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Interactive dynamics of two interfaces in a reaction diffusion system

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1 Introduction

We consider a bistable reaction-diffusion-advection system describing the growth of biological individuals which move by diffusion and chemotaxis [5]:

$$\begin{cases} u_t &= d_u\Delta u-\nabla(u\nabla\chi(w))+f(u) \\ w_t &= d_w\Delta w+u-\gamma w \end{cases} \tag{1}$$

where $f(u)=u(1-u)(u-a)$ $(0<a<1/2\ )$.
By the numerical simulations, there are many various patterns with static and dynamic properties [1], [5], [6]. Here, we mainly consider the dynamics of stripe and snaky patterns. In our model, the phase transition phenomena appears due to the bistable system and we call the boundary of two phases an interface. Therefore, it is enough to consider the dynamics of the movement of the interface understanding these pattern formations. Especially, we are concerned with the interactive dynamics of two interfaces far from equilibrium. To do so, we introduce the equation which describes the movement of the interface and discuss the interactive dynamics between two interfaces.

On the other hand, we already show the equation dominated the dynamics of the interfaces near the equilibrium and the stability of the planar standing pulse solution in the channel domain [5], [7]. In the case of two interfaces with same constant curvatures given in [6], [8], it is shown that the stability of the simple static snaky pattern and formally construction of the traveling solution with a triple junction.

2 Interactive dynamics of two interfaces

In this section, we formally introduce the equation described the dynamics of the interfaces with the interaction. To do so, we rewrite (1) as

$$\mathbf{u}_t = D\Delta\mathbf{u} - \mathbf{K}_2(\mathbf{u}) + F(\mathbf{u}) \tag{2}$$

where $\mathbf{K}_2(\mathbf{u}) = \mathrm{div}(u\nabla\chi, 0)^{\mathrm{T}}$ and $\mathbf{u} = (u, w)^{\mathrm{T}}$.

When $\Gamma(\sigma)$ is a curve corresponding to the interface with a parameter σ, the local coordinate is given by $(x, y) = \Gamma(\sigma) + \lambda\nu(\sigma), (\lambda, \sigma) = (\Lambda(x, y), \Sigma(x, y))$ with the normal vector $\nu(\sigma)$ at $\Gamma(\sigma)$. Then, it holds that

$$\nabla\chi(\Lambda, \Sigma) = (\Lambda_x\chi_\lambda + \Sigma_x\chi_\sigma, \Lambda_y\chi_\lambda + \Sigma_y\chi_\sigma)^{\mathrm{T}} = \chi_\lambda\nabla\Lambda + \chi_\sigma\nabla\Sigma,$$

$\nabla\Lambda = \nu,\ \nabla\Sigma = \frac{1}{1-\kappa\lambda}\Gamma_\sigma,\ |\nabla\Lambda|^2 = 1,\ |\Gamma_\sigma|^2 = 1,\ <\nabla\Lambda,\ \nabla\Sigma>= 0,\ \Delta\Lambda = -\frac{\kappa}{1-\kappa\lambda},$
$\Delta\Sigma = \frac{\kappa_\sigma\lambda}{(1-\kappa\lambda)^3}$
and

$$\begin{aligned} K_2(u,\chi) &= \mathrm{div}(u\nabla\chi) =< \nabla u, \nabla\chi > + u\Delta\chi \\ &=< u_\lambda\nabla\Lambda + u_\sigma\nabla\Sigma, \chi_\lambda\nabla\Lambda + \chi_\sigma\nabla\Sigma > + u\{\chi_{\lambda\lambda} - \tfrac{\kappa}{1-\kappa\lambda}\chi_\lambda + \tfrac{1}{1-\kappa\lambda}(\tfrac{1}{1-\kappa\lambda}\chi_\sigma)_\sigma\} \\ &= K_1(u,\chi) + \hat{K}(u,\chi) \end{aligned}$$

where $K_1(u,\chi) = u_\lambda\chi_\lambda + u\chi_{\lambda\lambda},\ \hat{K}(u,\chi) = \frac{1}{(1-\kappa\lambda)^2}u_\sigma\chi_\sigma - \frac{\kappa}{1-\kappa\lambda}u\chi_\lambda + \frac{u}{1-\kappa\lambda}(\frac{1}{1-\kappa\lambda}\chi_\sigma)_\sigma$.
Since $\mathbf{K}_2 = \mathbf{K}_1 + \hat{\mathbf{K}}$ with $\mathbf{K}_1 = (K_1(u,\chi), 0),\ \mathbf{K}_2 = (K_2(u,\chi), 0),\ \hat{\mathbf{K}} = (\hat{K}(u,\chi), 0)$, we have

$$\begin{aligned} \mathbf{u}_t &= D(\mathbf{u}_{\lambda\lambda} + \mathbf{K}\mathbf{u}) - \mathbf{K}_2(\mathbf{u}) + F(\mathbf{u}) \\ &= L_1(\mathbf{u}) + D\mathbf{K}\mathbf{u} - \mathbf{K}_1(\mathbf{u}) - \hat{\mathbf{K}}(\mathbf{u}) \end{aligned} \tag{3}$$

where $L_1(\mathbf{u}) = D\mathbf{u}_{\lambda\lambda} + F(\mathbf{u})$ and $\mathbf{K}\mathbf{u} = -\frac{\kappa}{1-\kappa\lambda}\mathbf{u}_\lambda + \frac{1}{1-\kappa\lambda}(\frac{1}{1-\kappa\lambda}\mathbf{u}_\sigma)_\sigma$.

The equation corresponding to (3) in $\mathbf{R}$ is given by

$$\mathbf{u}_t = D\mathbf{u}_{xx} - \hat{\mathbf{K}}_1(\mathbf{u}) + F(\mathbf{u}) \tag{4}$$

where $\hat{\mathbf{K}}_1(\mathbf{u}) = (u_x\chi_x + u\chi_{xx}, 0)^{\mathrm{T}}$.
Let $P_\pm$ be two roots of $F(\mathbf{u}) = 0$. Then, the boundary conditions of (3) are

$$\lim_{x\to-\infty}\mathbf{u}(t,x) = P_- = (0,0),\ \lim_{x\to\infty}\mathbf{u}(t,x) = P_+ = (p,q). \tag{5}$$

Assumption: Let $\lambda = x + ct$. There exists a traveling front solution of (4), (5) with the velocity c, which solution $S(\lambda) = (\Phi(\lambda), \Psi(\lambda))$ satisfies

$$\begin{cases} 0 = DS_{\lambda\lambda} - cS_\lambda - \mathbf{K}_1(S) + F(S),\ \lambda \in \mathbf{R} \\ S(\pm\infty) = P_\pm. \end{cases} \tag{6}$$

Remark 2.1 [4] *For suitable constants d_u and d_w of the diagonal elements of the matrix D, there exists a traveling front solution of (6).*

We treat two interface curves Γ_i $(i = 1,2)$ which have not any common point. Let $(x,y) = \Gamma_i(\sigma_i) + \lambda_i\nu_i(\sigma_i)$ be local coordinates in the neighborhood of each curve Γ_i. Then, we assume that the solution $\mathbf{u}(t,x,y)$ is expanded by

$$\mathbf{u}(t,x,y) = S(\Lambda_1(t,x,y)) + S(-\Lambda_2(t,x,y)) - P_+ + \mathbf{v}(t,x,y). \tag{7}$$

Assumption: Let ε be a small parameter. Then, it holds that for the curves Γ_i ($i = 1,2$)

curvatures $\hat{\kappa}_i$ of $\Gamma_i \sim O(\varepsilon)$,
$\mathbf{v} \sim \mathbf{v}(t,\lambda_i,\sigma_i)$ in the neighborhood of Γ_i with $\sigma_i = O(\varepsilon)$.

Let $\hat{\kappa}_i = \varepsilon\kappa_i$ and $\sigma_i = \varepsilon\ell_i$. Substituting (7) into (3), the left hand side is represented by

$$\mathbf{u}_t = \Lambda_{1t}S_\lambda(\Lambda_1) - \Lambda_{2t}S_\lambda(-\Lambda_2) + \mathbf{v}_t + \Lambda_{it}\mathbf{v}_{\lambda_i} + \varepsilon\Sigma_{it}\mathbf{v}_{\ell_i} \quad \text{in the n.b.h. of } \Gamma_i. \tag{8}$$

First, we consider the problem in the neighborhood of Γ_1. Then, $S(-\Lambda_2) - P_+ + \mathbf{v}$ becomes a remainder term in the neighborhood.

For simplicity, let $\lambda_1 \to \lambda$, $\ell_1 \to \ell$, $\kappa_1 \to \kappa$, $\Lambda_1 \to \Lambda$, $\Sigma_1 \to \Sigma$. Then, the right hand side is rewritten as

$$\begin{aligned} L_1(\mathbf{u}) + D\mathbf{K}\mathbf{u} - \mathbf{K}_1(\mathbf{u}) - \hat{\mathbf{K}}(\mathbf{u}) &= L_1(S) + \hat{L}(S(-\Lambda_2) - P_+ + \mathbf{v}) + D\mathbf{K}S \\ &\quad -\mathbf{K}_1(S) - \hat{\mathbf{K}}(S) + D\mathbf{K}(S(-\Lambda_2) - P_+) \\ &\quad +D\mathbf{K}\mathbf{v} - \mathbf{K}_1'(S)(S(-\Lambda_2) - P_+ + \mathbf{v}) \\ &\quad -\hat{\mathbf{K}}'(S)(S(-\Lambda_2) - P_+ + \mathbf{v}) \end{aligned} \tag{9}$$

where $\hat{L} = L_1'(S) = Dd^2/d\lambda^2 + F'(S)$.

Assumption: For the solution $S(\lambda)$ of (6), there is a positive constant α and vector $\mathbf{a}_+ = (a,b)^{\mathrm{T}}$ such that

$$S(\lambda) - P_+ \sim e^{-\alpha\lambda}\mathbf{a}_+ \text{ as } \lambda \to \infty. \tag{10}$$

Then, we remark that $D\mathbf{K}(S(-\Lambda_2) - P_+) = O(\varepsilon\kappa e^{\alpha\Lambda_2})$, $\Lambda_t\mathbf{v}_\lambda$ and $\varepsilon\Sigma_t\mathbf{v}_\ell$ are small with respect to small ε. Since $0 = L_1(S) - cS_\lambda - \mathbf{K}_1(S)$ and $D\mathbf{K}\mathbf{v} = O(\varepsilon^2 + \varepsilon\mathbf{v}_\lambda)$, it follows that

$$\begin{aligned} \mathbf{v}_t + \Lambda_t S_\lambda &\sim cS_\lambda + \hat{L}\mathbf{v} + \hat{L}(e^{\alpha\Lambda_2}\mathbf{a}_+) + D\mathbf{K}S - \hat{\mathbf{K}}(S) + D\mathbf{K}\mathbf{v} \\ &\quad -\mathbf{K}_1'(S)(e^{\alpha\Lambda_2}\mathbf{a}_+ + \mathbf{v}) - \hat{\mathbf{K}}'(S)(e^{\alpha\Lambda_2}\mathbf{a}_+ + \mathbf{v}) \\ &\sim cS_\lambda + \hat{L}\mathbf{v} + \hat{L}(e^{\alpha\Lambda_2}\mathbf{a}_+) + D\mathbf{K}S - \hat{\mathbf{K}}(S) \\ &\quad -\mathbf{K}_1'(S)(e^{\alpha\Lambda_2}\mathbf{a}_+ + \mathbf{v}) - \hat{\mathbf{K}}'(S)(e^{\alpha\Lambda_2}\mathbf{a}_+ + \mathbf{v}). \end{aligned}$$

By (6), (10), it holds that

$$0=\alpha^2De^{-\alpha\lambda}\mathbf{a}_+ + F'(P_+)e^{-\alpha\lambda}\mathbf{a}_+ + \alpha ce^{-\alpha\lambda}\mathbf{a}_+ - \mathbf{K}_1'(P_+)e^{-\alpha\lambda}\mathbf{a}_+ + O(e^{-2\alpha\lambda}). \quad (11)$$

Therefore, it follows from (11) that

$$\hat{L}(e^{-\alpha\lambda}\mathbf{a}_+) \sim (F'(S)-F'(P_+)-\alpha c+\mathbf{K}_1'(P_+))e^{-\alpha\lambda}\mathbf{a}_+ \quad (12)$$

by $\hat{L}(e^{-\alpha\lambda}\mathbf{a}_+)=(\alpha^2D+F'(S))e^{-\alpha\lambda}\mathbf{a}_+$.

Since $\chi_\sigma=\chi_{\sigma\sigma}=0$, we note that $\hat{\mathbf{K}}'(S)(e^{\alpha\Lambda_2}\mathbf{a}_+ + \mathbf{v}) \sim 0$, $\mathbf{K}S=-\frac{\varepsilon\kappa}{1-\varepsilon\kappa\lambda}S_\lambda \sim -\varepsilon\kappa S_\lambda$ and $\hat{\mathbf{K}}(S)\sim(-\varepsilon\kappa\Phi\chi_\lambda,0)$ with respect to small ε. By (9), (12), we have

$$\begin{aligned}\mathbf{v}_t+\Lambda_t S_\lambda \ \sim\ & (\hat{L}-\mathbf{K}_1'(S))\mathbf{v}-\varepsilon\kappa DS_\lambda+cS_\lambda+\begin{pmatrix}\varepsilon\kappa\Phi\chi(\Psi)_\lambda\\ 0\end{pmatrix}\\ & +(F'(S)-F'(P_+)+\mathbf{K}_1'(P_+)-\mathbf{K}_1'(S)-\alpha c)e^{\alpha\Lambda_2}\mathbf{a}_+\end{aligned}$$

where

$$\begin{aligned}\mathbf{K}_1(S)\ &=\begin{pmatrix}(\Phi\chi(\Psi)_\lambda)_\lambda\\ 0\end{pmatrix}=\begin{pmatrix}\Phi_\lambda\chi'(\Psi)\Psi_\lambda+\Phi\left\{\chi''(\Psi)\Psi_\lambda^2+\chi'(\Psi)\Psi_{\lambda\lambda}\right\}\\ 0\end{pmatrix}\\ &\sim\begin{pmatrix}\alpha^2\left\{\chi(q)ab+\chi''(q)pb^2\right\}e^{-2\alpha\lambda}+\alpha^2\chi'(q)pbe^{-\alpha\lambda}\\ 0\end{pmatrix}\\ &\sim\begin{pmatrix}\alpha^2\chi'(q)pb\\ 0\end{pmatrix}e^{-\alpha\lambda},\end{aligned}$$

$$\mathbf{K}_1'(P_+)\mathbf{a}_+=\begin{pmatrix}0 & \alpha^2p\chi'(q)\\ 0 & 0\end{pmatrix}\mathbf{a}_+,$$

$$\begin{aligned}\mathbf{K}_1'(S)e^{-\alpha\lambda}\mathbf{a}_+\ &=\begin{pmatrix}e^{-\alpha\lambda}a\chi(\Psi)_\lambda+\Phi(e^{-\alpha\lambda}\chi'(\Psi)b)_\lambda\\ 0\end{pmatrix}\\ &=e^{-\alpha\lambda}\begin{pmatrix}a\chi(\Psi)_\lambda+\Phi(\chi''(\Psi)\Psi_\lambda-\alpha\chi'(\Psi))b\\ 0\end{pmatrix}.\end{aligned}$$

Next, we will have the outward normal velocity V of the interface. Let $\varphi^*=(\varphi_1^*,\varphi_2^*)$ be an eigenfunction corresponding to 0 eigenvalue of the adjoint operator $(\hat{L}-\mathbf{K}_1'(S))^*$ of $\hat{L}-\mathbf{K}_1'(S)$ normalized by $< S_\lambda,\varphi^* >=1$. Then, it follows from the solvability condition (e. g. [3]) that

$$\begin{aligned}\Lambda_t=\ & -\varepsilon\kappa< DS_\lambda,\varphi^* >_{L^2}+\varepsilon\kappa< \Phi\chi(\Psi)_\lambda,\varphi_1^* >_{L^2}+c\\ & +\int_{-\infty}^{\infty}e^{\alpha\Lambda_2(x(\lambda),y(\lambda))}< (G(S(\lambda))-G(P_+)-\alpha c)\mathbf{a}_+,\varphi^*(\lambda) >\ d\lambda\end{aligned}$$

where $G(\Pi)=F'(\Pi)-\mathbf{K}_1'(\Pi)$ and $< \cdot,\cdot >_{L^2}$ means the $L^2(\mathbf{R})$ inner product. From $\Lambda_t=-V$, the velocity V_1 of Γ_1 is given by

$$V_1=\varepsilon\kappa_1(< DS_\lambda,\varphi^* >_{L^2}-< \Phi\chi(\Psi)_\lambda,\varphi_1^* >_{L^2})-c-g_1 \quad (13)$$

where

$$g_1 = \int_{-\infty}^{\infty} e^{\alpha\Lambda_2(x(\lambda),y(\lambda))} < (G(S(\lambda)) - G(P_+) - \alpha c)\mathbf{a}_+, \varphi^*(\lambda) > \; d\lambda.$$

On the other hand, let $\hat{S}(\lambda) = S(-\lambda)$, $\hat{\varphi}^*(\lambda) = \varphi^*(-\lambda)$ and so on. Then, the normal velocity $V_2 = \Lambda_{2t}$ of the interface Γ_2 is represented by

$$V_2 = -\varepsilon\kappa_2(< D\hat{S}_\lambda, \hat{\varphi}^* >_{L^2} - < \hat{\Phi}\chi(\hat{\Psi})_\lambda, \hat{\varphi}_1^* >_{L^2}) + c + g_2 \tag{14}$$

where

$$g_2 = \int_{-\infty}^{\infty} e^{-\alpha\Lambda_1(x(\lambda),y(\lambda))} < (G(\hat{S}(\lambda)) - G(P_+) - \alpha c)\mathbf{a}_+, \hat{\varphi}^*(\lambda) > \; d\lambda.$$

If the velocity c of the traveling front solution of (6) is of order ε, then the velocity of the interface depends on the curvature.

3 Application (1 dimensional problem)

Let $\ell_1(t)$, $\ell_2(t)$ ($\ell_1(t) < \ell_2(t)$) be interface positions in the line. It follows from (13), (14) that

$$\ell_{1t}(t) = V_1 = -c - e^{\ell_1(t)-\ell_2(t)}H$$

$$\ell_{2t}(t) = V_2 = c + e^{\ell_2(t)-\ell_1(t)}H$$

where $H = \int_{-\infty}^{\infty} e^{\alpha\lambda} < (G(S(\lambda)) - G(P_+) - \alpha c)\mathbf{a}_+, \varphi^*(\lambda) > \; d\lambda$.
As $|\ell_1(t) - \ell_2(t)| \gg 1$, these equations imply that

$$\begin{cases} c < 0 \Longrightarrow \text{two interfaces are attractive} \\ c > 0 \Longrightarrow \text{two interfaces are repulsive.} \end{cases}$$

As $c = 0$, it holds that

$$\begin{cases} H^* < 0 \Longrightarrow \text{two interfaces are attractive} \\ H^* > 0 \Longrightarrow \text{two interfaces are repulsive} \end{cases}$$

where $H^* = \int_{-\infty}^{\infty} e^{\alpha\lambda} < (G(S(\lambda)) - G(P_+))\mathbf{a}_+, \varphi^*(\lambda) > \; d\lambda$.

Remark 3.1 [4] *For suitable constants d_u, d_w and $\chi(w)$, there exists a 1-dim. standing front solution of (6), that is, the velocity is zero.*

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