \Psi) \in Tw(\mathcal{C})$ , the space of morphisms is given by

$$\text{Hom}_{Tw(\mathcal{C})}((X, \Phi), (Y, \Psi)) := \text{Hom}_{\mathcal{C}^\oplus}(X, Y).$$

The differential  $d_{Tw(\mathcal{C})}$  is given by

$$d_{Tw(\mathcal{C})}(\varphi) := d(\varphi) + m(\Phi, \varphi) - (-1)^{|\varphi|} m(\varphi, \Psi)$$

for  $\varphi \in \text{Hom}_{Tw(\mathcal{C})}((X, \Phi), (Y, \Psi))$ .

**Definition 5.6** (Triangulated category  $Tr(\mathcal{C})$ ). The triangulated category  $Tr(\mathcal{C})$  is defined by

$$Tr(\mathcal{C}) := H^0(Tw(\tilde{\mathcal{C}})).$$

**Remark 5.7.** Since the procedure of taking  $Tw$  for the original DG category  $\mathcal{C}$  corresponds to adding objects enough for the resulting category to be closed under the triangle (see the remark below eq.(65)). Thus, for the zero-th cohomology categories one has the following equivalence

$$H^0(Tw(Tw(\mathcal{C}))) \simeq H^0((Tw(\mathcal{C}))).$$

This construction of triangulated categories in the framework of DG-categories suggests also a noncommutative generalization of  $D^b(\text{coh}(M))$  as we shall discuss later.

The generalization of this construction to an  $A_\infty$ -category  $\mathcal{C}$  is parallel. For a given  $A_\infty$ -category  $\mathcal{C}$ , the additive completion  $\mathcal{C}^\oplus$  and the  $A_\infty$ -category  $\tilde{\mathcal{C}}$  are defined in the same way. For the  $A_\infty$ -category  $(\tilde{\mathcal{C}})^\oplus$ , the remaining procedures are presented in [49]. However, to simplify the signs, we work for  $A_\infty$ -categories in suspended notation (eq.(59)), and hence we consider degree zero elements for defining twisted complexes.

**Definition 5.8** (Twisted complex in the  $A_\infty$ -category  $\mathcal{C}^\oplus$ ). For a given the  $A_\infty$ -category  $\mathcal{C}$ , a *twisted complex*  $(X, \Phi)$  in  $\mathcal{C}^\oplus$  is a pair of an object  $X \in \mathcal{C}^\oplus$  and an element  $\Phi \in (\mathfrak{s}(\mathcal{C}^\oplus))^0(X, X)$  satisfying the Maurer-Cartan equation for the  $A_\infty$ -structure:

$$d(\Phi) + m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + \dots = 0$$

in  $\mathfrak{s}(\mathcal{C}^\oplus)$ . A twisted complex  $(X, \Phi)$  is called *one-sided* if  $(X, \Phi)$  is of the form:  $X = \bigoplus_{i=1}^l X_i$ ,  $X_i \in \mathcal{C}$ , and  $\Phi = \{\phi_{ij}\}_{i,j=1,\dots,l}$ ,  $\phi_{ij} \in (\mathfrak{s}(\mathcal{C}))^0(X_i, X_j)$ , such that  $\phi_{ij} = 0$  for  $i \geq j$ .

If the  $A_\infty$ -structure  $m_n$  can be nonzero for any large  $n$ , the Maurer-Cartan equation for  $\Phi$  is not well-defined since it contains the infinite sum. However, if  $\Phi$  is one-sided, the Maurer-Cartan equation is always well-defined since  $m_n(\Phi, \dots, \Phi) = 0$  for any sufficiently large  $n$ .

**Definition 5.9** ( $A_\infty$ -category  $Tw(\mathcal{C})$ ). For a given  $A_\infty$ -category  $\mathcal{C}$ , the  $A_\infty$ -category  $Tw(\mathcal{C})$  of one-side twisted complexes is defined as follows. The objects are one-sided complexes  $(X, \Phi)$  in  $\mathcal{C}$ . For any two one-sided complexes  $(X, \Phi), (Y, \Psi) \in Tw(\mathcal{C})$ , the space of morphisms is given by

$$(Tw(\mathcal{C}))((X, \Phi), (Y, \Psi)) := (\mathcal{C}^\oplus)(X, Y).$$

The  $A_\infty$ -structure is given by

$$m_n(\varphi_{a_1 a_2}, \dots, \varphi_{a_n a_{n+1}}) := \sum_{k_1, \dots, k_{n+1} \in \mathbb{Z}_{\geq 0}} m_{n+k_1+\dots+k_{n+1}}(\Phi_{a_1}^{k_1}, \varphi_{a_1 a_2}, \Phi_{a_2}^{k_2}, \dots, \Phi_{a_n}^{k_n}, \varphi_{a_n a_{n+1}}, \Phi_{a_{n+1}}^{k_{n+1}})$$

for  $a_i := (X_{a_i}, \Phi_{a_i}) \in Tw(\mathcal{C})$  and  $\varphi_{a_i a_{i+1}} \in (s(Tw(\mathcal{C}))(a_i, a_{i+1}))$ .

**Remark 5.10.** The  $A_\infty$ -category  $Tw(\mathcal{C})$  reduces to the DG-category in Definition 5.5 when the higher  $A_\infty$ -products of  $\mathcal{C}$  are absent  $m_3 = m_4 = \dots = 0$ . In particular, one sees that the product  $m_2$  in  $Tw(\mathcal{C})$  does not receive any correction by  $\Phi_{a_i}$  and is the same as  $m_2$  in  $(\mathcal{C})^\oplus$  because of the absence of the higher  $A_\infty$ -products.

**Definition 5.11** (The zero-th cohomology category  $H^0(\mathcal{C})$ ). For an  $A_\infty$ -category  $\mathcal{C}$ , the category  $H^0(\mathcal{C})$  is defined by  $\text{Ob}(H^0(\mathcal{C})) := \text{Ob}(\mathcal{C})$  and for any  $X, Y \in \mathcal{C}$  the space of morphisms is

$$\text{Hom}_{H^0(\mathcal{C})}(X, Y) := H^0(\mathcal{C}^\bullet(Y, X)).$$

The composition in  $H^0(\mathcal{C})$  is given by the one induced from  $m_2$  in  $\mathcal{C}$ .

**Remark 5.12.** In other words,  $H^0(\mathcal{C})$  is obtained by the degree zero part of the graded category obtained by forgetting higher  $A_\infty$ -products  $m_3, m_4, \dots$  of a minimal  $A_\infty$ -category of  $\mathcal{C}$  (see Definition 4.26 and below). Then, it is clear that the composition in  $H^0(\mathcal{C})$  is associative.

For any  $A_\infty$ -category  $\mathcal{C}$ , the triangulated category  $Tr(\mathcal{C})$  is defined as

$$Tr(\mathcal{C}) := H^0(Tw(\tilde{\mathcal{C}})).$$

A variant of this Bondal-Kapranov-Kontsevich construction is that, for  $\mathcal{C}$ , we do not pass  $\tilde{\mathcal{C}}$  and consider the twisted complex in  $\mathcal{C}^\oplus$ . The resulting category

$$H^0(Tw(\mathcal{C}))$$

is an extension closed full subcategory (of an abelian category), which is called an *exact category* (see [26, p10]).

The following lemmas should be known.

**Lemma 5.13.** *Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two  $A_\infty$ -categories which are homotopy equivalent to each other. Then,  $H^0(Tw(\mathcal{C})) \simeq H^0(Tw(\mathcal{C}'))$ .*

**Proof.** This essentially follows from the decomposition theorem for  $A_\infty$ -algebras (see [35]). The detail will be presented elsewhere.  $\square$

**Corollary 5.14.** *Let  $\mathcal{C} \simeq \mathcal{C}'$  be two homotopy equivalent  $A_\infty$ -categories. Then,  $Tr(\mathcal{C}) \simeq Tr(\mathcal{C}')$  as triangulated categories.*  $\square$

Note that the converse is not true in general. For two abelian categories which are not equivalent, the derived categories can be equivalent (see [26, 43]). We shall see similar phenomena in subsection 5.2. These hold true also in the setting with cyclic structures due to the decomposition theorem for cyclic  $A_\infty$ -structures [35, subsection 5.2].

**Lemma 5.15.** *Let  $\mathcal{C}$  be a cyclic  $A_\infty$ -category. Then, a cyclic structure is induced on the  $A_\infty$ -category  $\text{Tw}(\mathcal{C})$ .  $\square$*

**Lemma 5.16.** *Let  $\mathcal{C}$  and  $\mathcal{C}'$  be two cyclic  $A_\infty$ -categories which are homotopy equivalent to each other. Then,  $\text{Tw}(\mathcal{C}) \simeq \text{Tw}(\mathcal{C}')$  as cyclic  $A_\infty$ -categories.  $\square$*

For a triangulated category  $\mathcal{T}$ , if there exists an  $A_\infty$ -category  $\mathcal{C}$  such that  $\mathcal{T} \simeq \text{Tr}(\mathcal{C})$ , we say  $\mathcal{C}$  is a generator of  $\mathcal{T}$ . By construction, a generator  $\mathcal{C}$  is a full subcategory of  $\text{Tw}(\mathcal{C})$  or  $\text{Tw}(\tilde{\mathcal{C}})$ .

The point is that the notion of generators of a triangulated category  $\mathcal{T}$  does not exist without the help of DG or  $A_\infty$ -categories. For instance, let  $\mathcal{C}$  be a DG full subcategory  $\mathcal{C}' \subset \mathcal{C}$  of a DG-category  $\mathcal{C}$  such that  $H^0(\mathcal{C})$  forms a triangulated category. Then,  $\text{Tw}(\tilde{\mathcal{C}}')$  is also a full sub DG-category of  $\mathcal{C}$ . In particular,  $\text{Tr}(\mathcal{C}')$  is the smallest triangulated full subcategory of  $H^0(\mathcal{C})$  containing  $H^0(\mathcal{C}')$ , and hence  $H^0(\mathcal{C}')$  may be called a generator of  $\text{Tr}(\mathcal{C}')$ . However, now forget about  $H^0(\mathcal{C})$  and try to find a triangulated category  $\mathcal{T}'$  generated by  $H^0(\mathcal{C}')$ . One fails to do it since, for a morphism  $u : X \rightarrow Y$  in  $\text{Tr}(\mathcal{C}')$ , the axiom of triangles (see comments around eq.(65)) does not determine  $Z \in \mathcal{T}'$  uniquely (up to isomorphisms) so that  $(X, Y, Z; u, v, w)$  is a triangle in  $\mathcal{T}'$ . On the other hand, if we keep the DG-structure instead of  $H^0(\mathcal{C}')$ , the DG-category  $\mathcal{C}'$  generates the triangulated category  $\text{Tr}(\mathcal{C}')$  uniquely. This shows an essence of Bondal-Kapranov construction [4].

Corollary 5.14 then implies that, in order to show an equivalence of triangulated categories  $\mathcal{T} \simeq \mathcal{T}'$ , we may find generator  $A_\infty$ -categories  $\mathcal{C}$  and  $\mathcal{C}'$ ,  $\text{Tr}(\mathcal{C}) \simeq \mathcal{T}$ ,  $\text{Tr}(\mathcal{C}') \simeq \mathcal{T}'$ , such that  $\mathcal{C} \simeq \mathcal{C}'$  as  $A_\infty$ -categories.

On the other hand, a similar remark to Remark 5.7 applies to  $A_\infty$ -categories. Then, the left hand side of HMS (62) means

$$D^b(\text{Fuk}(\hat{M}, \omega)) := \text{Tr}(\text{Fuk}(\hat{M}, \omega)).$$

Namely, the Fukaya category  $\text{Fuk}(\hat{M}, \omega)$  may be already ‘large’ enough, this procedure guarantees that the result is closed under the triangles.

Then, one may think that a strategy to show HMS is to find a generator DG-category  $\mathcal{C}_{DG}$  of  $D^b(\text{coh}(M, I))$  and a generator  $A_\infty$ -category  $\mathcal{C}_{A_\infty}$  of  $D^b(\text{Fuk}(\hat{M}, \omega))$  such that  $\mathcal{C}_{A_\infty} \simeq \mathcal{C}_{DG}$ . However, as mentioned above, the Fukaya category  $\text{Fuk}(\hat{M}, \omega)$  is not still defined as an  $A_\infty$ -category. Thus, here is a modified version of HMS:

**Problem 5.17.** *For mirror pair manifolds  $(M, I)$  and  $(\hat{M}, \omega)$ , find a DG-category  $\mathcal{C}_{DG}$  and a Fukaya  $A_\infty$ -category  $\mathcal{C}_{A_\infty}$  on  $(\hat{M}, \omega)$  such that  $\text{Tr}(\mathcal{C}_{DG}) \simeq D^b(\text{coh}(M, I))$  and  $\mathcal{C}_{A_\infty} \simeq \text{DG}(M, I)$  as  $A_\infty$ -categories.*

Recall that the Fukaya category has (transversal) higher  $A_\infty$ -products associated to pseudo-holomorphic disks. Then, one hope is to obtain the Fukaya  $A_\infty$  category  $\mathcal{C}_{A_\infty}$  as a minimal, or smaller model of the DG-category  $\mathcal{C}_{DG}$  (see Remark 4.24). A strategy for the plan is proposed by Kontsevich-Soibelman [51] and then [37]. When one can do it for a class of mirror pair manifolds, by Lemma 5.13 and Corollary 5.14, one has an  $A_\infty$ -equivalence  $\text{Tw}(\mathcal{C}_{A_\infty}) \simeq \text{Tw}(\mathcal{C}_{DG})$  and then a triangulated equivalence  $\text{Tr}(\mathcal{C}_{A_\infty}) \simeq \text{Tr}(\mathcal{C}_{DG})$ , where  $\text{Tw}(\mathcal{C}_{A_\infty})$ , defined in half geometric and half algebraic ways, can be a candidate for a Fukaya category as an  $A_\infty$ -category. Then, when  $\text{Fuk}(\hat{M}, \omega)$  will be defined completely in a geometric way, to compare it with  $\text{Tw}(\mathcal{C}_{A_\infty})$  will be another interesting future problem.

**5.2. HMS for noncommutative two-tori.** Though HMS itself is still a difficult problem, we would like to discuss a noncommutative generalization of HMS. Since, the complex manifold

side,  $D^b(\text{coh}(M, I))$ , is defined in an algebraic way, it may be possible to consider a triangulated category on a noncommutative deformation of  $M$  via module categories over noncommutative algebras. Furthermore, if we can find a generator DG-category of the triangulated category, a minimal (or smaller)  $A_\infty$ -category homotopy equivalent to the generator DG-category may define a generator  $A_\infty$ -category for a noncommutative version of Fukaya  $A_\infty$ -categories.

This plan worked well for real noncommutative two-tori with complex structures, as did first in [31] (at the level without higher  $A_\infty$ -products). After combining many further results mentioned later, now we can answer the HMS, in the sense in Problem 5.17, and a noncommutative generalization of it for noncommutative two-tori, though the result does not depend greatly the noncommutativity in this case. In this subsection, we discuss this case of noncommutative tori; answer the (NC)HMS obtained by combining previous results in Theorem 5.20.

First, for the complex side  $(\mathcal{A}_\theta^2, \tau)$ , the triangulated category we consider is

$$\text{Tr}(\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2)).$$

Recall that, in the case of  $\mathcal{A}_\theta^2$ , the curved DG category  $\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2)$  is a DG category (see the end of subsection 4.3).

For the symplectic side, we consider a (commutative) flat complexified symplectic (Definition 3.13) torus  $(T^2, \omega, B)$  with  $(\omega, B)$  given by  $\tau$  via the mirror relation (41) [31, 32]. Note that, in this two-dimensional case, any one-dimensional submanifold becomes a Lagrangian submanifold as  $(d_a)^2 = 0$  for any  $\mathcal{E}_a \in \Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2)$ . We fix an irrational number  $\theta \in \mathbb{R}$ . Let  $\pi : \mathbb{R}^2 \rightarrow T^2$  is the universal cover of  $T^2$ , with coordinates  $(x, y) \in \mathbb{R}^2$  being identified by  $\pi$  as  $x \sim x + 1$  and  $y \sim y + 1$ .

Let  $\mathcal{Lag}$  be the set of quadruples  $(p, q, \alpha, \beta)$  such that  $(p, q) \in \mathbb{Z}^2$  are relatively prime integer satisfying  $q + p\theta > 0$  and  $(\alpha, \beta) \in \mathbb{R}^2$ . We label by  $a, b, \dots$  elements in  $\mathcal{Lag}$ . Each  $a := (p_a, q_a, \alpha_a, \beta_a) \in \mathcal{Lag}$  is associated with a geodesic cycle  $\pi(L_a) \in T^2$ ,

$$L_a : q_a y = p_a x + \alpha_a, \quad \alpha_a \in \mathbb{R},$$

where  $\beta_a$  is regarded as a constant defining a flat connection of a trivial line bundle:

$$\nabla_{a,1} = \frac{\partial}{\partial x} - \frac{2\pi i \beta_a}{q_a + p_a \theta}, \quad \beta_a \in \mathbb{R}.$$

The period of the cycle  $\pi(L_a)$  is  $q_a + p_a \theta$  in the coordinate  $x$ , so we call that  $\nabla_a$  and  $\nabla_{a'} := (Z_1)^{-n} \nabla_a (Z_1)^n$ ,  $Z_1 := \exp(2\pi i x / (q_a + p_a \theta))$ , are gauge equivalent for any  $n \in \mathbb{Z}$ . One has  $\beta_{a'} = \beta_a + n$ . For each object  $a \in \mathcal{Lag}$ , we again attach the number  $\mu_a$  by

$$\mu_a := \frac{p_a}{q_a + p_a \theta},$$

and, for any two objects  $a$  and  $b$ , we set  $\mu_{ab} := \mu_b - \mu_a$ . Note that by a  $SL(2, \mathbb{R})$  translation

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x^\theta \\ y^\theta \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

a line  $x + \theta y = 0$  becomes a vertical line  $x^\theta = 0$ ,  $\mu_a$  is the slope of  $L_a$  after the translation.

For a fixed  $n \geq 2$ , consider a collection  $\vec{a} := (a_1, \dots, a_{n+1})$  of objects  $a_1, \dots, a_{n+1} \in \mathcal{Lag}$  such that  $\mu_{a_i a_{i+1}} \neq 0$  for any  $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ , where we set  $a_i = a_{i+(n+1)}$ . We call such a collection  $\vec{a}$  a *transversal* collection.

For a transversal collection  $\vec{a}$ , let  $\vec{v} := (v_{a_1 a_2}, \dots, v_{a_n a_{n+1}}, v_{a_{n+1} a_1})$  be a collection of intersection points in  $T^2$  such that  $v_{a_i a_{i+1}} \in \pi(L_{a_i}) \cap \pi(L_{a_{i+1}})$ . We call  $\vec{v}$  *generic* if  $v_{a_{i-1} a_i} \neq v_{a_i a_{i+1}}$  for any  $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ .

We call an  $A_\infty$ -category satisfying the following axiom a *Fukaya  $A_\infty$ -category*  $\mathcal{C}$  on  $(T^2, \rho, \theta; \mathfrak{Lag})$ .

**Axiom 5.18.** (i) For any  $a, b \in \mathfrak{Lag}$  such that  $\mu_{ab} \neq 0$ , the space of morphisms is set to be

$$\mathcal{C}(a, b) := \bigoplus_{v_{ab} \in \pi(L_a) \cap \pi(L_b)} \mathbb{C}[v_{ab}],$$

where the degree  $||[v_{ab}]||$  is zero if  $\mu_{ab} > 0$  and one if  $\mu_{ab} < 0$ . There exists a nondegenerate inner product

$$\eta : \mathcal{C}(a, b) \otimes \mathcal{C}(b, a) \rightarrow \mathbb{C}$$

defined by  $\eta([v_{ab}], [v_{ba}]) = 1$  if  $v_{ab} = v_{ba}$  as points in  $T^2$  and zero otherwise. By this we define the dual basis  $[v_{ab}]^* \in \mathcal{C}(b, a)$  of a base  $[v_{ab}] \in \mathcal{C}(a, b)$ ; if  $v_{ab} = v_{ba}$ , we denote  $[v_{ba}] = [v_{ab}]^*$  and vice versa.

(ii) For any transversal collection  $\vec{a} = (a_1, \dots, a_{n+1})$ ,  $n \geq 2$ , express the  $A_\infty$ -product  $m_n$  as

$$m_n([v_{a_1 a_2}], \dots, [v_{a_n a_{n+1}}]) := \sum_{v_{a_{n+1} a_1} \in \pi(L_{a_1}) \cap \pi(L_{a_{n+1}})} c(\vec{v}) \cdot [v_{a_{n+1} a_1}]^*$$

with a constant  $c(\vec{v}) \in \mathbb{C}$ , where  $\vec{v} = (v_{a_1 a_2}, \dots, v_{a_n a_{n+1}}, v_{a_{n+1} a_1})$ . By degree counting,  $m_n$  can be nonzero only if two of the numbers  $\mu_{a_1 a_2}, \dots, \mu_{a_n a_{n+1}}, \mu_{a_{n+1} a_1}$  are positive, since  $|m_n| = (2 - n)$ .

When  $\vec{v}$  is generic, the constant  $c(\vec{v})$  is set to be

$$c(\vec{v}) := \sum_{\vec{v} \in CC(\vec{v})} \text{sign}(\vec{v}) \exp(2\pi i \rho A(\vec{v})) \exp(2\pi i \int \beta(\vec{v})), \quad (66)$$

where  $CC(\vec{v})$  is the subset of

$$\{\vec{v} = (\tilde{v}_{a_1 a_2}, \dots, \tilde{v}_{a_n a_{n+1}}, \tilde{v}_{a_{n+1} a_1}) \in (\pi^{-1}(v_{a_1 a_2}), \dots, \pi^{-1}(v_{a_n a_{n+1}}), \pi^{-1}(v_{a_{n+1} a_1}))\}$$

satisfying the following conditions:

- the geodesic interval  $(\tilde{v}_{a_{i-1} a_i}, \tilde{v}_{a_i a_{i+1}})$  is included in  $\pi^{-1}\pi(L_{a_i})$  for any  $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ ,
- $\vec{v}$  forms a clockwise convex  $(n+1)$ -gon in the universal cover  $\mathbb{R}^2$  of  $T^2$ .
- $\tilde{v}_{a_{n+1} a_1} = v_{a_{n+1} a_1} \in \mathbb{R}^2$ , where we fixed an inclusion of the fundamental domain  $T^2$  to the universal cover  $\mathbb{R}^2$  and denoted the image of  $v_{a_{n+1} a_1}$  also by  $v_{a_{n+1} a_1}$  itself.

The sign  $\text{sign}(\vec{v})$  is then defined by

$$\text{sign}(\vec{v}) := \begin{cases} \text{sign}(x^\theta(\tilde{v}_{a_1 a_2}) - x^\theta(\tilde{v}_{a_{n+1} a_1})), & (n : \text{odd}), \\ 1, & (n : \text{even}), \end{cases}$$

$A(\vec{v})$  is the area of the convex  $(n+1)$ -gon, and  $\int \beta(\vec{v})$  is given by

$$\sum_{i=1}^{n+1} (x^\theta(\tilde{v}_{a_i a_{i+1}}) - x^\theta(\tilde{v}_{a_{i-1} a_i})) \frac{\beta_{a_i}}{q_{a_i} + p_{a_i} \theta}.$$

**Remark 5.19.** This axiom is at least compatible with a (cyclic)  $A_\infty$ -structure. For transversal generic collections, the fact that only convex  $(n+1)$ -gons are ‘counted’ is equivalent to the fact that  $c(\vec{v})$  is nonzero only when  $\sum_{i=1}^{n+1} \deg(v_{a_i a_{i+1}}) = -2 + (n+1)$ . The  $A_\infty$ -relation follows from concentrating on a polygon which has one nonconvex vertex. There exist two ways to divide the polygon into two convex polygons. The corresponding terms then appear with opposite signs and cancel with each other in the  $A_\infty$ -relation. See [36], where these facts are explained for the case  $\mathbb{R}^2$ , which is enough to understand these facts for  $T^2$ .

**Theorem 5.20.** For fixed irrational number  $\theta$  and  $\rho, \tau \in \mathbb{C}$ ,  $\text{Im}(\tau) > 0$ , with  $\rho = -1/\tau$ , there exist homotopy equivalent cyclic  $A_\infty$ -categories  $\mathcal{C}_{A_\infty, \theta} \simeq \mathcal{C}_{DG, \theta}$  such that



- i)  $\mathcal{C}_{A_\infty, \theta}$  is a unital minimal cyclic  $A_\infty$  Fukaya category (Axiom 5.18) on  $(T^2, \rho, \theta; \mathcal{L}ag)$ ,
- ii)  $\mathcal{C}_{DG, \theta}$  is a unital cyclic DG-category such that  $Tr(\mathcal{C}_{DG, \theta}) \simeq Tr(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$ .

Here, we can set the generator DG category as  $\mathcal{C}_{DG, \theta} := \Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2)$  in Definition 4.15.<sup>10</sup> The existence of the unital minimal cyclic  $A_\infty$ -category  $\mathcal{C}_{A_\infty, \theta}$  homotopy equivalent to  $\mathcal{C}_{DG, \theta}$  follows from the explicit construction of it from  $\mathcal{C}_{DG, \theta}$  in the way done for  $\mathbb{R}^2$ , the covering space of  $T^2$ , with  $\theta = 0$  in [36], where the leading term  $f_1$  of the homotopy equivalence  $\mathcal{F} := \{f; f_1, f_2, \dots\} : \mathcal{C}_{A_\infty, \theta} \rightarrow \mathcal{C}_{DG, \theta}$  is obtained in [32]. For  $T^2$  with  $\theta = 0$  case, the unital minimal cyclic  $A_\infty$ -category  $\mathcal{C}_{A_\infty, \theta=0}$  is presented explicitly in [38] (see also [37, section 6]). The second statement (Theorem 5.20 ii)) is obtained due to a result by Polishchuk on classification of holomorphic vector bundles on noncommutative two-tori. In our notation:

**Theorem 5.21** (Polishchuk [62, Theorem 1.1]). *The exact category  $H^0(Tw(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2)))$  coincides with the category  $H^0(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$  of all holomorphic vector bundles over  $(\mathcal{A}_\theta^2, \tau)$ .*

Now, the DG-category  $Tw(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2))$  is a DG full subcategory of  $\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2)$ . For a DG-category  $\mathcal{C}$ , let us denote  $\widetilde{Tw}(\mathcal{C}) := Tw(\widetilde{\mathcal{C}})$ . Then,  $\widetilde{Tw}(Tw(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2)))$  is a DG full subcategory of  $\widetilde{Tw}(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$ . On the other hand,  $\widetilde{Tw}(Tw(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2)))$  is a DG full subcategory of  $Tw(\widetilde{Tw}(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2)))$  also. Thus, the following sequence of DG full subcategories is obtained,

$$\widetilde{Tw}(Tw(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2))) \subset Tw(\widetilde{Tw}(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2))) \subset \widetilde{Tw}(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2)),$$

where  $H^0(\widetilde{Tw}(Tw(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2)))) \simeq H^0(\widetilde{Tw}(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))) = Tr(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$  as triangulated category since the zero-th cohomology categories have the same objects (see Definition 5.1), and  $H^0(Tw(\widetilde{Tw}(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2)))) \simeq Tr(\Omega^\tau(\text{Pmod}^{st} - \mathcal{A}_\theta^2))$  (see Remark 5.7). This concludes the statement of Theorem 5.20 ii).

One sees that the final results, the triangulated categories  $Tr(\mathcal{C}_{A_\infty, \theta}) \simeq Tr(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$  do not depend on the irrational number  $\theta$ . This is because one sees:

**Proposition 5.22.** *For irrational numbers  $\theta \neq \theta'$ , one has a cyclic  $A_\infty$ -isomorphism  $\tilde{\mathcal{C}}_{A_\infty, \theta} \simeq \tilde{\mathcal{C}}_{A_\infty, \theta'}$ .  $\square$*

On the other hand, in [63], the equivalence of  $Tr(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$  with the derived category  $D^b(\text{coh}(T^2, \tau))$  of coherent sheaves on the (commutative) elliptic curve is discussed (without HMS above) using an analog of Fourier-Mukai transformation functor from  $Tr(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$  to  $D^b(\text{coh}(T^2, \tau))$  which can be defined naturally in the noncommutative tori framework. This also implies that  $Tr(\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2))$  is independent of  $\theta$ . Thus, Theorem 5.20 completes HMS for (commutative) tori in the sense of Problem 5.17. For partial results on HMS for two tori so far, see references in [37].

The  $\theta$  dependence of  $\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2)$  is interpreted as follows. First, recall that  $K_0(\mathcal{A}_\theta^2)$  is identified with the even lattice  $\wedge^{\text{even}}(D)$  (eq.(8)). For any  $E \in \text{Pmod} - \mathcal{A}_\theta^2$ , an element  $\mu(E) = \pm(q + p dx \wedge dy) \in K_0(\mathcal{A}_\theta^2)$  with positive trace,  $q + p\theta > 0$ , is defined. Attach  $\phi_\theta(E) \in \mathbb{R}$  by

$$-\frac{1}{2} < \phi_\theta(E) := \frac{1}{\pi} \text{Arg} \left( \frac{p}{q + p\theta} \right) < \frac{1}{2}.$$

<sup>10</sup>See Remark 2.18 for the reason of the minus sign for  $\theta$  here and in Theorem 5.20 ii). Instead of  $\Omega^\tau(\text{Pmod}^\nabla - \mathcal{A}_\theta^2)$ , one can construct a DG category consisting of left modules. Then, as DG-categories, the structure of left  $\mathcal{A}_\theta^2$ -modules is isomorphic to that of right  $\mathcal{A}_\theta^2$ -modules.

The surjection  $s : \text{Ob}(\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2)) \rightarrow \text{Ob}(\text{Pmod}\text{-}\mathcal{A}_\theta^2)$  obtained by forgetting the additional structures induces a map  $\phi_\theta : \text{Ob}(\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2)) \rightarrow \mathbb{R}$  by  $\phi_\theta(\mathcal{E}) := \phi_\theta(s(\mathcal{E}))$  for any  $\mathcal{E} \in \text{Ob}(\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2))$ . For objects  $\mathcal{E}[n]$ ,  $n \in \mathbb{Z}$ , in the shift functor completion (Definition 5.3) of  $\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2)$ , we set  $\phi_\theta(\mathcal{E}[n]) := \phi_\theta(\mathcal{E}) + n$ . On the other hand, the result of [63] further implies that the shift functor completion of  $H^0(\widetilde{\text{Tw}}(\text{Pmod}^{\text{st}}\text{-}\mathcal{A}_\theta^2))$  is equivalent to the triangulated categories  $H^0(\widetilde{\text{Tw}}(\text{Pmod}^{\text{st}}\text{-}\mathcal{A}_\theta^2)) \simeq D^b(\text{coh}(T^2, \tau))$ . This, together with Theorem 5.21, implies that the shift functor completion of  $H^0(\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2))$  is equivalent to the derived category  $D^b(\text{coh}(T^2, \tau))$  [62, Corollary 1.2]. The consequence is that the full subcategory of  $D^b(\text{coh}(T^2, \tau))$  consisting of objects with  $-1/2 < \phi_\theta < 1/2$  is equivalent to  $\Omega^\tau(\text{Pmod}^\nabla\text{-}\mathcal{A}_\theta^2)$  as categories. Since  $\theta$  is irrational, there does not exist an object with  $\phi_\theta = \pm 1/2$ , which simplifies the arguments.

This  $\theta$  dependence is interpreted [63] as the  $t$ -structure in the framework of Bridgeland-Douglas stability conditions [5]. In this framework, the full subcategory of  $D^b(\text{coh}(T^2, \tau))$  consisting of objects with  $-1/2 < \phi_\theta < 1/2$  is obtained as the heart of the  $t$ -structure, and hence forms an abelian category by [5, Proposition 5.3]. The real number  $\phi$  corresponds to what is called the *phase*. The *central charges* defining stability conditions and their physical interpretation are discussed in the last half of [32, section 2.3]. An interpretation of the mirror duality  $\rho = -1/\tau$  for noncommutative tori is also given there.

**5.3. On HMS for higher dimensional noncommutative tori.** The next problem may be the extension of the HMS for two-tori in the previous subsection to higher (even) dimensional tori. The extension is not quite straightforward; in this higher dimensional case, we do not know what kind of categories we should consider. On the other hand, the noncommutativity should correspond to a deformation in the sense of extended deformation by Barannikov-Kontsevich [1]; we should set up the problem (of constructing appropriate categories and of discussing their deformations) so that the correspondence would be described well.

In [34], a full subcategory  $\mathcal{C}$  of the curved DG category  $\Omega^\tau(\text{Pmod}^{\text{st}}\text{-}\mathcal{A}_\theta^{2n})$  is constructed explicitly in the sense in subsection 2.5 for an abelian variety defined by  $(\mathcal{A}_{\theta=0}^{2n}, \tau = \mathbf{i} \cdot \mathbf{1}_n)$  and its noncommutative deformations  $(\mathcal{A}_\theta^{2n}, \tau = \mathbf{i} \cdot \mathbf{1}_n)$ , where the full subcategory  $\mathcal{C}$  consists of modules over  $(\mathcal{A}_\theta^{2n}, \tau = \mathbf{i} \cdot \mathbf{1}_n)$  which corresponds to holomorphic line bundles when  $\theta = 0$ . The deformations discussed explicitly correspond to the noncommutativities of the following three cases:

$$\text{Type } \theta_1: \theta_2 = \theta_3 = 0, \quad \text{Type } \theta_2: \theta_1 = \theta_3 = 0, \quad \text{Type } \theta_3: \theta_1 = \theta_2 = 0,$$

for the skew-symmetric matrix  $\theta \in \text{Mat}_{2n}(\mathbb{R})$  defining noncommutativity

$$\theta := \begin{pmatrix} \theta_1 & -\theta_2 \\ \theta_2^\dagger & \theta_3 \end{pmatrix}.$$

Thus,  $\theta_1, \theta_3 \in \text{Mat}_n(\mathbb{R})$  are skew-symmetric and  $\theta_2 \in \text{Mat}_n(\mathbb{R})$  can be an arbitrary  $n$  by  $n$  matrix. The results are as follows. For Type  $\theta_1$  case, the category is deformed by any  $\theta_1$ . A parallel fact holds for Type  $\theta_3$  case. Then, for Type  $\theta_2$  case, the category is deformed by  $\theta_2$  iff  $\theta_2 - \theta_2^\dagger \neq 0$ . These results are discussed by observing the deformations of zero-th cohomology categories  $H^0(\mathcal{C}^{\hat{f}})$  of the full DG subcategories  $\mathcal{C}^{\hat{f}}$  of  $\Omega^\tau(\text{Pmod}^{\hat{f}}\text{-}\mathcal{A}_\theta^{2n})$  corresponding to  $\mathcal{C}^{\hat{f}} \in \Omega^\tau(\text{Pmod}^{\text{st}}\text{-}\mathcal{A}_\theta^{2n})$  for some  $\hat{f} \in \Lambda^2$ . It is also observed there that such DG-categories  $\mathcal{C}^{\hat{f}}$  includes infinitely many objects as in the case of noncommutative two-tori. Note that the algebraic structure of  $H^0(\mathcal{C}^{\hat{f}})$  is related to the addition formula of Riemann theta functions; morphisms between holomorphic line bundles are described by theta functions and the compositions of morphisms are given by the addition formula for commutative case. The noncommutative deformation of the addition

formula of theta functions is presented in [33]. The corresponding Fukaya categories may be defined as special minimal  $A_\infty$ -categories of  $\mathcal{C}^f$  via homological perturbation theory as discussed in [37], which should include the effect of the noncommutativities quite more nontrivially than the noncommutative two-tori case.

On the other hand, for Type  $\theta_2$  case with  $\theta_2 = \theta_2^t$ , the category, for instance,  $Tw(\tilde{\mathcal{C}})$  does not depend on  $\theta_2$ . The noncommutativity  $\theta_2 = \theta_2^t$  plays a similar role to  $\theta$  in two-tori case. (A composition of theta functions in this situation is discussed in [45].) This seems to imply that to take  $\theta_2$  generic, keeping the relation  $\theta_2 = \theta_2^t$ , may be convenient to discuss HMS for (commutative, higher dimensional) abelian varieties as in [16, 51].

These noncommutative deformation might be understood as generalized complex structure

$$\begin{pmatrix} \mathbf{1}_{2n} & \theta \\ \mathbf{0}_{2n} & \mathbf{1}_{2n} \end{pmatrix} \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^t \end{pmatrix} \begin{pmatrix} \mathbf{1}_{2n} & -\theta \\ \mathbf{0} & \mathbf{1}_{2n} \end{pmatrix} = \begin{pmatrix} I & -\theta I^t - I\theta \\ \mathbf{0} & -I^t \end{pmatrix}, \quad (67)$$

where  $I = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}$  now. This is often called a  $\beta$ -transformation of  $I$  (see [24, 2]), where  $\beta = \theta$ . In fact, one sees that  $I$  is not deformed if  $\theta_2 = \theta_2^t$ . However, we do not still observe in [34] a precise relation between Type  $\theta_1$  deformation and Type  $\theta_3$  deformation. Though a (noncommutative version of) holomorphic vector bundles and Lagrangian submanifolds can be discussed in the context of generalized geometry [24, 2], in the author's understanding at present, these seem not to suggest appropriate categories we should consider.

## 6. CONCLUDING REMARKS

Though we defined the curved DG category  $\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$  for noncommutative complex tori  $(\mathcal{A}_\theta^{2n}, \tau)$ , it is not clear at present whether considering triangulated categories from their sub DG-categories as explained in the previous subsection is a correct direction or not. Another possibility is to consider the DG-category of *non one-sided* twisted complexes in  $\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$ . They are defined as a straightforward generalization of the twisted complexes in an  $A_\infty$ -category (Definition 5.8) to a weak  $A_\infty$ -category (for instance, see [59]), where one notices that the analog of one-sided complexes does not exist except  $m_0 = 0$ . Instead of it, one can also define homotopy equivalence of weak  $A_\infty$ -categories at least formally, since an  $A_\infty$ -functor is thought of as a cochain map of DG coalgebra structures defining weak  $A_\infty$ -categories (see Remark 4.21). Then, one may classify the curved DG-categories  $\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$  by homotopy equivalence of  $A_\infty$ -categories. However, in both ways, one may see that they already require something beyond homological algebras as opposed to the case of one-sided twisted complexes in  $A_\infty$ -categories.

We should remark that a version of Fukaya category consisting of Lagrangian foliations is proposed in [15] for higher dimensional symplectic tori, which gives another candidate for the objects to form categories we should consider. The corresponding extension in complex geometry side may be the derived category of quasi-coherent sheaves instead of something obtained from  $\Omega^\tau(\text{Pmod}^{st}\text{-}\mathcal{A}_\theta^{2n})$ .

For more general noncommutative algebras  $\mathcal{A}$ , there still may not exist a general machinery to associate (weak) DG algebras  $\Omega^{0,\bullet}(\mathcal{A})$ . For our case  $\mathcal{A} = \mathcal{A}_\theta^d$ ,  $d = 2n$ , it exists because we have appropriate derivations  $\delta_1, \dots, \delta_d \in \text{Der}(\mathcal{A})$ . The space  $\text{Der}(\mathcal{A})$  of derivations is spanned over  $\mathbb{C}$  by  $\delta_1, \dots, \delta_d$  and inner derivations, as shown first by Takai for  $d = 2$  [79]. Similar procedure may work for noncommutative algebras obtained as deformation of commutative algebras. It seems to be more important at present to find and study good examples first to understand a *correct* direction of formulations. See [50] which includes some common interests with this article.

In any case, the construction of triangulated categories from  $A_\infty$ -category as in subsection 5.1 may provide not only a way of giving equivalence of triangulated categories but a way of formulating deformation of triangulated categories. Since deformation of (weak)  $A_\infty$ -categories can be defined as the deformation of codifferential  $m$  such that  $(m)^2 = 0$  (for instance, see [37, Remark 2.3] for  $A_\infty$ -algebras). Namely, deformation of a triangulated category  $\mathcal{T} \simeq \text{Tr}(\mathcal{C})$  may be defined as deformation of the generator  $A_\infty$ -category  $\mathcal{C}$ .

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