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Stationary patterns for a cooperative model with nonlinear diffusion

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1 Introduction

In this article we study positive steady-state solutions of the following strongly coupled reaction-diffusion system:

$$
\text{(P)} \begin{cases} u_t = \Delta \left[\left(1 + \dfrac{\alpha}{\mu + v} \right) u \right] + u(a - u + cv) & \text{in } \Omega \times (0,T), \\ v_t = \Delta v + v(-b + du - v) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot), \quad v(\cdot,0) = v_0(\cdot) & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$; $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$; a, b, c, d and μ are all positive constants; α is a non-negative constant; u_0 and v_0 are given non-negative functions which are not identically zero. System (P) is a Lotka-Volterra cooperative model with a density-dependent diffusion term of a fractional type; unknown functions u and v represent population densities of two cooperative species, respectively; a and $-b$ denote the intrinsic growth rates of the respective species; c and d denote interaction coefficients. When $\alpha = 0$, (P) is reduced to a classical Lotka-Volterra cooperative model with diffusion. See [6] and [13] for such a cooperative model.

In the first equation of (P), the nonlinear diffusion term $\alpha\Delta\{u/(\mu+v)\}$ describes a situation where species u tends to leave low-density areas of species v. This situation is natural because relations between u and v are cooperative. A population model with density-dependent diffusion was first proposed by Shigesada, Kawasaki and Teramoto [14] to investigate the habitat segregation phenomena between two competing species. Since their work, many mathematicians have studied population models with density-dependent diffusion. However, population models including density-dependent diffusion terms of a fractional type have appeared in recent years; for example, see [5], [16] for cooperative models with Dirichlet boundary conditions; [2], [3] for prey-predator models with Dirichlet boundary conditions; [12], [15] for three-species prey-predator models with Neumann boundary conditions. See also the monograph of Okubo and Levin [11] for the biological background.

The stationary problem associated with (P) is

$$(\mathrm{SP})\begin{cases} \Delta\left[\left(1+\dfrac{\alpha}{\mu+v}\right)u\right]+u(a-u+cv)=0 & \text{in } \Omega, \\ \Delta v+v(-b+du-v)=0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}=\dfrac{\partial v}{\partial n}=0 & \text{on } \partial\Omega. \end{cases}$$

Our main purpose is to study the existence of stationary patterns (i.e. positive non-constant solutions) for (SP) with the weak cooperative condition

$$\frac{a}{b}>\frac{1}{d}>c. \tag{1.1}$$

From now on, we will always assume (1.1). It is well known that, if $\alpha=0$, then every solution of (P) converges to a unique positive constant steady-state

$$(u^*,v^*):=\left(\frac{a-bc}{1-cd},\frac{ad-b}{1-cd}\right)$$

uniformly as $t\to\infty$; see [6]. This implies the following proposition.

Proposition 1.1. *Let $\alpha=0$. Then (u^*,v^*) is a unique positive solution of* (SP).

Proposition 1.1 means that no stationary pattern exists in the linear diffusion case. However, the presence of density-dependent diffusion enables us to construct stationary patterns of (SP). We focus on α to show the emergence of stationary patterns for (SP).

Let $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ denote eigenvalues of $-\Delta$ with the homogeneous Neumann boundary condition on $\partial\Omega$ and let m_i denote the algebraic multiplicity of λ_i. Then we have the following theorem.

Theorem 1.1. *Suppose that $\{v^*(b-\mu)\}/(\mu+v^*)\in(\lambda_l,\lambda_{l+1})$ for some $l\geq 1$ and that $\sum_{i=1}^{l} m_i$ is odd. Then there exists a positive constant $\alpha^*=\alpha^*(a,b,c,d,\mu)$ such that* (SP) *has at least one positive non-constant solution for each $\alpha>\alpha^*$.*

We are also interested in the limiting patterns of (SP) as $\alpha\to\infty$. Under the restriction $N\leq 3$, we obtain the following limiting system as $\alpha\to\infty$.

Theorem 1.2. *Suppose $N \leq 3$ and $b > \mu$. Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ be any sequence such that $\lim_{i\to\infty} \alpha_i = \infty$ and positive functions (u_i, v_i) satisfy* (SP) *with $\alpha = \alpha_i$. Then, by passing to a subsequence if necessary, it holds that*

$$\lim_{i\to\infty}(u_i, v_i) = (\tau(\mu + \bar{v}), \bar{v}) \quad \text{in } C^1(\bar{\Omega}) \times C^1(\bar{\Omega}),$$

where τ is a positive constant satisfying $1 < d\tau < b/\mu$, $\bar{v}$ is a positive function in Ω and $(\tau, \bar{v})$ satisfies

$$\begin{cases} \Delta\bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 & \text{in } \Omega, \\ \dfrac{\partial \bar{v}}{\partial n} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega (\mu + \bar{v})\{a - \tau\mu + (c - \tau)\bar{v}\}dx = 0. & \end{cases} \tag{1.2}$$

We expect that the limiting system (1.2) may give much information on profiles of stationary patterns of (SP) for large α. We will give some remarks about (1.2) in the last section.

Throughout the article, the usual norms of $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\bar{\Omega})$ are defined by

$$\|\psi\|_p := \left(\int_\Omega |\psi(x)|^p dx\right)^{1/p} \quad \text{and} \quad \|\psi\|_\infty := \max_{x\in\bar{\Omega}} |\psi(x)|,$$

respectively.

2 Stability of the constant solution (u^*, v^*)

In this section, we will analyze the linearized stability of the constant stationary solution (u^*, v^*) for (P).

The linearized eigenvalue problem of (P) at (u^*, v^*) is given by

$$\begin{cases} -\left(1 + \dfrac{\alpha}{\mu + v^*}\right)\Delta h + \dfrac{\alpha u^*}{(\mu + v^*)^2}\Delta k + u^* h - cu^* k = \eta h & \text{in } \Omega, \\ -\Delta k - dv^* h + v^* k = \eta k & \text{in } \Omega, \\ \dfrac{\partial h}{\partial n} = \dfrac{\partial k}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We know that (u^*, v^*) is linearly stable when $\alpha = 0$. Using the expansions of h and k in terms of eigenfunctions of $-\Delta$, one can see that η is an eigenvalue of (2.1) if and only if

$$\det\begin{pmatrix} -\eta + \left(1 + \frac{\alpha}{\mu + v^*}\right)\lambda_i + u^* & -\frac{\alpha u^*}{(\mu + v^*)^2}\lambda_i - cu^* \\ -dv^* & -\eta + \lambda_i + v^* \end{pmatrix} = 0$$

for some $i \geq 0$. In particular, $\eta = 0$ is an eigenvalue of (2.1) if and only if

$$\frac{\lambda_i}{(\mu+v^*)^2}\{(\mu+v^*)(\lambda_i+v^*)-du^*v^*\}\alpha+(\lambda_i+u^*)(\lambda_i+v^*)-cdu^*v^*=0$$

for some $i \geq 0$. Note that $(\lambda_i+u^*)(\lambda_i+v^*)-cdu^*v^* > 0$ for all $i \geq 0$ because of (1.1). Thus it is easy to see that the linearized stability of (u^*, v^*) changes as α increases in (P) if and only if

$$\begin{aligned}(\mu+v^*)(\lambda_1+v^*)-du^*v^* &= (\mu+v^*)\lambda_1+v^*(\mu+v^*-du^*)\\ &= (\mu+v^*)\lambda_1+v^*(\mu-b)\\ &< 0.\end{aligned}$$

Therefore, $b > \mu$ is necessary for the linearized stability of (u^*, v^*) to change (and so we do not discuss the case $b \leq \mu$, especially, $-b \geq 0$). This means that the difference in the intrinsic growth rates between two species u and v contributes to creating stationary patterns in (SP).

3 Proof of Theorem 1.1

3.1 Reduction to the semilinear system

Our method of the proof of Theorem 1.1 will be based on the Leray-Schauder degree theory (see e.g., [9]) and some a priori estimates. We first introduce a new unknown function U by

$$U=\left(1+\frac{\alpha}{\mu+v}\right)u. \tag{3.1}$$

Clearly, there exists a one-to-one correspondence between $(u, v) > 0$ and $(U, v) > 0$. As far as we discuss positive solutions, (SP) is rewritten in the following equivalent form:

$$\text{(EP)}\begin{cases}\Delta U+\dfrac{\mu+v}{\mu+v+\alpha}U\left(a-\dfrac{\mu+v}{\mu+v+\alpha}U+cv\right)=0 & \text{in } \Omega,\\ \Delta v+v\left(-b+d\dfrac{\mu+v}{\mu+v+\alpha}U-v\right)=0 & \text{in } \Omega,\\ \dfrac{\partial U}{\partial n}=\dfrac{\partial v}{\partial n}=0 & \text{on } \partial\Omega.\end{cases}$$

Thus, it is sufficient to show the existence of positive non-constant solutions of (EP).

3.2 A priori estimates

In this subsection, we will give some a priori estimates for positive solutions of (EP). Before stating the a priori estimates, we recall the following maximum principle due to Lou and Ni [7].

Lemma 3.1. *Suppose that* $g \in C(\bar{\Omega} \times \mathbb{R})$.
(i) *If* $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies*

$$\Delta w(x) + g(x, w(x)) \geq 0 \quad in \ \Omega, \quad \frac{\partial w}{\partial n} \leq 0 \quad on \ \partial\Omega,$$

and $w(x_0) = \max_{\bar{\Omega}} w$, *then* $g(x_0, w(x_0)) \geq 0$.
(ii) *If* $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *satisfies*

$$\Delta w(x) + g(x, w(x)) \leq 0 \quad in \ \Omega, \quad \frac{\partial w}{\partial n} \geq 0 \quad on \ \partial\Omega,$$

and $w(x_0) = \min_{\bar{\Omega}} w$, *then* $g(x_0, w(x_0)) \leq 0$.

Now we can derive the following a priori estimates.

Lemma 3.2. *Let* ζ *be any fixed positive number. Then there exist two positive constants* $C_*(\zeta) = C_*(\zeta, a, b, c, d, \mu) < C^*(\zeta) = C^*(\zeta, a, b, c, d, \mu)$ *such that, if* $\alpha \leq \zeta$, *then any positive solution* (U, v) *of* (EP) *satisfies*

$$a \leq U(x) \leq C^*(\zeta) \quad and \quad C_*(\zeta) \leq v(x) \leq C^*(\zeta) \quad for\ all\ x \in \bar{\Omega}.$$

Proof. Let $U(x_0) = \max_{\bar{\Omega}} U$ and $v(y_0) = \max_{\bar{\Omega}} v$ with $x_0, y_0 \in \bar{\Omega}$. Applying Lemma 3.1 to (EP), we have

$$\max_{\bar{\Omega}} U \leq \frac{\mu + v(x_0) + \alpha}{\mu + v(x_0)}(a + cv(x_0))$$

and

$$\max_{\bar{\Omega}} v \leq -b + d\frac{\mu + v(y_0)}{\mu + v(y_0) + \alpha}U(y_0) \leq -b + d\max_{\bar{\Omega}} U. \tag{3.2}$$

Thus

$$\begin{aligned} \max_{\bar{\Omega}} U &\leq a + cv(x_0) + \zeta\frac{a + cv(x_0)}{\mu + v(x_0)} \\ &\leq a + c(-b + d\max_{\bar{\Omega}} U) + \zeta \max\left\{\frac{a}{\mu}, c\right\}. \end{aligned}$$

Therefore, we see

$$\max_{\bar{\Omega}} U \leq \frac{a - bc + \zeta\max\{a/\mu, c\}}{1 - cd}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\max_{\bar{\Omega}} v \leq -b + \frac{d(a - bc + \zeta \max\{a/\mu, c\})}{1 - cd} = \frac{ad - b + \zeta d \max\{a/\mu, c\}}{1 - cd}. \tag{3.4}$$

Hence we have obtained the desired upper bound of (U, v).

Let $U(z_0) = \min_{\bar{\Omega}} U$ with some $z_0 \in \bar{\Omega}$. Using Lemma 3.1 to the first equation of (EP), we get

$$\min_{\bar{\Omega}} U \geq \frac{\mu + v(z_0) + \alpha}{\mu + v(z_0)}(a + cv(z_0)) \geq a. \tag{3.5}$$

Thus we have obtained the desired lower bound of U.

Finally, we derive a lower bound of v by contradiction. Suppose that there exist a certain positive constant ζ_0 and a sequence $\{(U_i, v_i, \alpha_i)\}_{i=1}^{\infty}$ such that $\alpha_i \leq \zeta_0$ for all $i \in \mathbb{N}$, $\lim_{i\to\infty} \alpha_i = \alpha_\infty$ for some non-negative constant α_∞,

$$\lim_{i\to\infty} \min_{\bar{\Omega}} v_i = 0 \tag{3.6}$$

and positive functions (U_i, v_i) satisfy

$$\begin{cases} \Delta U_i + \dfrac{\mu + v_i}{\mu + v_i + \alpha_i} U_i \left(a - \dfrac{\mu + v_i}{\mu + v_i + \alpha_i} U_i + cv_i\right) = 0 & \text{in } \Omega, \\ \Delta v_i + v_i \left(-b + d\dfrac{\mu + v_i}{\mu + v_i + \alpha_i} U_i - v_i\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial U_i}{\partial n} = \dfrac{\partial v_i}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

By using the regularity theory for elliptic equations (see e.g., [1]) to the second equation of (3.7), it follows from (3.3) and (3.4) that

$$\|v_i\|_{W^{2,p}(\Omega)} \leq C(\zeta_0)$$

with some positive constant $C(\zeta_0) = C(\zeta_0, a, b, c, d, \mu)$ independent of i. If $p > N$, then Sobolev's embedding theorem implies $\{v_i\}_{i=1}^{\infty}$ is compact in $C^1(\bar{\Omega})$. Consequently, there exists a subsequence, which is still denoted by $\{v_i\}_{i=1}^{\infty}$, such that

$$\lim_{i\to\infty} v_i = v_\infty \quad \text{in } C^1(\bar{\Omega}) \tag{3.8}$$

with some non-negative function $v_\infty \in C^1(\bar{\Omega})$. Similarly, there exists a non-negative function $U_\infty \in C^1(\bar{\Omega})$ such that

$$\lim_{i\to\infty} U_i = U_\infty \quad \text{in } C^1(\bar{\Omega}). \tag{3.9}$$

Therefore, v_∞ satisfies

$$\Delta v_\infty + v_\infty \left(-b + d\frac{\mu + v_\infty}{\mu + v_\infty + \alpha_\infty} U_\infty - v_\infty\right) = 0 \quad \text{in } \Omega, \quad \frac{\partial v_\infty}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in a weak sense. By standard elliptic regularity theory we have $v_\infty \in C^2(\bar{\Omega})$ and thus v_∞ is a classical solution of the above equation. Then it follows from (3.6),(3.8) and the strong maximum principle that $v_\infty \equiv 0$ in $\bar{\Omega}$. We can easily see from the above argument that U_∞ satisfies

$$\Delta U_\infty + \frac{\mu}{\mu+\alpha_\infty} U_\infty \left(a - \frac{\mu}{\mu+\alpha_\infty} U_\infty \right) = 0 \quad \text{in } \Omega, \quad \frac{\partial U_\infty}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in the classical sense. Then by the strong maximum principle and Lemma 3.1, either $U_\infty \equiv a(\mu+\alpha_\infty)/\mu$ or $U_\infty \equiv 0$ in $\bar{\Omega}$. Combining (3.5) and (3.9), we can conclude $U_\infty \equiv a(\mu+\alpha_\infty)/\mu$ in $\bar{\Omega}$. Hence

$$\lim_{i\to\infty} \left(-b + d\frac{\mu+v_i}{\mu+v_i+\alpha_i} U_i - v_i \right) = ad - b > 0 \quad \text{uniformly in } \Omega$$

by (1.1) and this means

$$v_i \left(-b + d\frac{\mu+v_i}{\mu+v_i+\alpha_i} U_i - v_i \right) > 0 \quad \text{in } \Omega$$

for sufficiently large $i \in \mathbb{N}$ because $v_i > 0$ in Ω. On the other hand, from the second equation of (3.7), we have

$$\int_\Omega v_i \left(-b + d\frac{\mu+v_i}{\mu+v_i+\alpha_i} U_i - v_i \right) dx = -\int_\Omega \Delta v_i dx = -\int_{\partial\Omega} \frac{\partial v_i}{\partial n} d\sigma = 0$$

for all $i \in \mathbb{N}$. This is a contradiction; thus our proof is complete. □

3.3 Completion of the proof of Theorem 1.1

Set $X = C(\bar{\Omega}) \times C(\bar{\Omega})$. For each $\alpha \geq 0$, define an operator F_α by

$$F_\alpha \begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} (-\Delta + I)^{-1} \left[U + \frac{\mu+v}{\mu+v+\alpha} U \left(a - \frac{\mu+v}{\mu+v+\alpha} U + cv \right) \right] \\ (-\Delta + I)^{-1} \left[v + v \left(-b + d\frac{\mu+v}{\mu+v+\alpha} U - v \right) \right] \end{pmatrix},$$

where I is the identity map from $C(\bar{\Omega})$ into itself, and $(-\Delta + I)^{-1}$ is the inverse operator of $-\Delta + I$ subject to the homogeneous Neumann boundary condition on $\partial\Omega$. It is easy to see that $F_\alpha : X \to X$ is well-defined, and that by elliptic regularity theory and Sobolev's embedding theorem, F_α is a continuous and compact operator for each $\alpha \geq 0$. From these observations, one can define the Leray-Schauder degree of $I - F_\alpha$ at 0 in a suitable open set. Furthermore, (U, v) is a positive solution of $(I - F_\alpha)(U, v) = 0$ if and only if (U, v) is a positive solution of (EP).

In view of (3.1), we set

$$U_\alpha^* = \left(1 + \frac{\alpha}{\mu + v^*} \right) u^*.$$

Hence (U_α^*, v^*) is a zero point of $I - F_\alpha$. Then we can calculate the index of $I - F_0$ at (u^*, v^*) and the index of $I - F_\alpha$ at (U_α^*, v^*) for sufficiently large α, which are denoted by $\operatorname{index}(I - F_0, (u^*, v^*))$ and $\operatorname{index}(I - F_\alpha, (U_\alpha^*, v^*))$, respectively. We refer to [10] for the proofs of Lemmas 3.3 and 3.4.

Lemma 3.3. *It holds that* $\operatorname{index}(I - F_0, (u^*, v^*)) = 1$.

Lemma 3.4. *Suppose that* $\{v^*(b - \mu)\}/(\mu + v^*) \in (\lambda_l, \lambda_{l+1})$ *for some* $l \geq 1$. *Then there exists a positive constant* $\alpha^* = \alpha^*(a, b, c, d, \mu)$ *such that, if* $\alpha > \alpha^*$, *then*

$$\operatorname{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^{l} m_i},$$

where m_i *denotes the algebraic multiplicity of* λ_i *defined in Section 1.*

By virtue of Lemmas 3.3 and 3.4, we are ready to prove Theorem 1.1. In the proof of Theorem 1.1, we represent (EP) as $(\mathrm{EP})_\alpha$ to indicate the dependence on α.

Proof of Theorem 1.1. Fix any $\alpha > \alpha^*$, where α^* is a constant given in Lemma 3.4. It follows from Lemma 3.2 that there exist two positive constants $C_*(\alpha) = C_*(\alpha, a, b, c, d, \mu) < C^*(\alpha) = C^*(\alpha, a, b, c, d, \mu)$ such that

$$a \leq U(x) \leq C^*(\alpha) \quad \text{and} \quad C_*(\alpha) \leq v(x) \leq C^*(\alpha) \quad \text{for all } x \in \bar{\Omega}$$

for any positive solution (U, v) of $(\mathrm{EP})_\nu$ with any $\nu \in [0, \alpha]$. We define

$$S = \left\{(U, v) \in X \mid \frac{a}{2} \leq U \leq 2C^*(\alpha),\ \frac{C_*(\alpha)}{2} \leq v \leq 2C^*(\alpha) \text{ in } \bar{\Omega}\right\};$$

so that $I - F_\nu$ has no zero point on the boundary of S for any $\nu \in [0, \alpha]$. Note that $I - F_0$ has a unique zero point (u^*, v^*) in S. On account of the homotopy invariance of the Leray-Schauder degree and Lemma 3.3, we have

$$\deg(I - F_\alpha, S, 0) = \deg(I - F_0, S, 0) = \operatorname{index}(I - F_0, (u^*, v^*)) = 1. \tag{3.10}$$

Suppose that $(\mathrm{EP})_\alpha$ has no positive non-constant solution, i.e. $I - F_\alpha$ has a unique zero point (U_α^*, v^*) in S. Then from the assumption $\sum_{i=1}^{l} m_i$ being odd and Lemma 3.4, it follows that

$$\deg(I - F_\alpha, S, 0) = \operatorname{index}(I - F_\alpha, (U_\alpha^*, v^*)) = (-1)^{\sum_{i=1}^{l} m_i} = -1,$$

which contradicts (3.10). Thus we complete the proof. □

4 Proof of Theorem 1.2

We first state some a priori estimates independent of α.

Lemma 4.1. *Suppose that $N \leq 3$. Then there exists a positive constant $C_0 = C_0(a, b, c, d, \mu)$ independent of α such that any positive solution (u, v) of* (SP) *satisfies*

$$\|u\|_\infty \leq C_0 \quad \text{and} \quad \|v\|_\infty \leq C_0.$$

Lemma 4.1 can be proved by combining the L^2-estimates for positive solutions of (SP) (independent of α and N) with Harnack inequality (due to Lin, Ni and Takagi [4], and Lou and Ni [8]). We refer to [10] for the proof of Lemma 4.1.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\{(u_i, v_i, \alpha_i)\}_{i=1}^\infty$ be any sequence such that $\lim_{i\to\infty} \alpha_i = \infty$ and positive functions (u_i, v_i) satisfy (SP) with $\alpha = \alpha_i$. Set

$$\psi_i = \left(\frac{1}{\alpha_i} + \frac{1}{\mu + v_i}\right) u_i.$$

Note that positive functions (ψ_i, v_i) satisfy

$$\begin{cases} \Delta\psi_i + \dfrac{u_i(a - u_i + cv_i)}{\alpha_i} = 0 & \text{in } \Omega, \\ \Delta v_i + v_i(-b + du_i - v_i) = 0 & \text{in } \Omega, \\ \dfrac{\partial\psi_i}{\partial n} = \dfrac{\partial v_i}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and that $\{\psi_i\}_{i=1}^\infty$ is bounded independently of i by Lemma 4.1. Then by the compactness argument as in the proof of (3.8), there exists a subsequence, which is still denoted by $\{\psi_i\}_{i=1}^\infty$, such that

$$\lim_{i\to\infty} \psi_i = \tau \quad \text{in } C^1(\bar{\Omega})$$

for a non-negative function $\tau \in C^1(\bar{\Omega})$. Similarly, we see

$$\lim_{i\to\infty} v_i = \bar{v} \quad \text{in } C^1(\bar{\Omega}) \tag{4.1}$$

for a non-negative function $\bar{v} \in C^1(\bar{\Omega})$. Therefore, we obtain

$$\lim_{i\to\infty} u_i = \lim_{i\to\infty} \frac{\psi_i}{1/\alpha_i + 1/(\mu + v_i)} = \tau(\mu + \bar{v}) \quad \text{in } C^1(\bar{\Omega}). \tag{4.2}$$

We will show that τ is a positive constant. Observe that τ satisfies

$$\Delta\tau = 0 \quad \text{in } \Omega, \quad \frac{\partial \tau}{\partial n} = 0 \quad \text{on } \partial\Omega$$

in a weak sense. A standard elliptic regularity theory ensures $\tau \in C^2(\bar{\Omega})$; so that τ must be a non-negative constant. Let $v_i(x_i) = \max_{\bar{\Omega}} v_i$ with some $x_i \in \bar{\Omega}$. It follows from Lemma 3.1 that

$$u_i(x_i) \geq \frac{b + v_i(x_i)}{d} > \frac{b}{d} \ (> 0)$$

for all $i \in \mathbb{N}$. This fact, together with (4.2), yields $\tau > 0$.

We next prove $(\tau, \bar{v})$ satisfies (1.2). Note that $\bar{v}$ satisfies

$$\Delta\bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 \quad \text{in } \Omega, \quad \frac{\partial \bar{v}}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{4.3}$$

in a weak sense. In the standard manner, one can see that $\bar{v} \in C^2(\bar{\Omega})$ and $\bar{v}$ is a classical nonnegative solution of (4.3). It follows from the strong maximum principle that either $\bar{v} \equiv 0$ or $\bar{v} > 0$ in Ω. We show $\bar{v} > 0$ in Ω by contradiction. Suppose that $\bar{v} \equiv 0$ in Ω. Then it follows from (4.1) and (4.2) that

$$\lim_{i\to\infty} a - u_i + cv_i = a - \tau\mu \quad \text{and} \quad \lim_{i\to\infty} -b + du_i - v_i = -b + d\tau\mu$$

uniformly in Ω. On the other hand,

$$\int_\Omega u_i(a - u_i + cv_i)dx = \int_\Omega v_i(-b + du_i - v_i)dx = 0 \tag{4.4}$$

for all $i \in \mathbb{N}$. Consequently, $a - \tau\mu = -b + d\tau\mu = 0$ because of $u_i > 0$ and $v_i > 0$ in Ω and thus $ad - b = 0$. This contradicts (1.1). Therefore $\bar{v} > 0$ in Ω.

By (4.1), (4.2) and (4.4), it is clear that

$$\int_\Omega (\mu + \bar{v})\{a - \tau\mu + (c - \tau)\bar{v}\}dx = \int_\Omega (\mu + \bar{v})\{a - \tau(\mu + \bar{v}) + c\bar{v}\}dx = 0.$$

Hence it only remains to show $1 < d\tau < b/\mu$. By the assumption of Theorem 1.2,

$$-b + d\tau\mu < -\mu + d\tau\mu = \mu(d\tau - 1).$$

It thus follows from Lemma 3.1 and (4.3) that if $d\tau - 1 \leq 0$, then $\max_{\bar{\Omega}} \bar{v} \leq 0$ and this contradicts $\bar{v} > 0$ in Ω. Therefore, $d\tau > 1$. Using Lemma 3.1 and $\bar{v} > 0$ in Ω again, we obtain $d\tau < b/\mu$. Hence we complete the proof. □

5 Remarks about the limiting system (1.2)

We easily see that $(\tau, \bar{v}) = (u^*/(\mu+v^*), v^*)$ is the only positive constant solution of (1.2). So our concern is about positive non-constant solutions of (1.2). We discuss the differential equations without the integral constraint in (1.2) under the restriction $N \leq 3$:

$$\begin{cases} \Delta\bar{v} + \bar{v}\{-b + d\tau\mu + (d\tau - 1)\bar{v}\} = 0 & \text{in } \Omega, \\ \dfrac{\partial\bar{v}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Set

$$w = \frac{d\tau - 1}{b - d\tau\mu}\bar{v},$$

where $1 < d\tau < b/\mu$. Then (5.1) is rewritten in the following equivalent form:

$$\begin{cases} \dfrac{1}{b - d\tau\mu}\Delta w - w + w^2 = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

We note that, if $(0 <)b - d\tau\mu \ll 1$, then (5.2) has no positive non-constant solution (see [4]). Therefore, $b \gg 1$ is necessary for (1.2) to have positive non-constant solutions. We will study (1.2) in detail in the future.

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