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LOCAL SOLVABILITY OF A CLASS OF NONSTATIONARY SEMILINEAR SOBOLEV TYPE EQUATIONS

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In the paper local existence and uniqueness of a solution of the Cauchy problem and of the generalized Showalter problem for a class of first order nonstationary semilinear Sobolev type equations is shown by means of methods of the theory of degenerate operator semigroups. Sufficient conditions of the existence of twice differentiable solution of semilinear evolution equation is obtained for this aim. Abstract result is illustrated on an example of modified phase field system.

Let consider for Sobolev type equation

$$L\,\dot{u}(t) = Mu(t) + N(t,u(t)), \quad t \in (t_0,T), \tag{1}$$

the Cauchy problem

$$u(t_0) = u_0 \tag{2}$$

and the generalized Showalter problem

$$Pu(t_0) = u_0. \tag{3}$$

They are abstract forms of initial-boundary-value problems for various partial differential equations and systems of equations modeling real processes [1 – 4]. Here $\mathfrak{U}$ and $\mathfrak{F}$ are Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $\ker L \neq \{0\}$, $M \in Cl(\mathfrak{U};\mathfrak{F})$. Nonlinear operator $N : U \to \mathcal{F}$, that is defined on a set $U \subset \mathbb{R}\times\mathfrak{U}$, will satisfy some regularity properties and compliment properties that will be formulated below. It is supposed that operator M is strongly (L,p)-sectorial, then there exists a degenerate analytic semigroup of the equation $L\dot{u}(t) = Mu(t)$. Operator P in the condition (3) is an identity of the operator semigroup.

If there exists the operator $L^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$ then the equation (1) can be rewritten in the form

$$\dot{u}(t) = L^{-1}Mu(t) + L^{-1}N(t,u(t)), \quad t \in (t_0,T). \tag{4}$$

The goal of this work to apply known results on local solvability of the problem (2), (4) with operator $L^{-1}M$ generating analytic operator semigroup and results of the theory of degenerate analytic semigroups [5 – 8] to the research of the problems (1), (2) and (1), (3) in the case of strong (L,p)-sectoriality of operator M, $\ker L \neq \{0\}$. In this case the equation (1) can be reduced to the system of two equations with two independent unknown functions on to mutually complementary subspaces, i. e. on the kernel and on the image of the resolving semigroup of the linear part of the original equation. Under the condition of the independence of nonlinear operator N on the function $(I-P)u$ the system of equations has a simple form that is accessible for analysis. Main problem of such analysis is necessity of $(p+1)$-multiple differentibility of solution of semilinear equation solvable with respect to the derivative. This problem in the paper is resolved for $p=1$.

Local solvability of the problem (1), (2) with smooth operator $N(t,u) \equiv N(u)$ in the sence of Frechet was studied in the works of G.A.Sviridyuk and his coathors (see, for example, [1, 2, 4] and references there). In contrast to those results in this paper found local solutions are not a quasistationary trajectories.

Obtained abstract result is illustrated on an example of initial-boundary-value problem for a modified phase field system of equations.

1. Regularity of solutions of nondegenerate evolution equation

Let $\mathfrak{V}$ be Banach space. Denote by $\mathcal{L}(\mathfrak{V})$ the Banach space of linear continuous operators from $\mathfrak{V}$ to $\mathfrak{V}$. The set of linear closed operators with dence in $\mathfrak{V}$ domains acting to this space will be denoted by $Cl(\mathfrak{V})$.

Operator $A \in Cl(\mathfrak{V})$ is called sectorial if

$$\exists a \in \mathbb{R} \quad \exists \theta \in (\pi/2, \pi) \quad S_{a,\theta} \equiv \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta\} \subset \rho(A);$$

$$\exists K \in \mathbb{R}_+ \quad \forall \mu \in S_{a,\theta} \quad \|(\mu I - A)^{-1}\|_{\mathcal{L}(\mathfrak{V})} \leq \frac{K}{|\mu - a|}.$$

As it is known, operator A is sectorial if and only if it generates continuous at zero analytic semigroup $\{V(t) \in \mathcal{L}(\mathfrak{V}) : |\arg t| < \theta - \pi/2\}$, $V(0) = I$.

Take $b > a$, $A_1 \equiv bI - A$ and define, as in [9, §1.4], the subspace $\mathfrak{V}_\alpha \equiv \mathrm{dom} A_1^\alpha \subset \mathfrak{V}$ with the norm $\|v\|_\alpha = \|A_1^\alpha v\|_{\mathfrak{V}}$, where $\alpha \geq 0$.

Suppose that operator B maps open set $W \subset \mathbb{R} \times \mathfrak{V}_\alpha$ for some $\alpha \in [0,1)$ to $\mathfrak{V}$, it is locally Hölder with respect to t and is locally Lipschitz with respect to v on W. In other words, for every $(t_1, v_1) \in W$ there exists its neighborhood $O \subset W$, and for all $(t,v), (s,w) \in O$

$$\|B(t,v) - B(s,w)\|_{\mathfrak{V}} \leq C(|t-s|^\theta + \|v - w\|_\alpha)$$

for some $C, \theta \in \mathbb{R}_+$.

DEFINITION 1. For $(t_0, v_0) \in W$ function $v \in C([t_0,T); \mathfrak{V}_\alpha) \cap C^1((t_0,T); \mathfrak{V})$ is called the solution of the Cauchy problem

$$v(t_0) = v_0 \tag{5}$$

for the equation

$$\dot{v}(t) = Av(t) + B(t, v(t)) \tag{6}$$

on (t_0,T) if it satisfies the condition (5), and for all $t \in (t_0,T)$ correlation $(t, v(t)) \in W$ holds, $v(t) \in \mathrm{dom} A$, function v satisfies the differential equation (6).

Theorem 1 [9, §3.3]. *Let an operator A be sectorial, an operator $B : W \to \mathfrak{V}$ be locally Hölder with respect to t and locally Lipschitz with respect to v on an open set $W \subset \mathbb{R} \times \mathfrak{V}_\alpha$, $\alpha \in [0,1)$. Then for every $(t_0, v_0) \in W$ there exists such $T = T(t_0, v_0) > t_0$ that the problem* (5), (6) *has a unique solution on* (t_0, T).

PROOF. Under the Theorem 3.3.3 it is sufficiently to prove the continuity of a unique solution in the norm of $\mathfrak{V}_\alpha$ at the point t_0. Since $v_0 \in \mathfrak{V}_\alpha$ then

$$\|v(t)-v_0\|_\alpha \le \|(V(t-t_0)-I)A_1^\alpha v_0\|_{\mathfrak{V}} + \left\| \int_{t_0}^{t} A_1^\alpha V(t-s)B(s,v(s))ds \right\|_{\mathfrak{V}} \le$$

$$\|(V(t-t_0)-I)A_1^\alpha v_0\|_{\mathfrak{V}} + C\int_{t_0}^{t}(t-s)^{-\alpha}ds \to 0$$

as $t \to t_0$. □

The aim of this paragraph is obtaining of sufficient conditions for the existence of a solution of the problem (5), (6) from the class $C^2((t_0,T);\mathfrak{V})$. Denote by $\mathcal{D}_A$ the Banach space dom$A$ with the norm $\|\cdot\|_A = \|\cdot\|_{\mathfrak{V}} + \|A\cdot\|_{\mathfrak{V}}$.

Lemma 1. *Let an operator A be sectorial, $f \in C([0,T);\mathcal{D}_A)\cap C^1((0,T);\mathfrak{V})$, the derivative $\dot f$ be locally Hölder in $\mathfrak{V}$ and there exists such $\rho>0$ that*

$$\int_0^\rho \|\dot f(s)\|_{\mathfrak{V}}ds < \infty, \quad F(t) \equiv \int_0^t V(t-s)f(s)ds$$

for $t\in[0,T)$. Then

$$F \in C([0,T);\mathcal{D}_A)\cap C((0,T);\mathcal{D}_{A^2})\cap C^1([0,T);\mathfrak{V})\cap C^2((0,T);\mathfrak{V}),$$

$$\ddot F(t) = A^2F(t) + Af(t) + \dot f(t), \quad t\in(0,T), \tag{7}$$

$$F(0)=0, \quad \dot F(0) = f(0).$$

PROOF. For $\rho\in(0,T)$, define $F_\rho(t)=0$ when $t\in[0,\rho]$, and if $t\in[\rho,T)$ then $F_\rho(t) = \int_0^{t-\rho} V(t-s)f(s)ds$. Put $f(s)=0$ for $s<0$. In [9, Lemma 3.2.1] it is shown that $F_\rho(t)\in D(A)$ for $t\in[0,T)$, $F_\rho(t)$ is differentiable when $t>\rho$,

$$\dot F_\rho(t) = AF_\rho(t) + V(\rho)f(t-\rho), \quad t\in(\rho,T), \tag{8}$$

$\lim\limits_{\rho\to 0+} F_\rho(t) = F(t)$, $\lim\limits_{t\to 0+} F(t) = 0$, $F\in C^1((0,T);\mathfrak{V})$, $F(t)\in D(A)$ for $t\in(0,T)$, $\dot F(t) = AF(t)+f(t)$.

Under the conditions of the present lemma

$$\|AF(t)\|_{\mathfrak{V}} = \left\| \int_0^t \frac{d}{ds}[V(t-s)]f(s)ds \right\|_{\mathfrak{V}} \le$$

$$\|f(t)-V(t)f(0)\|_{\mathfrak{V}} + \left\|\int_0^t V(t-s)\dot f(s)ds\right\|_{\mathfrak{V}} \le$$

$$\|f(t)-f(0)\|_{\mathfrak{V}}+\|f(0)-V(t)f(0)\|_{\mathfrak{V}}+C\int\limits_0^t\|\dot f(s)\|_{\mathfrak{V}}ds\to 0$$

when $t\to 0+$. Therefore $\lim\limits_{t\to 0+}\dot F(t)=f(0)$.

For $h>0$ we have the inequalities

$$\|AF(t+h)-AF(t)\|_{\mathfrak{V}}\le\left\|\int\limits_0^{t+h}AV(t+h-s)f(s)ds-\int\limits_0^{t}AV(t-s)f(s)ds\right\|_{\mathfrak{V}}\le$$

$$\left\|\int\limits_0^{t}(V(h)-I)AV(t-s)f(s)ds\right\|_{\mathfrak{V}}+\left\|\int\limits_t^{t+h}AV(t+h-s)f(s)ds\right\|_{\mathfrak{V}}\le$$

$$C_1h^{\theta_1/2}\int\limits_0^t\|A_1^{1+\theta_1/2}V(t-s)\|_{\mathcal{L}(\mathfrak{V})}\|f(s)-f(t)\|_{\mathfrak{V}}ds+$$

$$\int\limits_t^{t+h}\|AV(t+h-s)\|_{\mathcal{L}(\mathfrak{V})}\|f(s)-f(t+h)\|_{\mathfrak{V}}ds+\left\|\int\limits_0^{t}A(V(h)-I)V(t-s)f(t)ds\right\|_{\mathfrak{V}}+$$

$$+\left\|\int\limits_t^{t+h}AV(t+h-s)f(t+h)ds\right\|_{\mathfrak{V}}\le$$

$$C_1h^{\theta_1/2}\int\limits_0^t(t-s)^{\theta_1/2-1}ds+C_2\int\limits_t^{t+h}(t+h-s)^{\theta_2-1}ds+$$

$$\|(V(h)-I)(V(t)-I)f(t)\|_{\mathfrak{V}}+\|(V(h)-I)f(t+h)\|_{\mathfrak{V}}\le$$

$$Ch^{\theta}+\|(V(h)-I)(V(t)-I)f(t)\|_{\mathfrak{V}}+$$

$$\|(V(h)-I)(f(t+h)-f(t))\|_{\mathfrak{V}}+\|(V(h)-I)f(t)\|_{\mathfrak{V}}\le$$

$$Ch^{\theta}+\|(V(h)-I)(V(t)-I)f(t)\|_{\mathfrak{V}}+C_3\|f(t+h)-f(t)\|_{\mathfrak{V}}+\|(V(h)-I)f(t)\|_{\mathfrak{V}}\to 0,$$

when $h\to 0+$. It was utilized that function f on some segment $[t_0,t_1]$ containing the points $t, t+h$ has the Hölder property and that inequalities

$$\|A_1^{\theta}Az\|_{\mathfrak{V}}\le(b+1)\|A_1^{\theta+1}z\|_{\mathfrak{V}},\quad \|A_1^{\theta}V(t)z\|_{\mathfrak{V}}\le e^{bt}\|A_1^{\theta}V_1(t)z\|_{\mathfrak{V}}\le\frac{e^{bt}C\|z\|_{\mathfrak{V}}}{t^{\theta}},$$

$$\|(V(h)-I)AV(t-s)z\|_{\mathfrak{V}}\le C_1h^{\theta}\|A_1^{1+\theta}V(t-s)z\|_{\mathfrak{V}}\le\frac{C_2h^{\theta}e^{b(t-s)}\|z\|_{\mathfrak{V}}}{(t-s)^{1+\theta}}$$

holds, where $\theta>0$, $V_1(\cdot)$ is semigroup that is generating by sectorial operator $-A_1$. Therefore function $AF(\cdot)$ is right-hand side continuous. For the proof of its left-hand side continuity we have for $h>0$ the inequalities

$$\left\|\int\limits_0^{t-h}AV(t-h-s)f(s)ds-\int\limits_0^{t}AV(t-s)f(s)ds\right\|_{\mathfrak{V}}\le$$

$$\left\|\int\limits_0^{t-h}(V(h)-I)AV(t-h-s)f(s)ds\right\|_{\mathfrak{V}}+\left\|\int\limits_{t-h}^{t}AV(t-s)f(s)ds\right\|_{\mathfrak{V}}\leq$$

$$C_1h^{\theta_1/2}\int\limits_0^{t-h}\|A_1^{1+\theta_1/2}V(t-h-s)\|_{\mathcal{L}(\mathfrak{V})}\|f(s)-f(t-h)\|_{\mathfrak{V}}ds+$$

$$\int\limits_{t-h}^{t}\|AV(t-s)\|_{\mathcal{L}(\mathfrak{V})}\|f(s)-f(t)\|_{\mathfrak{V}}ds+$$

$$\left\|\int\limits_0^{t-h}A(V(h)-I)V(t-h-s)f(t-h)ds\right\|_{\mathfrak{V}}+\left\|\int\limits_{t-h}^{t}AV(t-s)f(t)ds\right\|_{\mathfrak{V}}\leq$$

$$Ch^{\theta}+\|(V(t-h)-I)(V(h)-I)f(t-h)\|_{\mathfrak{V}}+\|(V(h)-I)f(t)\|_{\mathfrak{V}}\leq$$
$$Ch^{\theta}+C_2\|f(t-h)-f(t)\|_{\mathfrak{V}}+C_3\|(V(h)-I)f(t)\|_{\mathfrak{V}}\to 0,$$

when $h\to 0+$. So we have $AF\in C([0,T);\mathfrak{V})$.

From the definition of the integral and from the enclosing $\operatorname{im}V(t)\subset\bigcap\limits_{k\in\mathbb{N}}\operatorname{im}A^k$ for $t>0$ it follows that $F_\rho(t)\in D(A^2)$ for $t\in[0,T)$,

$$A^2F_\rho(t)=\int\limits_0^{t-\rho}A^2V(t-s)f(s)ds=-A\int\limits_0^{t-\rho}\frac{d}{ds}[V(t-s)]f(s)ds=$$

$$-AV(\rho)f(t-\rho)+AV(t)f(0)+\int\limits_0^{t-\rho}AV(t-s)\dot{f}(s)ds=$$

$$-AV(\rho)f(t-\rho)+AV(t)f(0)+\int\limits_0^{t-\rho}AV(t-s)(\dot{f}(s)-\dot{f}(t))ds-(V(\rho)-V(t))\dot{f}(t).$$

We have

$$\|AV(t-s)\|_{\mathcal{L}(\mathfrak{V})}=O(|t-s|^{-1}),\quad \|\dot{f}(s)-\dot{f}(t)\|_{\mathfrak{V}}=O(|t-s|^{\theta}),\ \theta>0,$$

as $s\to t-$ because the semigroup is analytic and the derivative $\dot{f}$ has the local Hölder property. Then the last integral has a limit, when $\rho\to 0+$. Thus for $\rho\to 0+$ we have

$$A^2F_\rho(t)\to -Af(t)+AV(t)f(0)+\int\limits_0^{t}AV(t-s)(\dot{f}(s)-\dot{f}(t))ds-(I-V(t))\dot{f}(t).$$

From the closure of the operator A^2 it follows that $F(t)\in\operatorname{dom}A^2$ for $t\in(0,T)$.

Besides,

$$A^2F(t) = -A\int_0^t \frac{d}{ds}[V(t-s)]f(s)ds =$$

$$-Af(t) + AV(t)f(0) + \int_0^t AV(t-s)(\dot{f}(s) - \dot{f}(t))ds - (I - V(t))\dot{f}(t),$$

$$\|A^2F_\rho(t) - A^2F(t)\|_{\mathfrak{X}} \le \|Af(t) - AV(\rho)f(t-\rho)\|_{\mathfrak{X}} +$$

$$+\int_{t-\rho}^t \|AV(t-s)\|_{\mathcal{L}(\mathfrak{X})} \cdot \|\dot{f}(s) - \dot{f}(t)\|_{\mathfrak{X}}ds + \|(V(\rho) - I)\dot{f}(t)\|_{\mathfrak{X}} \le$$

$$C_1\|Af(t-\rho) - Af(t)\|_{\mathfrak{X}} + \|(V(\rho) - I)Af(t)\|_{\mathfrak{X}} + C_2\rho^\theta + \|(V(\rho) - I)\dot{f}(t)\|_{\mathfrak{X}} \to 0$$

as $\rho \to 0+$ uniformly with respect to $t \in [t_0, t_1] \subset (0, T)$. Really from the uniform continuity, for example, of the function Af on the segment $[t_0, t_1]$

$$\forall \varepsilon > 0 \ \exists s_1, \ldots, s_n \in [t_0, t_1] \ \forall t \in [t_0, t_1] \ \exists k \in \{1, \ldots, n\} \ \|Af(t) - Af(s_k)\|_{\mathfrak{X}} < \varepsilon.$$

Also we have

$$\forall k \in \{1, \ldots, n\} \ \exists \delta_k > 0 \ \forall \rho \in (0, \delta_k) \ \|V(\rho)Af(s_k) - Af(s_k)\|_{\mathfrak{X}} < \varepsilon.$$

Therefore for all $t \in [t_0, t_1]$ and for $\rho \in (0, \min\{\delta_1, \ldots, \delta_n\})$

$$\|V(\rho)Af(t) - Af(t)\|_{\mathfrak{X}} \le$$

$$\|V(\rho)A(f(t) - f(s_k))\|_{\mathfrak{X}} + \|(V(\rho) - I)Af(s_k)\|_{\mathfrak{X}} + \|Af(s_k) - Af(t)\|_{\mathfrak{X}} < C\varepsilon.$$

When $t > \rho$ we have

$$\frac{d}{dt}(AF_\rho(t)) = \frac{d}{dt}\left(-V(\rho)f(t-\rho) + V(t)f(0) + \int_0^{t-\rho} V(t-s)\dot{f}(s)ds\right) =$$

$$AV(t)f(0) + \int_0^{t-\rho} AV(t-s)(\dot{f}(s) - \dot{f}(t))ds + (V(t) - V(\rho))\dot{f}(t) =$$

$$A\dot{F}_\rho(t) = A^2F_\rho(t) + AV(\rho)f(t-\rho)$$

because the operator A is closed and the equality (8) holds. Then

$$\ddot{F}_\rho(t) = A^2F_\rho(t) + AV(\rho)f(t-\rho) + V(\rho)\dot{f}(t-\rho).$$

Reasoning as before we have

$$\lim_{\rho\to 0+} V(\rho)\dot{f}(t-\rho) = \dot{f}(t)$$

uniformly with respect to $t \in [t_0, t_1] \subset (0, T)$. Then the equality (7) holds. Let show that $F \in C^2((0,T); \mathfrak{V})$. Because the equality (7) holds it is sufficient to show that the function A^2F is continuous on $(0,T)$. Really for $h > 0$

$$\|A^2F(t+h) - A^2F(t)\|_{\mathfrak{V}} \le \|Af(t+h) - Af(t)\|_{\mathfrak{V}} + \|(V(t+h) - V(t))Af(0)\|_{\mathfrak{V}} +$$

$$\left\| \int_0^{t+h} AV(t+h-s)\dot{f}(s)ds - \int_0^{t} AV(t-s)\dot{f}(s)ds \right\|_{\mathfrak{V}} .$$

The first two terms in right-hand side of the last inequality tend to zero as $h \to 0$. For the last term we can repeat previous similar reasoning in this proof with function f instead of $\dot{f}$ under the local Hölder property of $\dot{f}$. □

Lemma 2. *Let an operator A be sectorial, an operator B map an open set $W \subset \mathbb{R} \times \mathfrak{V}_\alpha$ for some $\alpha \in [0,1)$ to the set $\mathrm{dom}A \subset \mathfrak{V}$, $AB \in C(W;\mathfrak{V})$, $B \in C^1(W;\mathfrak{V})$, operators $A_1^\alpha B : W \to \mathfrak{V}$, $\frac{\partial B}{\partial t} : W \to \mathfrak{V}$, $\frac{\partial B}{\partial v}u : W \to \mathfrak{V}$ be locally Hölder with respect to t and locally Lipschitz with respect to v on W for all $u \in \mathfrak{V}_\alpha$. Besides, let a function $v \in C([t_0,T);\mathfrak{V}_\alpha)$ and for $t \in [t_0,T)$ the correlation $(t, v(t)) \in W$ holds,*

$$v(t) = V(t-t_0)v_0 + \int_{t_0}^{t} V(t-s)B(s,v(s))ds. \tag{9}$$

Then $v \in C((t_0,T);\mathcal{D}_A) \cap C^2((t_0,T);\mathfrak{V})$. Besides, if $v_0 \in \mathrm{dom}A$ then the function $v \in C([t_0,T);\mathcal{D}_A) \cap C^1([t_0,T);\mathfrak{V})$, $\dot{v}(t_0) = Av_0 + B(t_0,v_0)$.

PROOF. Under the Lemma 3.3.2 [9] the function $v : (t_0,T) \to \mathfrak{V}$ is differentiable and satifies the conditions (5), (6). Besides,

$$\dot{v}(t) = AV(t-t_0)v_0 + \int_{t_0}^{t} AV(t-s)B(s,v(s))ds + B(t,v(t)) =$$

$$AV(t-t_0)v_0 + V(t-t_0)B(t_0,v_0) + \int_{t_0}^{t} V(t-s)\frac{d}{ds}B(s,v(s))ds. \tag{10}$$

Then for $h > 0$ we have the inequalities

$$\|\dot{v}(t+h) - \dot{v}(t)\|_\alpha \le \|(V(h) - I)AV(t-t_0)v_0\|_\alpha +$$

$$\|(V(h) - I)V(t-t_0)B(t_0,v_0)\|_\alpha +$$

$$\int_{t_0}^{t} \left\|(V(h)-I)V(t-s)\frac{d}{ds}B(s,v(s))\right\|_\alpha ds + \int_{t}^{t+h} \left\|\dot{V}(t+h-s)\frac{d}{ds}B(s,v(s))\right\|_\alpha ds \le$$

$$C_1 \left\|(V(h)-I)V\left(\frac{t-t_0}{2}\right)A_1^{1+\alpha}V\left(\frac{t-t_0}{2}\right)v_0\right\|_{\mathfrak{V}} +$$

$$C_1\|(V(h)-I)V(t-t_0)A_1^{\alpha}B(t_0,v_0)\|_{\mathfrak{V}}+C_4h^{\delta}\int\limits_{t_0}^{t}(t-s)^{-(\delta+\alpha)}\left\|\frac{d}{ds}B(s,v(s))\right\|_{\mathfrak{V}}ds$$

$$+C_5\cdot\max_{s\in[t_1,t_2]}\left\|\frac{d}{ds}B(s,v(s))\right\|_{\mathfrak{V}}\int\limits_{t}^{t+h}(t+h-s)^{-\alpha}ds\le Ch^{\theta},$$

where $t,t+h\in[t_1,t_2]\subset(t_0,T)$. Here the local Hölder property of the function $A_1^{\alpha}B(\cdot,v(\cdot))$ follows from the local Hölder property of the function v (see. [9, Lemma 3.3.2]), continuous differentibility of the operator semigroup in unform topology on the semiaxis $(0,+\infty)$ is utilized. The number δ is chosen from the interval $(0,1-\alpha)$.

Then from the conditions of the Lemma it follows that

$$\left\|\frac{d}{ds}B(s+h,v(s+h))-\frac{d}{ds}B(s,v(s))\right\|_{\mathfrak{V}}\le$$

$$\|B_t'(s+h,v(s+h))-B_t'(s,v(s))\|_{\mathfrak{V}}+\|B_v'(s+h,v(s+h))\dot v(s+h)-B_v'(s,v(s))\dot v(s)\|_{\mathfrak{V}}\le$$

$$C_1(h^{\theta_1}+\|v(s+h)-v(s)\|_{\alpha})+\|B_v'(s+h,v(s+h))\dot v(s+h)-B_v'(s+h,v(s+h))\dot v(s)\|_{\mathfrak{V}}+$$

$$+\|B_v'(s+h,v(s+h))\dot v(s)-B_v'(s,v(s))\dot v(s)\|_{\mathfrak{V}}\le$$

$$C_2(h^{\theta_2}+\|v(s+h)-v(s)\|_{\alpha}+\|\dot v(s+h)-\dot v(s)\|_{\alpha})\le Ch^{\theta},$$

where $B_t'=\frac{\partial B}{\partial t}$, $B_v'=\frac{\partial B}{\partial v}$. Thereby the local Hölder property of the function $\frac{d}{dt}B(t,v(t))$ is proved. Then from the Lemma 1 with the function $f(t)=B(t,v(t))$ and from the analiticity of the operator semigroup we have $v\in C((t_0,T);\mathcal{D}_A)\cap C^2((t_0,T);\mathfrak{V})$.

For $v_0\in \mathrm{dom}A$ under the same Lemma and the equality (9) we have $v\in C([t_0,T);\mathcal{D}_A)$ and from the convergence of the integral in the equality (10) it follows that

$$\lim_{t\to t_0+}\dot v(t)=\lim_{t\to t_0+}V(t-t_0)(Av_0+B(t_0,v_0))=Av_0+B(t_0,v_0). \qquad \square$$

Now we can formulate the main result of this paragraph.

Theorem 2. *Let an operator A be sectorial, an operator B map an open set $W\subset\mathbb{R}\times\mathfrak{V}_{\alpha}$ for some $\alpha\in[0,1)$ to the set* $\mathrm{dom}A\subset\mathfrak{V}$*, besides, $AB\in C(W;\mathfrak{V})$, $B\in C^1(W;\mathfrak{V})$, operators $A_1^{\alpha}B:W\to\mathfrak{V}$, $\frac{\partial B}{\partial t}:W\to\mathfrak{V}$, $\frac{\partial B}{\partial v}u:W\to\mathfrak{V}$ be locally Hölder with respect to t and locally Lipschitz with respect to v on W for all $u\in\mathfrak{V}_{\alpha}$. Then for every $(t_0,v_0)\in W$ there exists such $T=T(t_0,v_0)>t_0$ that the problem (5), (6) has a unique solution $v\in C([t_0,T);\mathfrak{V}_{\alpha})\cap C((t_0,T);\mathcal{D}_A)\cap C^2((t_0,T);\mathfrak{V})$ on the interval (t_0,T), besides, if $v_0\in$* $\mathrm{dom}A$ *then $v\in C([t_0,T);\mathcal{D}_A)\cap C^1([t_0,T);\mathfrak{V})$, $\dot v(t_0)=Av_0+B(t_0,v_0)$.*

Proof. Under the Theorem 1 there exists a unique solution

$$v\in C([t_0,T);\mathfrak{V}_{\alpha})\cap C^1((t_0,T);\mathfrak{V})$$

of the problem (5), (6), satisfying to the integral equation (9). From the Lemma 2 the assertion of the present theorem follows. $\square$

2. Local solvability of Sobolev type equation

Let us formulate some results that obtaining before in [6 – 8] and will be utilized in this work.

Let $\mathfrak{U}$, $\mathfrak{F}$ be Banach spaces. Denote by $\mathcal{L}(\mathfrak{U};\mathfrak{F})$ the Banach space of linear continuous operators, acting from $\mathfrak{U}$ to $\mathfrak{F}$. The set of linear closed operators with dense domains in $\mathfrak{U}$, acting to $\mathfrak{F}$, will be denoted by $\mathcal{C}l(\mathfrak{U};\mathfrak{F})$.

Everywhere we suppose that operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$. Denote $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$, $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$, $R^L_{(\mu,p)}(M) = \prod\limits_{k=0}^{p} R^L_{\mu_k}(M)$, $L^L_{(\mu,p)}(M) = \prod\limits_{k=0}^{p} L^L_{\mu_k}(M)$.

DEFINITION 2. Operator M is called *strongly (L,p)-sectorial*, if

(i) $\exists a \in \mathbb{R}\ \exists \theta \in (\pi/2,\pi)\ S_{a,\theta} \equiv \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta\} \subset \rho^L(M)$;

(ii) $\exists K \in \mathbb{R}_+\ \forall \mu = (\mu_0,\mu_1,\dots,\mu_p) \in (S_{a,\theta})^{p+1}$

$$\max\left\{\|R^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{U})}, \|L^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{F})}\right\} \le \frac{K}{\prod\limits_{k=0}^{p}|\mu_k - a|};$$

(iii) there exists a dense in $\mathfrak{F}$ subspace $\overset{\circ}{\mathfrak{F}}$ such that

$$\|M(\lambda L - M)^{-1}L^L_{(\mu,p)}(M)f\|_{\mathfrak{F}} \le \frac{\mathrm{const}(f)}{|\lambda - a|\prod\limits_{k=0}^{p}|\mu_k - a|} \quad \forall f \in \overset{\circ}{\mathfrak{F}}$$

for all $\lambda,\mu_0,\mu_1,\dots,\mu_p \in S_{a,\theta}$;

(iv) for all $\lambda,\mu_0,\mu_1,\dots,\mu_p \in S_{a,\theta}$

$$\|R^L_{(\mu,p)}(M)(\lambda L - M)^{-1}\|_{\mathcal{L}(\mathfrak{F};\mathfrak{U})} \le \frac{K}{|\lambda - a|\prod\limits_{k=0}^{p}|\mu_k - a|}.$$

Denote by $\mathfrak{U}^0$ ($\mathfrak{F}^0$) the kernel $\ker R^L_{(\mu,p)}(M)$ ($\ker L^L_{(\mu,p)}(M)$) and by $\mathfrak{U}^1$ ($\mathfrak{F}^1$) the closure of subspace $\mathrm{im} R^L_{(\mu,p)}(M)$ ($\mathrm{im} L^L_{(\mu,p)}(M)$) in the sense of the norm of the space $\mathfrak{U}$ ($\mathfrak{F}$). By M_k (L_k) denote the restriction of the operator M (L) on $\mathrm{dom} M_k = \mathfrak{U}^k \cap \mathrm{dom} M$ ($\mathfrak{U}^k$), $k = 0,1$.

Theorem 3 (see [6, 7]). *Let operator M be strongly (L,p)-sectorial. Then*

(i) $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$, $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$;

(ii) $L_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$, $M_k \in \mathcal{C}l(\mathfrak{U}^k;\mathfrak{F}^k)$, $k = 0,1$;

(iii) *there exist operators* $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$, $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$;

(iv) *the operator* $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ *is nilpotent with degree not greater than* p;

(v) *there exists continuous at zero analytical semigroup* $\{U(t) \in \mathcal{L}(\mathfrak{U}) : |\arg t| < \theta - \pi/2\}$ *of the equation* $L\dot{u} = Mu$;

(vi) *the infinitesimal generator of the semigroup* $\{U_1(t) = U(t)\Big|_{\mathfrak{U}^1} \in \mathcal{L}(\mathfrak{U}^1) : |\arg t| < \theta - \pi/2\}$, $U_1(0) = I$ *is the operator* $L_1^{-1}M_1 \in \mathcal{C}l(\mathfrak{U}^1)$.

REMARK 1. The projector along $\mathfrak{U}^0$ on $\mathfrak{U}^1$ (along $\mathfrak{F}^0$ on $\mathfrak{F}^1$) will be denoted by P (Q). Under the conditions of the Theorem 3 the equalities $QL = LP$, $QMu = MPu$ for $u \in \mathrm{dom}M$ hold. They are utilized for the proof of the assertion (ii).

REMARK 2. Under the assertion (vi) of the Theorem 3 and under the Yosida theorem the operator $L_1^{-1}M_1 \in Cl(\mathfrak{U}^1)$ is sectorial.

Let define the solution of the Cauchy problem

$$u(t_0) = u_0, \tag{11}$$

for the Sobolev type equation

$$L\,\dot{u}(t) = Mu(t) + N(t, u(t)), \quad t \in (t_0, T). \tag{12}$$

DEFINTION 3. Let operator $N : U \to \mathfrak{F}$ be defined on the set $U \subset \mathbb{R} \times \mathfrak{U}$, a function $u \in C([t_0,T);\mathfrak{U}) \cap C^1((t_0,T);\mathfrak{U})$ satisfies the condition (11) and for all $t \in (t_0,T)$ the relations $(t, u(t)) \in U$ and $u(t) \in \mathrm{dom}M$ hold. If u satisfies the differential equation (12) then it is called *the solution* of the problem (11), (12) on the interval (t_0, T).

This paper is devoted to the research of local solvability of the Cauchy problem (11) for a class of nonstationary semilinear Sobolev type equation (12). One class of such equations with strongly (L,p)-sectorial operator M and with $\mathrm{im}N \subset \mathfrak{U}^1$ completely investigated before in [10].

Another class of semilinear nonstationary Sobolev type equation with nonlinear operator depending only on the projection Pu of phase function u was investigated in [10] in the case of strongly $(L,0)$-sectorial operator M. The main problem of this paper is studying of local solvability of this class equations in the case of strongly $(L,1)$-sectorial operator M. The main difficulty in this case is obtaining of a twice defferentiable solution of the problem (5), (6). It was resolved in the previous paragraph. This fact allows to prove the main result of the paper.

As before for sectorial operator $A = L_1^{-1}M_1 \in Cl(\mathfrak{U}^1)$ let construct an operator $A_1 = bI - A$, $b > a$, its degrees A_1^α for $\alpha \geq 0$ and subspaces $\mathfrak{U}^1_\alpha \equiv \mathrm{dom}A_1^\alpha$ of the space $\mathfrak{U}^1$ with norms $\|u\|_\alpha = \|A_1^\alpha u\|_{\mathfrak{U}}$.

Theorem 4. *Let operator M be strongly $(L,1)$-sectorial, operator N map an open set $U \subset \mathbb{R} \times \mathfrak{U}^0 \oplus \mathfrak{U}^1_\alpha$ for some $\alpha \in [0,1)$ to the set $\mathfrak{F}^0 \dot{+} L_1[\mathrm{dom}M_1] \subset \mathfrak{F}$, $L_1^{-1}QN \in C(U;\mathcal{D}_{M_1}) \cap C^1(U;\mathfrak{U})$, $M_0^{-1}(I-Q)N \in C^2(U;\mathfrak{U})$, operators $A_1^\alpha L_1^{-1}QN : U \to \mathfrak{U}$, $\frac{\partial(QN)}{\partial t} : U \to \mathfrak{F}$, $\frac{\partial(QN)}{\partial u}v : U \to \mathfrak{F}$ be locally Hölder with respect to t and locally Lipschitz with respect to u on U for all $v \in \mathfrak{U}^1_\alpha$. Besides, suppose that for all $(t,u) \in U$, $w \in \mathfrak{U}^0$ the relations $(t, u+w) \in U$, $N(t,u) = N(t, u+w)$ hold. Then for every $(t_0, u_0) \in U$ such that $Pu_0 \in \mathrm{dom}M$,*

$$(I-P)u_0 = -M_0^{-1}(I-Q)N(t_0, Pu_0) - H\frac{\partial}{\partial t}[M_0^{-1}(I-Q)N(t,Pu)]\bigg|_{t=t_0} -$$

$$-H\frac{\partial}{\partial(Pu)}[M_0^{-1}(I-Q)N(t,Pu)]\bigg|_{t=t_0}(L_1^{-1}M_1Pu_0 + L_1^{-1}QN(t_0,Pu_0)), \tag{13}$$

there exists such $T = T(t_0, u_0) > t_0$ that the problem (11), (12) has a unique solution on (t_0, T).

PROOF. Let act on the equality (12) by the operator $L_1^{-1}Q$ then under the Remark 1 the equation

$$\dot{v} = L_1^{-1}M_1 v + L_1^{-1}QN(t, v + w), \tag{14}$$

holds where $Pu(t) = v(t)$, $(I - P)u(t) = w(t)$, $u(t) = v(t) + w(t)$. Acting on the equation (12) by the operator $M_0^{-1}(I - Q)$ we obtain

$$H\,\dot{w} = w + M_0^{-1}(I - Q)N(t, v + w). \tag{15}$$

Thus the problem (11), (12) is reduced to the Cauchy problem $v(t_0) = Pu_0$, $w(t_0) = (I - P)u_0$ for the system of equations (14), (15).

Operators $A = L_1^{-1}M_1$, $B(t, v) = L_1^{-1}QN(t, v)$ satisfy the conditions of the Theorem 2 (with the space $\mathfrak{U}^1 = \mathfrak{V}$) under the Remark 2 and the conditions of the present theorem. Since $(t_0, u_0) \in U$, $-(I - P)u_0 \in \mathfrak{U}^0$, $Pu_0 = u_0 - (I - P)u_0$, then $(t_0, Pu_0) \in U$ and under the Theorem 2 for some T depending on (t_0, u_0), there exists a unique solution $v \in C([t_0, T); \mathfrak{U}^1_\alpha) \cap C^1([t_0, T); \mathfrak{U}^1) \cap C^2((t_0, T); \mathfrak{U}^1)$ of the Cauchy problem $v(t_0) = Pu_0$ for the equation (14) on the interval (t_0, T), besides, $\dot{v}(t_0) = L_1^{-1}M_1Pu_0 + L_1^{-1}QN(t_0, Pu_0)$.

Since $-(I - P)u \in \mathfrak{U}^0$ then for every $(t, u) \in U$ relation $(t, Pu) = (t, u - (I - P)u) \in U$ holds. Therefore $N(t, u) \equiv N(t, Pu)$. Thus the equation (15) has the form

$$H\dot{w} = w + M_0^{-1}(I - Q)N(t, v), \tag{16}$$

where the function v is already known. If there exists a solution of the equation (16) then the right-hand side of the equation is differentiable because the operator N is continuously differentiable in the sense of Frechet. Therefore the left-hand side of the equation is differentiable also. Ater differentiation of the equation (16) and acting on it by the operator H we obtain

$$w(t) = \left(H\frac{d}{dt}\right)^2 w(t) - M_0^{-1}(I - Q)N(t, v) - H\frac{d}{dt}[M_0^{-1}(I - Q)N(t, v)] =$$

$$-M_0^{-1}(I - Q)N(t, v) - H\frac{\partial}{\partial t}[M_0^{-1}(I - Q)N(t, v)] - H\frac{\partial}{\partial v}[M_0^{-1}(I - Q)N(t, v)]\dot{v}, \tag{17}$$

because from the continuity and nilpotency of the first degree of the operator H it follows that the equality $\left(H\frac{d}{dt}\right)^2 w(t) = \frac{d^2}{dt^2}H^2 w(t) \equiv 0$ holds. From the relations $M_0^{-1}(I - Q)N \in C^2(U; \mathfrak{U})$, $v \in C^1([t_0, T); \mathfrak{U}^1) \cap C^2((t_0, T); \mathfrak{U}^1)$ we have $w \in C([t_0, T); \mathfrak{U}^0) \cap C^1((t_0, T); \mathfrak{U}^0)$. Thus the uniqueness of a solution of the equation (16) is proved. His existence can be proved by the replacement of the function w from (17) to the equation.

From the form of the solution (17) of the equation (16) it follows that it is the solution of the Cauchy problem $w(0) = (I - P)u_0$ if it satisfies the condition (13). Note that for all $t \in (t_0, T)$ under the Theorem 2 we have $(t, v(t)) \in U$, $v(t) \in \mathrm{dom}M_1$ and therefore under the conditions of present theorem $(t, v(t) + w(t)) \in U$. It is obviously that $w(t) \in \mathrm{dom}M$ for all $t \in (t_0, T)$. □

REMARK 3. Analogous reasoning as in the proof of the Theorem 4 can be utilized in the case of strongly (L, p)-sectorial operator M for all $p \in \mathbb{N}$. But for the obtaining of result we need such conditions on nonlinear operator that is sufficient for the existence of a solution of the problem (5), (6) from the class $C^p([t_0,T);\mathfrak{V})\cap C^{p+1}((t_0,T);\mathfrak{V})$.

REMARK 4. In the works of G.A.Sviridyuk and his coathors Sobolev type equations with the same linear part as in this paper and with independent on t nonlinear operator N were considered (see, for example, [1, 2, 4]). In contrast to mentioned works in the Theorem 4 a solution of nonlinear Sobolev type equation is not a quasistationary trajectory, i. e. the equation $H(I-P)\dot{u}(t)\equiv 0$ is not satisfied on it.

If instead of the Cauchy problem naturally arising for Sobolev type equations generalized Showalter problem [11]

$$Pu(t_0)=u_0 \tag{18}$$

will be considered (see also [12]), then similar to the Theorem 4 result will be obtained. But the assertion will be true for every $(t_0,u_0)\in U\cap\mathbb{R}\times\mathrm{dom}M_1$ and the concordance condition (13) will be absent.

Теорема 5. *Let operator M be strongly $(L,1)$-sectorial, operator N map an open set $U\subset\mathbb{R}\times\mathfrak{U}^0\oplus\mathfrak{U}^1_\alpha$ for some $\alpha\in[0,1)$ to the set $\mathfrak{F}^0\dot{+}L_1[\mathrm{dom}M_1]\subset\mathfrak{F}$, $L_1^{-1}QN\in C(U;\mathcal{D}_{M_1})\cap C^1(U;\mathfrak{U})$, $M_0^{-1}(I-Q)N\in C^2(U;\mathfrak{U})$, operators $A_1^\alpha L_1^{-1}QN: U\to\mathfrak{U}$, $\frac{\partial(QN)}{\partial t}:U\to\mathfrak{F}$, $\frac{\partial(QN)}{\partial u}v:U\to\mathfrak{F}$ be locally Hölder with respect to t and locally Lipschitz with respect to u on U for all $v\in\mathfrak{U}^1_\alpha$. Besides, suppose that for all $(t,u)\in U$, $w\in\mathfrak{U}^0$ the relations $(t,u+w)\in U$, $N(t,u)=N(t,u+w)$ hold. Then for every $(t_0,u_0)\in U\cap\mathbb{R}\times\mathrm{dom}M_1$ there exists such $T=T(t_0,u_0)>t_0$ that the problem (12), (18) has a unique solution on (t_0,T).*

3. Example of a problem with not quasistationary trajectories

Let $a,b,\alpha,\beta,\lambda\in\mathbb{R}$, $a<b$. Denote $Aw=w_{xx}$, $A:\mathrm{dom}A\to L_2(a,b)$,

$$\mathrm{dom}A=H^2_{\frac{\partial}{\partial n}}(a,b)\equiv\left\{w\in H^2(a,b):\frac{\partial}{\partial x}w(a)=\frac{\partial}{\partial x}w(b)=0\right\}\subset L_2(a,b).$$

Let choose an orthonormal basis $\{\varphi_k:k\in\mathbb{N}\}$ of eigenfinctions of the operator A in the space $L_2(\Omega)$, where the functions φ_k correspond to eigenvalues λ_k of the operator, that numbered in the nonincreasing order taking into account of their multiplicity.

Consider the problem

$$\left(\beta+\frac{d^2}{dx^2}\right)u(x,t_0)=\left(\beta+\frac{d^2}{dx^2}\right)u_0(x),\quad x\in(a,b), \tag{19}$$

$$u_x(a,t)=u_x(b,t)=v_x(a,t)=v_x(b,t)=0,\quad t\in(t_0,T), \tag{20}$$

$$u_t=u_{xx}-v_{xx}+\int\limits_a^x f\left(t,\xi,\sum_{\lambda_k\neq-\beta}\langle u,\varphi_k\rangle\varphi_k(\xi)\right)d\xi,\quad (x,t)\in(a,b)\times(t_0,T), \tag{21}$$

$$v_{xx}+\beta v+\alpha u+g\left(t,x,\sum_{\lambda_k\neq-\beta}\langle u,\varphi_k\rangle\varphi_k(x)\right)=0,\quad (x,t)\in(a,b)\times(t_0,T). \tag{22}$$

Functions $u(x,t)$, $v(x,t)$ are unknown in the problem.

REMARK 5. The system (21) – (22) with $f \equiv g \equiv 0$ are obtained by linear replacement of unknown functions in the linearized system of phase field describing phase transitions of first kind [13].

Put $\mathfrak{U} = \mathfrak{F} = (L_2(a,b))^2$,

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x^2} \\ \alpha & \beta + \frac{\partial^2}{\partial x^2} \end{pmatrix}, \quad \mathrm{dom} M = \left(H^2_{\frac{\partial}{\partial n}}(\Omega)\right)^2.$$

Before [14] it was shown that in the case of $-\beta \notin \sigma(A)$ the operator M is strongly $(L,0)$-sectorial (see also [15]). In present paper we will reject this condition on β.

Theorem 6. *Let $\alpha \neq 0$, $-\beta \in \sigma(A) \setminus \{0\}$. Then the operator M is strongly $(L,1)$-sectorial.*

PROOF. The equation $-\beta\mu + (\alpha+\beta-\mu)\lambda_k + \lambda_k^2 = 0$ has a solution $\mu = \delta_k = \frac{(\alpha+\beta+\lambda_k)\lambda_k}{\beta+\lambda_k}$ in the case of $-\beta \neq \lambda_k$. If for some $k \in \mathbb{N}$ the equality $-\beta = \lambda_k$ holds then it follows from the equation that $\alpha\beta = 0$. It is not difficult to verify that if $\beta = 0 \in \sigma(A)$ or $\alpha = 0$, $-\beta \in \sigma(A)$ then $\rho^L(M) = \emptyset$ because for every eigenfunction φ_k corresponding to eigenvalue $\lambda_k = -\beta$ the equality $\mu L\varphi_k = M\varphi_k$ will hold for all $\mu \in \mathbb{C}$. The conditions of present theorem such facilities exclude therefore

$$(\mu L - M)^{-1} = \begin{pmatrix} \sum\limits_{k=1}^{\infty} \frac{(-\beta-\lambda_k)\langle\cdot,\varphi_k\rangle\varphi_k}{-\beta\mu+(\alpha+\beta-\mu)\lambda_k+\lambda_k^2} & \sum\limits_{k=1}^{\infty} \frac{-\lambda_k\langle\cdot,\varphi_k\rangle\varphi_k}{-\beta\mu+(\alpha+\beta-\mu)\lambda_k+\lambda_k^2} \\ \sum\limits_{k=1}^{\infty} \frac{\alpha\langle\cdot,\varphi_k\rangle\varphi_k}{-\beta\mu+(\alpha+\beta-\mu)\lambda_k+\lambda_k^2} & \sum\limits_{k=1}^{\infty} \frac{(\mu-\lambda_k)\langle\cdot,\varphi_k\rangle\varphi_k}{-\beta\mu+(\alpha+\beta-\mu)\lambda_k+\lambda_k^2} \end{pmatrix} =$$

$$\begin{pmatrix} \sum\limits_{\lambda_k\neq-\beta} \frac{\langle\cdot,\varphi_k\rangle\varphi_k}{\mu-\delta_k} & \sum\limits_{\lambda_k\neq-\beta} \frac{\lambda_k\langle\cdot,\varphi_k\rangle\varphi_k}{(\beta+\lambda_k)(\mu-\delta_k)} - \frac{1}{\alpha}\sum\limits_{\lambda_k=-\beta}\langle\cdot,\varphi_k\rangle\varphi_k \\ \sum\limits_{\lambda_k\neq-\beta} \frac{-\alpha\langle\cdot,\varphi_k\rangle\varphi_k}{(\beta+\lambda_k)(\mu-\delta_k)} - \frac{1}{\beta}\sum\limits_{\lambda_k=-\beta}\langle\cdot,\varphi_k\rangle\varphi_k & \sum\limits_{\lambda_k\neq-\beta} \frac{(\mu-\lambda_k)\langle\cdot,\varphi_k\rangle\varphi_k}{-(\beta+\lambda_k)(\mu-\delta_k)} - \frac{\mu+\beta}{\alpha\beta}\sum\limits_{\lambda_k=-\beta}\langle\cdot,\varphi_k\rangle\varphi_k \end{pmatrix}.$$

Denote $\mathbb{M} = \{k \in \mathbb{N} : \lambda_k \neq -\beta\}$ then $a = \max\limits_{k\in\mathbb{M}} \delta_k < \infty$ and for all $\theta \in (\pi/2, \pi)$, $\mu \in S^L_{a,\theta}(M)$ the operator $(\mu L - M)^{-1}$ is continuous under the boundedness of sequences $\frac{\lambda_k}{\beta+\lambda_k}$, $\frac{\alpha}{\beta+\lambda_k}$, $\frac{\mu-\lambda_k}{\beta+\lambda_k}$. Then

$$R^L_\mu(M) = \begin{pmatrix} \sum\limits_{\lambda_k\neq-\beta} \frac{\langle\cdot,\varphi_k\rangle\varphi_k}{\mu-\delta_k} & 0 \\ \sum\limits_{\lambda_k\neq-\beta} \frac{-\alpha\langle\cdot,\varphi_k\rangle\varphi_k}{(\beta+\lambda_k)(\mu-\delta_k)} - \frac{1}{\beta}\sum\limits_{\lambda_k=-\beta}\langle\cdot,\varphi_k\rangle\varphi_k & 0 \end{pmatrix},$$

$$R^L_{\mu_0}(M)R^L_{\mu_1}(M) = \begin{pmatrix} \sum\limits_{\lambda_k\neq-\beta} \frac{\langle\cdot,\varphi_k\rangle\varphi_k}{(\mu_0-\delta_k)(\mu_1-\delta_k)} & 0 \\ \sum\limits_{\lambda_k\neq-\beta} \frac{-\alpha\langle\cdot,\varphi_k\rangle\varphi_k}{(\beta+\lambda_k)(\mu_0-\delta_k)(\mu_1-\delta_k)} & 0 \end{pmatrix},$$

$$L^L_\mu(M) = \begin{pmatrix} \sum\limits_{\lambda_k\neq-\beta} \frac{\langle\cdot,\varphi_k\rangle\varphi_k}{\mu-\delta_k} & \sum\limits_{\lambda_k\neq-\beta}^{\infty} \frac{\lambda_k\langle\cdot,\varphi_k\rangle\varphi_k}{(\beta+\lambda_k)(\mu-\delta_k)} - \frac{1}{\alpha}\sum\limits_{\lambda_k=-\beta}\langle\cdot,\varphi_k\rangle\varphi_k \\ 0 & 0 \end{pmatrix},$$

$$L^L_{\mu_0}(M)L^L_{\mu_1}(M) = \begin{pmatrix} \sum\limits_{\lambda_k\neq-\beta} \frac{\langle\cdot,\varphi_k\rangle\varphi_k}{(\mu_0-\delta_k)(\mu_1-\delta_k)} & \sum\limits_{\lambda_k\neq-\beta}^{\infty} \frac{\lambda_k\langle\cdot,\varphi_k\rangle\varphi_k}{(\beta+\lambda_k)(\mu_0-\delta_k)(\mu_1-\delta_k)} \\ 0 & 0 \end{pmatrix},$$

$$R^L_{\mu_0}(M)R^L_{\mu_1}(M)(\gamma L - M)^{-1} =$$

$$\begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{(\mu_0-\delta_k)(\mu_1-\delta_k)(\gamma-\delta_k)} & \sum\limits_{\lambda_k \neq -\beta} \frac{\lambda_k \langle \cdot, \varphi_k \rangle \varphi_k}{(\beta+\lambda_k)(\mu_0-\delta_k)(\mu_1-\delta_k)(\gamma-\delta_k)} \\ \sum\limits_{\lambda_k \neq -\beta} \frac{-\alpha \langle \cdot, \varphi_k \rangle \varphi_k}{(\beta+\lambda_k)(\mu_0-\delta_k)(\mu_1-\delta_k)(\gamma-\delta_k)} & \sum\limits_{\lambda_k \neq -\beta} \frac{-\alpha\lambda_k \langle \cdot, \varphi_k \rangle \varphi_k}{(\beta+\lambda_k)^2(\mu_0-\delta_k)(\mu_1-\delta_k)(\gamma-\delta_k)} \end{pmatrix},$$

$$M(\gamma L - M)^{-1} L^L_{\mu_0}(M) L^L_{\mu_1}(M) =$$

$$\begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \frac{\delta_k \langle \cdot, \varphi_k \rangle \varphi_k}{(\mu_0-\delta_k)(\mu_1-\delta_k)(\gamma-\delta_k)} & \sum\limits_{\lambda_k \neq -\beta} \frac{\delta_k \lambda_k \langle \cdot, \varphi_k \rangle \varphi_k}{(\beta+\lambda_k)(\mu_0-\delta_k)(\mu_1-\delta_k)(\gamma-\delta_k)} \\ 0 & 0 \end{pmatrix},$$

Take

$$K = \frac{1}{\sin^3\theta} \max_{k \in \mathbf{M}} \left\{ 1, \left| \frac{\alpha}{\beta + \lambda_k} \right|, \left| \frac{\lambda_k}{\beta + \lambda_k} \right|, \frac{|\alpha\lambda_k|}{(\beta + \lambda_k)^2} \right\},$$

$$\operatorname{const}(f) = \|f\|_{H^2(\Omega)} \max_{k \in \mathbf{M}} \left\{ K, \left| \frac{\alpha + \beta + \lambda_k}{\beta + \lambda_k} \right|, \frac{|(\alpha + \beta + \lambda_k)\lambda_k|}{(\beta + \lambda_k)^2} \right\}$$

then for every $f \in \overset{\circ}{\mathfrak{F}} = \operatorname{dom} M$ and for all $\mu_0, \mu_1, \gamma \in S^L_{a,\theta}(M)$ the inequalities

$$\max\left\{ \|R^L_{(\mu,1)}(M)\|_{\mathcal{L}(\mathfrak{U})}, \|L^L_{(\mu,1)}(M)\|_{\mathcal{L}(\mathfrak{U})} \right\} \leq \frac{K}{|\mu_0 - a||\mu_1 - a|},$$

$$\|R^L_{(\mu,1)}(M)(\gamma L - M)^{-1}\|_{\mathcal{L}(\mathfrak{U})} \leq \frac{K}{|\gamma - a||\mu_0 - a||\mu_1 - a|},$$

$$\|M(\gamma L - M)^{-1} L^L_{(\mu,1)}(M) f\|_{\mathfrak{F}} \leq \frac{\operatorname{const}(f)}{|\gamma - a||\mu_0 - a||\mu_1 - a|}$$

hold. □

The projector P under the conditions of the Theorem 6 has a form

$$P = s\text{-}\lim_{\mu \to +\infty} \left(\mu R^L_\mu(M)\right)^2 = \begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \langle \cdot, \varphi_k \rangle \varphi_k & 0 \\ \sum\limits_{\lambda_k \neq -\beta} \frac{-\alpha \langle \cdot, \varphi_k \rangle \varphi_k}{\beta + \lambda_k} & 0 \end{pmatrix},$$

$$Q = s\text{-}\lim_{\mu \to +\infty} \left(\mu L^L_\mu(M)\right)^2 = \begin{pmatrix} \sum\limits_{\lambda_k \neq -\beta} \langle \cdot, \varphi_k \rangle \varphi_k & \sum\limits_{\lambda_k \neq -\beta} \frac{\lambda_k \langle \cdot, \varphi_k \rangle \varphi_k}{\beta + \lambda_k} \\ 0 & 0 \end{pmatrix}.$$

Denote $\mathcal{W} = \operatorname{span}\{\varphi_k : \lambda_k = -\beta\}$, $\mathcal{Z} = \overline{\operatorname{span}}\{\varphi_k : \lambda_k \neq -\beta\}$, where the overline means the closure in the sense of the space $L_2(a,b)$, P_1 is the projector in the space $L_2(a,b)$ on $\mathcal{Z}$ along $\mathcal{W}$. We have $\mathfrak{U}^0 = \ker P = \mathcal{W} \times L_2(a,b)$, $\mathfrak{U}^1 = \operatorname{im} P = \{(u, -\alpha(\beta + A)^{-1}u) \in (L_2(a,b))^2 : u \in \mathcal{Z}\}$ is isomorphic to $\mathcal{Z} \times \{0\}$, $\mathfrak{F}^0 = \ker Q = \{(u,v) \in (L_2(a,b))^2 : P_1 u = -A(\beta + A)^{-1} P_1 v, u, v \in L_2(a,b)\} = \mathcal{W} \times L_2(a,b)$, $\mathfrak{F}^1 = \operatorname{im} Q = \{(u + A(\beta + A)^{-1}v, 0) \in \mathfrak{U} : (u,v) \in \mathcal{Z}\} = \mathcal{Z} \times \{0\}$.

For the sectorial operator A let us construct the operator $A_1 = -A$ and the subspaces $\mathcal{H}^\gamma = \operatorname{dom} A_1^{\gamma/2}$, $\gamma \geq 0$.

Theorem 7. *Let $\alpha \neq 0$, $-\beta \in \sigma(A) \setminus \{0\}$, functions $f, g \in C^2(\mathbb{R} \times [a,b] \times \mathbb{R}; \mathbb{R})$, $f(\cdot, a, \cdot) \equiv f(\cdot, b, \cdot) \equiv 0$. Then for every $(t_0, u_0) \in \mathbb{R} \times \mathcal{H}^1$ there exists such $T = T(t_0, u_0) > t_0$ that the problem* (19) – (22) *has a unique solution on* (t_0, T).

PROOF. The problem (19) – (22) can be reduced to the problem (12), (18) with the operators L, M that is given above and with the operator

$$N(t,u,v)(x) = \begin{pmatrix} \int\limits_a^x f(t,\xi,P_1u(\xi))\,d\xi \\ g(t,x,P_1u(x)) \end{pmatrix},$$

that is defined on the set $U = \mathbb{R}\times\mathcal{H}^1\times L_2(a,b)$.

Let verify the conditions of the Theorem 5. It is evidently that for every $u_0 \in L_2(a,b)$ function $\left(\beta+\frac{\partial^2}{\partial x^2}\right)u_0 \in \mathcal{Z}$ set the pair $(u_0, -\alpha(\beta+A)^{-1}u_0) \in \mathfrak{U}^1$. A function $u\in\mathcal{H}^1$ is continuous therefore for $(t,u,v)\in U$ under the Hölder inequality we have

$$\|N(t,u,v)\|^2_{(L_2(a,b))^2} \le (b-a)^2\int\limits_a^b f^2(t,\xi,P_1u(\xi))d\xi + \int\limits_a^b g^2(t,x,P_1u(x))dx < \infty.$$

So $N : U \to \mathfrak{F}$. Besides, the functions

$$QN(t,u,v)(x) = \int\limits_a^x P_1f(t,\xi,P_1u(\xi))\,d\xi,$$

$$iA_1^{1/2}L_1^{-1}QN(t,u,v) = \frac{\partial}{\partial x}QN(t,u,v) = P_1f(t,x,P_1u(x))$$

is continuously differentiable with respect to t,u and v, therefore is differentiable with respect to (t,u,v). Also we have

$$\left.\frac{\partial}{\partial x}QN(t,u,v)\right|_{x=a} = \left.\frac{\partial}{\partial x}QN(t,u,v)\right|_{x=b} = 0$$

under the conditions of the Theorem on a function f. Since for $u\in\mathcal{H}^1$ there exists the derivative

$$\frac{d}{dx}f(t,x,P_1u(x)) = \frac{\partial f}{\partial x}(t,x,P_1u(x)) + \frac{\partial f}{\partial u}(t,x,P_1u(x))P_1u'(x) \in L_2(a,b), \quad (23)$$

then $\mathrm{im}QN \subset \mathrm{dom}A = L_1[\mathrm{dom}M_1]$. Also from (23) it follows that the operator $M_1L_1^{-1}QN$ is continuous with respect to (t,u,v).

The remaining conditions of the Theorem 5 on the nonlinear operator follow from the conditions of smothness of functions f,g and they can be verified directly.

The list of references

1. *Sviridyuk G.A.* Semilinear equations of Sobolev type with a relatively sectorial operator. Dokl. Akad. Nauk, 1993, v. 329, no. 3, p. 274–277. (Russian)
2. *Sviridyuk G. A., Sukacheva T. G.* On the solvability of a nonstationary problem describing the dynamics of an incompressible viscoelastic fluid. Math. Notes, 1998, v. 63, no. 3, p. 388–395.
3. *Sveshnikov A. G., Alshin A. B., Korpusov M. O., Pletner Yu. D.* Linear and Nonlinear Sobolev Type Equations. Fizmatlit, Moscow, 2007. (Russian)
4. *Sviridyuk G. A., Kazak V. O.* The phase space of one generalized model by Oskolkov. Sib. Math. J., 2003, v. 44, no. 5, p. 877–882.
5. *Sviridyuk G. A., Fedorov V. E.* Analytic semigroups with kernels and linear equations of Sobolev type, Sib. Math. J., 1995, v. 36, no. 5, p. 873–987.
6. *Sviridyuk G. A., Fedorov V. E.* On the identities of analytic semigroups of operators with kernels. Sib. Math. J., 1998, v. 39, no. 3, p. 522–533.
7. *Sviridyuk G. A., Fedorov V. E.* Linear Sobolev Type Equations and Degenerate Semigroups of Operators. VSP, Utrecht, Boston, 2003.
8. *Fedorov V. E.* Holomorphic solution semigroups for Sobolev-type equations in locally convex spaces. Sb. Math., 2004, v. 195, no. 8, p. 1205–1234.
9. *Henry D.* Geometric Theory of Semilinear Parabolic Equations. Springer-Verlag, Berlin, Heidelberg, New York, 1981.
10. *Fedorov V. E.* On some nonstationary nonlinear Sobolev type equations. Nonclassical equations of mathematical physics. Coll. sc. articles, Sobolev Math. Inst. of SB RAS, Novosibirsk, 2007, p. 307–314. (Russian)
11. *Showalter R. E.* Partial differential equations of Sobolev – Galperin type. Pacific J. Math., 1963, v. 31, no. 3, p. 787–793.
12. *Plekhanova M. V., Fedorov V. E.* An optimal control problem for a class of degenerate equations. J. of Computer and System Sciences International, 2004, v. 43, no. 5, p. 698–702.
13. *Plotnikov P. I., Klepacheva A. V.* The phase field equations and gradient flows of marginal functions. Sib. Math. J, 2001, v. 42, no. 3, p. 551–567.
14. *Fedorov V. E., Urazayeva A. V.* Inverse problem for a class of singular linear operator-differential equations. Coll. sc. articles, Voronezh St. Univ., Voronezh, 2004, p.161–172. (Russian)
15. *Fedorov V. E., Sagadeyeva M. A.* Bounded solutions of linearized system of phase field equations. Nonclassical equations of mathematical physics. Coll. sc. articles, Sobolev Math. Inst. of SB RAS, Novosibirsk, 2005, p. 275–284. (Russian)

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