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Rate of approach of two solutions for a semilinear heat equation with power nonlinearity

Masaki Hoshino
Mathematical Institute, Tohoku University, 980-8578, Sendai, Japan
(E-mail address: sa2m28@math.tohoku.ac.jp)

1 Introduction

The aim of this paper is to review recent progress on semilinear parabolic equations that are obtained by joint work with Eiji Yanagida (Tohoku University). In this paper, we investigate the behavior of solutions of the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, Δ is the Laplace operator with respect to x, $p > 1$ and $u_0 \not\equiv 0$ is a given continuous function on $\mathbb{R}^N$ that decays to zero as $|x| \to \infty$. The problem (1.1) has been studied in many papers, since Fujita studied the blow-up problem [6]. Among them, the stability problem of stationary solutions is one of the most important problems and we study the problem (1.1) along this line.

It is known that there exist critical exponents p that govern the structure of solutions. The exponent

$$p_S = \begin{cases} \frac{N+2}{N-2} & \text{for } N > 2, \\ \infty & \text{for } N \leq 2, \end{cases}$$

is well known as the Sobolev exponent that is critical for the existence of positive stationary solution of (1.1). Namely, there exists a classical positive radial solution φ of

$$\Delta\varphi + \varphi^p = 0, \qquad x \in \mathbb{R}^N,$$

if and only if $p \geq p_S$ [1, 2, 8]. We denote the solution by $\varphi = \varphi_\alpha(r), r = |x|, \alpha > 0$, where $\varphi_\alpha(0) = \alpha$. Then $\varphi_\alpha(r)$ satisfies the initial value problem

$$\begin{cases} \varphi_{\alpha,rr} + \frac{N-1}{r}\varphi_{\alpha,r} + \varphi_\alpha^p = 0, \\ \varphi_\alpha(0) = \alpha, \quad \varphi_{\alpha,r}(0) = 0. \end{cases}$$

For each $\alpha > 0$, the solution φ_α is strictly decreasing in $|x|$ and satisfies $\varphi_\alpha \to 0$ as $|x| \to \infty$. We extend the solution by setting $\varphi_\alpha = -\varphi_{-\alpha}$ for $\alpha < 0$ and $\varphi_0 = 0$. Then the set $\{\varphi_\alpha; \alpha \in \mathbb{R}\}$ forms a one-parameter family of radial stationary solutions.

The exponent

$$p_c = \begin{cases} \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{for } N > 10, \\ \infty & \text{for } N \leq 10, \end{cases}$$

is another important exponent which appeared first in [15]. It is known that for $p_S \leq p < p_c$, any pair of positive stationary solutions intersects each other. For $p \geq p_c$, Wang [20] showed that the family of stationary solutions forms a simply ordered set, that is, φ_α is strictly increasing in α for each x. We call it the ordering property of $\{\varphi_\alpha\}$. Moreover, φ_α satisfies

$$\lim_{\alpha\to 0} \varphi_\alpha(|x|) = 0, \quad \lim_{\alpha\to\infty} \varphi_\alpha(|x|) = \varphi_\infty(|x|),$$

for each x, where $\varphi_\infty(|x|)$ is a singular stationary solution given by

$$\varphi_\infty(|x|) = L|x|^{-m}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

with

$$m = \frac{2}{p-1}, \quad L = \{m(N-2-m)\}^{1/(p-1)}.$$

It was also shown in [12] that each positive stationary solution has the expansion

$$\varphi_\alpha(|x|) = \begin{cases} L|x|^{-m} - a_\alpha |x|^{-m-\lambda_1} + \text{h.o.t.} & p > p_c, \\ L|x|^{-m} - a_\alpha |x|^{-m-\lambda_1} \log|x| + \text{h.o.t.} & p = p_c, \end{cases}$$

as $|x| \to \infty$, where λ_1 is a positive constant given by

$$\lambda_1 = \lambda_1(N,p) := \frac{N-2-2m-\sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

and $a_\alpha = a(\alpha)$ is a positive number that is monotone decreasing in α. Note that λ_1 is a smaller root of the quadratic equation

$$h(\lambda) := \lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0.$$

We define by

$$\lambda_2 = \lambda_2(N,p) := \frac{N-2-2m+\sqrt{(N-2-2m)^2-8(N-2-m)}}{2},$$

a larger root of the quadratic equation.

Concerning the stability problem, Gui, Ni and Wang [12, 13] showed that any regular positive radial stationary solution is unstable in any reasonable sense if $p_S < p < p_c$ and "weakly asymptotically stable" in a weighted L^∞ norm if $p \geq p_c$. For $p > p_c$, Poláčik and Yanagida [18, 19] improved the above results and proved that the solutions approach a set of stationary solutions for a wide class of the initial data. As a by-product, they also showed the existence of global unbounded solutions. We note that the study of global unbounded solutions of (1.1) [3, 5] is closely related to our problem on bounded solutions mentioned later.

Recently, Fila, Winkler and Yanagida [4] carried out the further investigation about the convergence of solutions of (1.1). They studied the following more general problem: Let u and $\tilde{u}$ denote solutions of (1.1) with initial data u_0, $\tilde{u}_0$ respectively. They considered how fast these two solutions approach each other as $t \to \infty$. In particular, in the case of $\tilde{u}_0 = \varphi_\alpha(|x|)$, then the rate of approach corresponds to the convergence rate to the stationary solution. More precisely, they showed that if $p > p_c$, $m+\lambda_1 < l < m+\lambda_2$ and u_0, $\tilde{u}_0$ satisfy

(H1) $$|u_0|,\ |\tilde{u}_0| \leq \varphi_\alpha(|x|), \quad x \in \mathbb{R}^N$$

and

(H2) $$|u_0 - \tilde{u}_0| \leq c_1(1+|x|)^{-l}, \quad x \in \mathbb{R}^N$$

with some constants $\alpha > 0$ and $c_1 > 0$, then $\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty}$ decays faster in time than the rate $t^{-(l-m-\lambda_1)/2}$.

The above result is no longer valid for large l and in fact they found a universal lower bound for the rate of approach which applies to any initial data. More precisely, they showed that if $p \geq p_c$ and $0 \leq \tilde{u}_0(x) < u_0(x) \leq \varphi_\infty(|x|)$ then $\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty}$ decays more slowly in time than the rate $t^{-(N-m-\lambda_1)/2}$. We note that there exists a gap of the convergence rate between the rate $t^{-(\lambda_2-\lambda_1)/2}$ which is obtained for the case $l = m+\lambda_2$ and a universal lower bound of the rate $t^{-(N-m-\lambda_1)/2}$.

On the other hand, for the grow-up problem which can be regarded as a stability problem of singular stationary solution, a sharp universal upper bound of the grow-up rate was found by Mizoguchi [17], and optimal lower bound of the grow-up rate was found by Fila, Winkler and Yanagida [3]. The results on the grow-up problem strongly suggest that the above result of the convergence rate is not optimal.

The main purpose of this study is to obtain a sharp bound of the convergence rate in the case of $l > m + \lambda_2$ which leads to its optimal convergence rate. In fact, we improve the results in [4] as follows.

Theorem 1.1 **([14])** *Let $p > p_c$. Suppose that u_0 and $\tilde{u}_0$ satisfy* (H1) *and* (H2).

(i) *If $m + \lambda_1 < l < m + \lambda_2 + 2$, then there exists constant $C > 0$ such that*

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^\infty} \leq C(1+t)^{-(l-m-\lambda_1)/2}$$

for all $t > 0$.

(ii) *If $l \geq m + \lambda_2 + 2$, then for any small $\varepsilon > 0$ there exists constant $C > 0$ such that*

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^\infty} \leq C(1+t)^{-(\lambda_2-\lambda_1+2)/2+\varepsilon}$$

for all $t > 0$.

Our next theorem shows that if $|u_0 - \tilde{u}_0|$ decays faster in space, then we have a slightly better estimate than in Theorem 1.1(ii).

Theorem 1.2 **([14])** *Let $p > p_c$. Suppose that u_0 and $\tilde{u}_0$ satisfy* (H1) *and*

$$|u_0 - \tilde{u}_0| \leq c_1 \exp(-\nu|x|^2), \quad x \in \mathbb{R}^N$$

with some constants $c_1 > 0$ and $\nu > 0$. Then there exists constant $C > 0$ such that

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^\infty} \leq C(1+t)^{-(\lambda_2-\lambda_1+2)/2}$$

for all $t > 0$.

Remark 1.3 *Let $p > p_c$ and $m+\lambda_1 < l < N-2$. In Theorem1.2 of* [4], *it was shown that if u_0 and $\tilde{u}_0$ satisfy $\varphi_\alpha(|x|) \leq \tilde{u}_0 < u_0 \leq \varphi_\infty(|x|)$ and $u_0 - \tilde{u}_0 \geq c_2(1+|x|)^{-l}$*

with some constants $\alpha > 0$ and $c_2 > 0$, then $\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty}$ decays more slowly in time than the rate $t^{-(l-m-\lambda_1)/2}$. Since we can show that $m + \lambda_2 + 2 < N - 2$ by direct computation, we find that Theorem 1.1 yields a sharp estimate of the convergence rate in the case of $\tilde{u}_0 = \varphi_\alpha(|x|)$.

The next result shows that there exists a universal lower bound for the rate of approach which applies to any two initial data. This lower bound implies that the convergence rate of Theorem 1.1(i) can not be extended to the range $l > m + \lambda_2 + 2$.

Theorem 1.4 **([14])** *Let $p > p_c$. Suppose that u_0 and $\tilde{u}_0$ satisfy*

$$\varphi_\alpha(|x|) \le \tilde{u}_0(x) < u_0(x) \le \varphi_\infty(|x|), \quad x \in \mathbb{R}^N \setminus \{0\}$$

with some $\alpha > 0$. Then for any $\varepsilon > 0$ there exists constant $C > 0$ such that

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \ge C(1+t)^{-(\lambda_2-\lambda_1+2)/2-\varepsilon}$$

for all $t > 0$.

On the other hand, we investigate the behavior of solutions of the Cauchy problem with singular nonlinear absorption term

$$\begin{cases} u_t = \Delta u - u^{-q}, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $u = u(x,t)$, $q > 0$ and $u_0 > 0$ is a given continuous function on $\mathbb{R}^N$ that grows to infinity as $|x| \to \infty$. The problem similar to (1.2) which includes singular nonlinear term has been studied in many papers, since Kawarada studied the quenching problem [16].

We also study the problem (1.2) concerning the stability of stationary solutions and use the same notation as in the problem (1.1) here. Namely, we denote the positive radial stationary solution by $\varphi = \varphi_\alpha(r), r = |x|, \alpha > 0$, where $\varphi_\alpha(0) = \alpha$. We see that $\varphi_\alpha(r)$ satisfies the initial value problem

$$\begin{cases} \varphi_{\alpha,rr} + \frac{N-1}{r}\varphi_{\alpha,r} - \varphi_\alpha^{-q} = 0, \\ \varphi_\alpha(0) = \alpha, \quad \varphi_{\alpha,r}(0) = 0. \end{cases}$$

Then we can find similar structure for (1.2) as that in (1.1). For example, the solution φ_α is strictly increasing in $|x|$ for each $\alpha > 0$ and satisfies $\varphi_\alpha \to \infty$ as $|x| \to \infty$. The exponent

$$q_c = \begin{cases} \frac{(N-2)^2-4N+8\sqrt{N-1}}{(N-2)(N-10)} & \text{for } 3 \le N < 10, \\ \infty & \text{for } N \ge 10, \end{cases}$$

is an important exponent for the problem (1.2) which appeared already in the problem (1.1). It is known that for $q > q_c$, any pair of positive stationary solutions intersects each other. For $0 < q \le q_c$, Guo and Wei [9] showed that the family of stationary solutions forms a simply ordered set, that is, φ_α is strictly increasing in α for each x. We also call it the ordering property of $\{\varphi_\alpha\}$. Moreover, φ_α satisfies

$$\lim_{\alpha\to\infty} \varphi_\alpha(|x|) = \infty, \quad \lim_{\alpha\to 0} \varphi_\alpha(|x|) = \varphi_0(|x|),$$

for each x, where $\varphi_0(|x|)$ is a singular stationary solution given by

$$\varphi_\infty(|x|) = L_q|x|^{m_q}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

with

$$m_q = \frac{2}{q+1}, \quad L_q = \{m_q(N-2+m_q)\}^{1/(q+1)}.$$

It was also shown in [9] that each positive stationary solution has the expansion

$$\varphi_\alpha(|x|) = \begin{cases} L_q|x|^{m_q} + b_\alpha|x|^{m_q-\lambda_3} + \text{h.o.t.} & 0 < q < q_c, \\ L_q|x|^{m_q} + b_\alpha|x|^{m_q-\lambda_3}\log|x| + \text{h.o.t.} & q = q_c, \end{cases}$$

as $|x| \to \infty$, where λ_3 is a positive constant given by

$$\lambda_3 = \lambda_3(N,q) := \frac{N-2+2m_q - \sqrt{(N-2+2m_q)^2 - 8(N-2+m_q)}}{2},$$

and $b_\alpha = b(\alpha)$ is a positive constant that is monotone increasing in α. Note that λ_3 is a smaller root of the quadratic equation

$$h_q(\lambda) := \lambda^2 - (N-2+2m_q)\lambda + 2(N-2+m_q) = 0.$$

We denote by

$$\lambda_4 = \lambda_4(N,q) := \frac{N-2+2m_q + \sqrt{(N-2+2m_q)^2 - 8(N-2+m_q)}}{2},$$

a larger root of the quadratic equation.

In the previous papers, Guo and Wei [9, 10, 11] studied the problem (1.2). Concerning the stability problem, they showed that any regular positive radial stationary solution is unstable in any reasonable sense if $q > q_c$ and "weakly asymptotically stable" in a weighted L_q^∞ norm if $0 < q \leq q_c$ in [10]. Building on the results in [9, 10], for $0 < q < q_c$, they improved the above results that showed global attractive properties of the stationary solutions and the solutions approach a set of stationary solutions for a wide class of the initial data in [11]. As a by-product, they also showed the existence of global quenching solutions.

Our concern in this stage is to find similarity between the problem (1.1) and (1.2) as Theorem 1.1 and so on (cf.[4, 14]). Namely, we want to obtain a sharp bound of the convergence rate for (1.2) which leads to its optimal convergence rate. In fact, we have some similar results again. For example, we obtain Theorem 1.5 corresponding to Theorem 1.1 as follows.

Theorem 1.5 *Let $0 < q < q_c$. Let $u_0, \tilde{u}_0$ be two initial data and u and $\tilde{u}$ denote the corresponding solutions of* (1.2). *Suppose that u_0 and $\tilde{u}_0$ satisfy*

$$\text{(H3)} \qquad u_0,\ \tilde{u}_0 \geq \varphi_\alpha(r) \quad \text{for} \quad r > 0$$

and

$$\text{(H4)} \qquad |u_0 - \tilde{u}_0| \leq c_q(1+r)^{-l} \quad \text{for} \quad r > 0$$

with some $\alpha > 0$ and $c_q > 0$.

(i) *If $\lambda_3 - m < l < \lambda_4 - m + 2$, then there exists constant $C_q > 0$ such that*

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \leq C_q(1+t)^{-(l+m-\lambda_3)/2}$$

for all $t > 0$.

(ii) *If $l \geq \lambda_4 - m + 2$, then for any small $\varepsilon > 0$ there exists constant $C_q > 0$ such that*

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \leq C_q(1+t)^{-(\lambda_4-\lambda_3+2)/2+\varepsilon}$$

for all $t > 0$.

In this article, we mainly focus our attention on the problem (1.1) in the following, and omit the details for the problem (1.2) here.

Proofs of these theorems are obtained by a comparison technique that is based on matched asymptotics expansion. This expansion consists of two parts which are called the inner expansion and the outer expansion. The inner expansion is used to approximate the behavior of solutions near the origin and the outer expansion is used to approximate the behavior of solutions near the spatial infinity. The inner expansion is the same as in [4] and the key of our proof is a precise description of the outer expansion. In fact, we will find a solution which behaves in a self-similar way near the spatial infinity. Then we construct suitable super and subsolutions by matching these inner and outer solutions.

This paper is organized as follows. In section 2, we recall preliminary results in [3] and [4]. The formal analysis in this section will give the idea of constructing super and subsolutions, and a matching condition of these two expansions leads to the exact convergence rate. In section 3, we derive an upper bound of the convergence rate. In section 4, we derive a universal lower bound of the convergence rate.

2 Preliminary results on the linearized equation

In this section, we summarize previous results on the linear equations that are needed in subsequent sections. For proofs of the results, see [3, 4].

We consider radial solutions $U = U(r,t)$, $r = |x|$, of the linearized equation of (1.1) at φ_α. Namely, let P_α be the linearized operator defined by

$$P_\alpha U := U_{rr} + \frac{N-1}{r} U_r + p{\varphi_\alpha}^{p-1} U$$

and let $U(r,t)$ be a solution of

$$\begin{cases} U_t = P_\alpha U, \quad r > 0, \quad t > 0, \\ U_r(0,t) = 0, \quad t > 0, \\ U(r,0) = U_0(r), \quad r \geq 0, \end{cases} \tag{2.1}$$

where U_0 is a continuous function that decays to zero as $r \to \infty$. From the maximum principle, we see that $U(\cdot,t) > 0$ for all $t > 0$ if $U_0 \geq 0$ and $U_0 \not\equiv 0$. We will describe some fundamental properties for the solution of (2.1).

2.1 Comparison principle

Let u and $\tilde{u}$ be solutions of (1.1) with initial data u_0 and $\tilde{u}_0$ respectively. We recall some comparison results for $u - \tilde{u}$ and the solution U of (2.1), which comes from the ordering property and the convexity of nonlinearity.

Lemma 2.1 ([4]) *Let $p \geq p_c$. Suppose that u_0 and $\tilde{u}_0$ satisfy* (H1). *If*

$$|u_0(x) - \tilde{u}_0(x)| \leq U_0(|x|), \quad x \in \mathbb{R}^N,$$

then

$$|u(x,t) - \tilde{u}(x,t)| \leq U(|x|,t), \quad x \in \mathbb{R}^N$$

for all $t > 0$.

Lemma 2.2 ([4]) *Let $p \geq p_c$. Suppose that u_0 and $\tilde{u}_0$ satisfy*

$$\varphi_\alpha(|x|) \leq \tilde{u}_0(x) \leq u_0(x) \leq \varphi_\infty(|x|), \quad x \in \mathbb{R}^N \setminus \{0\}$$

with some $\alpha > 0$. If

$$0 \leq U_0(|x|) \leq u_0(x) - \tilde{u}_0(x), \quad x \in \mathbb{R}^N,$$

then

$$0 \leq U(|x|,t) \leq u(x,t) - \tilde{u}(x,t), \quad x \in \mathbb{R}^N$$

for all $t > 0$.

2.2 Formal matched asymptotics

By the above comparison results, we may only consider the convergence of radial solution of the linearized equation (2.1). In the following, we recall the asymptotic analysis, which is only formal but will be useful in the rigorous analysis in subsequent sections.

First, following Fila, Winkler and Yanagida [4], the formal expansion of a solution of (2.1) near the origin is given by

$$U(r,t) = \sigma(t)\psi(r,t) + \sigma_t(t)\Psi(r,t) + \text{h.o.t.}, \tag{2.2}$$

where, $\sigma(t) = U(0,t)$, ψ and Ψ satisfy

$$\begin{cases} P_\alpha \psi = 0, & r > 0, \\ \psi(0) = 1, & \psi_r(0) = 0 \end{cases} \tag{2.3}$$

and

$$\begin{cases} P_\alpha \Psi = \psi, & r > 0, \\ \Psi(0) = 0, & \Psi_r(0) = 0, \end{cases} \tag{2.4}$$

respectively (see also [4] and [7] for details). We recall some results in [4] on the above linear differential equations (2.3) and (2.4) in the following.

Lemma 2.3 ([4]) *For all $\alpha > 0$ and $r \geq 0$, $\alpha \mapsto \varphi_\alpha(r)$ is differentiable and*

$$\psi(r) := \frac{\partial}{\partial \alpha} \varphi_\alpha$$

satisfies (2.3). *Moreover, if $p \geq p_c$, then $\psi(r)$ is positive and satisfies*

$$\psi(r) = c_\alpha r^{-m-\lambda_1} + o(r^{-m-\lambda_1}) \quad \text{as} \quad r \to \infty,$$

where c_α is a constant given by $c_\alpha = \frac{a_1 \lambda_1}{m} \alpha^{-\frac{m+\lambda_1}{m}}$ and $a_1 = a(1)$ is a constant independent of α.

Remark 2.4 *The function ψ defined in Lemma 2.3 satisfies $\psi_r < 0$ for all $r > 0$. Indeed, we see from* (2.3) *that ψ does not attain a positive local minimum by the positivity of φ_α and ψ.*

Lemma 2.5 ([4]) *If $p \geq p_c$, then the solution Ψ of* (2.4) *has the following properties:*

(i) *$\Psi(r)/\psi(r)$ is strictly increasing in $r > 0$. In particular, Ψ is positive for all $r > 0$.*

(ii) *Ψ satisfies*

$$\Psi(r) = C_\alpha r^{-m-\lambda_1+2} + o(r^{-m-\lambda_1+2}) \quad \text{as} \quad r \to \infty,$$

where

$$C_\alpha = \frac{c_\alpha}{g(m + \lambda_1 - 2)}, \quad g(\mu) := h(\mu - m).$$

Next, let us consider the expansion of a solution of (2.1) near $r = \infty$. By the expansion of $\varphi_\alpha(r)$ near $r = \infty$, $U(r,t)$ satisfies approximately

$$U_t = U_{rr} + \frac{N-1}{r}U_r + \frac{pL^{p-1}}{r^2}U, \quad r \simeq \infty. \tag{2.5}$$

Following [5], we assume that U is of a self-similar form

$$U(r,t) = t^{-l/2}F(\eta), \quad \eta = t^{-1/2}r. \tag{2.6}$$

Substituting this in (2.5), we see that F satisfies

$$F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + \frac{\eta}{2}F_\eta + \frac{l}{2}F + \frac{pL^{p-1}}{\eta^2}F = 0. \tag{2.7}$$

In order that the outer expansion matches with the inner solution (2.2), $F(\eta)$ must satisfy

$$\lim_{\eta\to 0}\eta^{m+\lambda_1}F(\eta) = a_0$$

in view of the spatial order of Lemma 2.3, where a_0 is an arbitrary constant depending on initial data. Matching the inner expansion (2.2) and the outer expansion (2.6), and using Lemma 2.3, we obtain

$$\begin{aligned}\sigma &\simeq r^{m+\lambda_1}t^{-(l/2)}F(\eta)\\ &= r^{m+\lambda_1}t^{-(m+\lambda_1)/2}t^{(m+\lambda_1)/2}t^{-(l/2)}F(\eta)\\ &= t^{-(l-m-\lambda_1)/2}\eta^{m+\lambda_1}F(\eta)\\ &\simeq t^{-(l-m-\lambda_1)/2}.\end{aligned}$$

This gives the exact convergence rate given in Theorem 1.1 (i).

2.3 Properties of self-similar solutions

In this subsection, we recall the behavior of solutions of (2.7) satisfying

$$\lim_{\eta\to 0}\eta^{m+\lambda_1}F(\eta) = a_0 > 0,$$

where $a_0 > 0$ is an arbitrary constant. To this end, we set

$$f(\eta) = \eta^{m+\lambda_1}F(\eta).$$

Substituting this in (2.7), we see that f satisfies

$$\begin{cases} f_{\eta\eta} + \frac{N-1-2(m+\lambda_1)}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{l-m-\lambda_1}{2} f = 0, & \eta > 0, \\ f(0) = a_0 > 0, \quad f_\eta(0) = 0. \end{cases} \tag{2.8}$$

The following lemma characterizes the behavior of f as $\eta \to \infty$, and explains why $l = m + \lambda_2 + 2$ is critical.

Lemma 2.6 ([3]) *Let f be the solution of* (2.8).

(i) *If $l \in (m+\lambda_1, m+\lambda_2+2)$, then $f > 0$ and $f_\eta < 0$ for all $\eta > 0$. Moreover, for each $\eta_0 > 0$, there exist $d_-(\eta_0) > 0$ such that*

$$f(\eta) \geq d_-(\eta_0)\eta^{-(l-m-\lambda_1)} \quad \textit{for} \quad \eta \geq \eta_0,$$

and $d_+ > 0$ such that

$$f(\eta) \leq d_+\eta^{-(l-m-\lambda_1)} \quad \textit{for all} \quad \eta > 0.$$

(ii) *If $l = m + \lambda_2 + 2$, then $f(\eta)$ is given explicitly by $f(\eta) = a_0 e^{-\eta^2/4}$.*

(iii) *If $l > m + \lambda_2 + 2$, then $f(\eta)$ vanishes at some finite η.*

3 Upper bound

Throughout this and the following sections, we assume $p > p_c$. The aim of this section is to derive an upper bound of the convergence rate. In the case $m+\lambda_1 < l < m+\lambda_2+2$, we will show that any solution of (2.1) with $0 \leq U_0 \leq (1+r)^{-l}$ decays faster in time than the rate $t^{-(l-m-\lambda_1)/2}$. To this end, we construct a suitable supersolution U^+ of (2.1):

$$\begin{cases} U_t^+ - P_\alpha U^+ \geq 0, & r > 0, \quad t > 0, \\ U_r^+(0,t) = 0, & t > 0. \end{cases}$$

3.1 Outer supersolution

First, we give an outer supersolution as follows.

Lemma 3.1 ([14]) *If $m+\lambda_1 < l < m+\lambda_2+2$, then*

$$U_{\text{out}}(r,t) := (t+\tau)^{-\frac{l}{2}}F(\eta), \quad \eta = (t+\tau)^{-1/2}r,$$

is a supersolution of (2.1), *where τ is an arbitrary positive constant.*

Proof. Using (2.7), we have

$$\begin{aligned} U_{\text{out},t} - P_\alpha U_{\text{out}} &= -(t+\tau)^{-\frac{l}{2}-1}\left(\frac{l}{2}F + \frac{\eta}{2}F_\eta + F_{\eta\eta} + \frac{N-1}{\eta}F_\eta + p\varphi_\alpha^{p-1}(t+\tau)F\right) \\ &= p(t+\tau)^{-\frac{l}{2}}(\varphi_\infty^{p-1} - \varphi_\alpha^{p-1})\eta^{-(m+\lambda_1)}f. \end{aligned}$$

Then by the ordering property and the positivity of f from Lemma 2.6, we obtain

$$U_{\text{out},t} - P_\alpha U_{\text{out}} \geq 0$$

for all $r, t > 0$. □

3.2 Inner supersolution and matching

Next, we construct an inner supersolution $U_{\text{in}}(r,t)$ in the same way as [4].

Lemma 3.2 ([14]) *Let $l > m+\lambda_1$ and set*

$$U_{\text{in}}(r,t) := (t+\tau)^{-q}\psi(r) - q(t+\tau)^{-q-1}\Psi(r),$$

where $q = (l-m-\lambda_1)/2$. If $\tau > 0$ is sufficiently large, then there exist constants $B > 0$ and $c > 0$ such that the following inequalities hold :

(i) $U_{\text{in},t} \geq P_\alpha U_{\text{in}}$ *for all $r > 0$ and $t > 0$.*

(ii) $U_{\text{in}}(r,t) > 0$ *for all $t > 0$ and $r \in [0, B(t+\tau)^{\frac{1}{2}}]$.*

(iii) $U_{\text{in}}(r,t) > c\,U_{\text{out}}(r,t)$ *at $r = B(t+\tau)^{\frac{1}{2}}$ for all $t > 0$.*

Proposition 3.3 ([14]) *Suppose that* $m + \lambda_1 < l < m + \lambda_2 + 2$ *and*

$$0 < U_0(r) \le c_1(1+r)^{-l}, \quad r \ge 0$$

with some constant $c_1 > 0$. *Then there exists constant* $C > 0$ *such that the solution of* (2.1) *satisfies*

$$\|U(\cdot,t)\|_{L^\infty} \le C(1+t)^{-(l-m-\lambda_1)/2} \quad \textit{for all} \quad t > 0.$$

Proof. Let U_{in} and U_{out} be as given in Lemmas 3.2 and 3.1 respectively, and define

$$U^+(r,t) := \begin{cases} U_{\text{in}}(r,t) & \text{for } r < r^*(t), \\ cU_{\text{out}}(r,t) & \text{for } r \ge r^*(t), \end{cases}$$

where $c > 0$ is given in Lemma 3.2 and $r^*(t)$ is defined by

$$r^*(t) := \sup\{r > 0 \mid U_{\text{in}}(\rho,t) < cU_{\text{out}}(\rho,t) \text{ for } \rho \in [0,r)\}.$$

Then by the comparison principle, we obtain

$$0 < U(r,t) \le C_1 U^+(r,t), \quad r \ge 0, \quad t > 0$$

with some constant $C_1 > 0$ and we see that U^+ satisfies

$$\|U^+(r,t)\|_{L^\infty} \le C_2(1+t)^{-(l-m-\lambda_1)/2} \quad \text{for all } t > 0$$

with some constant $C_2 > 0$. The proof is now complete. □

Proposition 3.4 ([14]) *Suppose that*

$$0 < U_0(r) \le c_1 \exp(-\nu r^2), \quad r \ge 0$$

with some constants $c_1 > 0$ *and* $\nu > 0$. *Then there exists constant* $C > 0$ *such that the solution of* (2.1) *satisfies*

$$\|U(\cdot,t)\|_{L^\infty} \le C(1+t)^{-(\lambda_2-\lambda_1+2)/2} \quad \textit{for all} \quad t > 0.$$

Proof. The proof is similar to the procedure in the previous Proposition 3.3. □

Now, let us complete the proofs of Theorem 1.1 – 1.2.

Proof of Theorem 1.1 (i). Taking

$$U_0(r) = c_1(1+r)^{-l},$$

we have by assumption

$$|u_0(x) - \tilde{u}_0(x)| \leq U_0(|x|), \quad x \in \mathbb{R}^N.$$

By Lemma 2.1 and Proposition 3.3, this implies

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \leq \|U(r,t)\|_{L^\infty} \leq C(1+t)^{-(l-m-\lambda_1)/2}$$

for all $t > 0$ with some constant $C > 0$. □

Proof of Theorem 1.1 (ii). Given any small $\varepsilon > 0$, we set

$$\hat{l} := m + \lambda_2 + 2 - 2\varepsilon,$$

and define

$$\hat{U}_0(r) := c_1(1+r)^{-\hat{l}}.$$

We denote the solution of (2.1) with initial data $\hat{U}_0$ by $\hat{U}$. Then $\hat{U}_0 > U_0$ and it follows from the comparison principle that $\hat{U}(r,t) > U(r,t)$ for all $r, t > 0$. On the other hand by Theorem 1.1 (i), $\hat{U}(r,t)$ satisfies

$$\|U(r,t)\|_{L^\infty} \leq \|\hat{U}(r,t)\|_{L^\infty} \leq C(1+t)^{-(\lambda_2-\lambda_1+2)/2+\varepsilon}$$

for all $t > 0$ with some constant $C > 0$. □

Proof of Theorem 1.2. Taking

$$U_0(r) = c_1 \exp(-\nu|x|^2),$$

we have by assumption

$$|u_0(x) - \tilde{u}_0(x)| \leq U_0(|x|), \quad x \in \mathbb{R}^N.$$

By Lemma 2.1 and Proposition 3.4, this implies

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \leq \|U(r,t)\|_{L^\infty} \leq C(1+t)^{-(\lambda_2-\lambda_1+2)/2}$$

for all $t > 0$ with some constant $C > 0$. □

4 Universal lower bound

In this section, we prove that there exists a universal lower bound of the convergence rate which applies to any initial data from above or below to a stationary solution. Our key idea is to modify the outer solution. We construct a suitable subsolution U^- of (2.1):

$$\begin{cases} U_t^- - P_\alpha U^- \leq 0, & r > 0, \quad t > 0, \\ U_r^-(0,t) = 0, & t > 0. \end{cases}$$

4.1 Outer subsolution

In this subsection, we construct a suitable subsolution of (2.1) with a vanishing property; U_{out} is identically equal to 0 near $\eta = 0$ and $\eta = \infty$.

First, we recall the initial value problem (2.8):

$$\begin{cases} f_{\eta\eta} + \frac{n-1}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{\beta}{2} f = 0, & \eta > 0, \\ f(0) = a_0 > 0, \quad f_\eta(0) = 0, \end{cases}$$

where $n = N - 2(m + \lambda_1)$, $\beta = l - m - \lambda_1$, and throughout this section, l is fixed to $l = m + \lambda_2 + 2 + \varepsilon$, with an arbitrarily constant $\varepsilon > 0$. We note that f vanishes at some finite η_0 and $f > 0$ for $0 < \eta < \eta_0$ by Lemma 2.6.

Next, we modify this initial value problem as follows:

$$\begin{cases} \tilde{f}_{\eta\eta} + \frac{n-1}{\eta} \tilde{f}_\eta + \frac{\eta}{2} \tilde{f}_\eta + \frac{\tilde{\beta}}{2} \tilde{f} = 0, & \eta > 0, \\ \tilde{f}(\eta_0/2) = f(\eta_0/2), \quad \tilde{f}_\eta(\eta_0/2) = f_\eta(\eta_0/2), \end{cases} \tag{4.1}$$

where $\tilde{\beta} = l - m - \lambda_1 + \delta$ with any constant $\delta > 0$. Then, we see that the solution of (4.1) has a desired vanishing property as follows.

Lemma 4.1 ([14]) *There exist two vanishing points of $\tilde{f}$ (denoted by η_1 and η_2) such that $0 < \eta_1 < \eta_0/2 < \eta_2 < \eta_0$ and $0 < \tilde{f}(\eta) < f(\eta)$ for $\eta_1 < \eta < \eta_2$.*

Lemma 4.2 ([14]) *Let $\tilde{\varepsilon}$ be a positive constant satisfying $\tilde{\varepsilon} > \delta > 0$, and define*

$$U_{\mathrm{out}}(r,t) := \begin{cases} (t+\tau)^{-\frac{l+\tilde{\varepsilon}}{2}} \tilde{F}(\eta) & \eta_1 \leq \eta \leq \eta_2, \\ 0 & \textit{otherwise}, \end{cases}$$

with $\eta = (t+\tau)^{-1/2}r$, *where* $\tilde{F}(\eta) = \eta^{-(m+\lambda_1)}\tilde{f}(\eta)$. *If* $\tau > 0$ *is sufficiently large, then* U_{out} *is a subsolution of* (2.1).

Proof. It is trivial that $U \equiv 0$ is a subsolution of (2.1). Thus, we only need to verify the case of $\eta_1 \leq \eta \leq \eta_2$. We can check U_{out} becomes a subsolution for sufficient large $\tau > 0$ by straight forward computation. □

4.2 Inner subsolution and matching

We use the same inner subsolution as in [4].

Lemma 4.3 ([4]) *For any* $q > 0$,

$$U_{\mathrm{in}}(r,t) := (t+\tau)^{-q}\psi(r)$$

is a subsolution of (2.1) *for all* $t > 0$.

Since the subsolution as above decays too slowly as $r \to \infty$, we shall only use it in an inner region $0 \leq r \leq r^*(t)$ with suitable positive function $r^*(t)$.

In the outer region, we shall work with a different class of subsolutions defined in Lemma 4.2 instead of the subsolution defined in Lemma 4.3.

Proposition 4.4 ([14]) *Suppose* $U_0(r) > 0$ *for all* $r > 0$. *Then for any small* $\varepsilon > 0$, *there exists constant* $C > 0$ *such that the solution of* (2.1) *satisfies*

$$U(0,t) \geq C(1+t)^{-(\lambda_2-\lambda_1+2)/2-\varepsilon} \quad \textit{for all } t > 0.$$

Proof. The proof is similar to the procedure in the previous Proposition 3.3. See [14] for details. □

Proof of Theorem 1.4. We take

$$U_0(r) := \min_{|x|=r} |u_0(x) - \tilde{u}_0(x)| > 0, \quad r \geq 0.$$

Then by Lemma 2.2 and Proposition 4.4, we have

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \geq U(0,t) \geq C(1+t)^{-(\lambda_2-\lambda_1+2)/2-\varepsilon}$$

for all $t > 0$ with some constant $C > 0$. □

Remark 4.5 *We can relax the condition of initial data. In fact, this theorem holds for the case* $U_0 = 0$ *for sufficient large* $|x| > 0$.

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