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Existence of Solutions with Moving Singularities for a Semilinear Parabolic Equation

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Abstract

We study the Cauchy problem for a semilinear parabolic equation with a power nonlinearity. It is known that in some parameter range, the equation has a singular steady state. Our concern is a solution with a moving singularity that is obtained by perturbing the singular steady state. By the formal expansion, it turns out that the correction term must satisfy the heat equation with inverse-square potential near the singular point. From the well-posedness of this equation, we see that there appears a critical exponent. Paying attention to this exponent, given a motion of the singular point and suitable initial data, we establish the time-local existence result.

1 Introduction

We study singular solutions of the semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + u^p & \text{in} \quad \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $p > 1$ is a parameter and $u_0 \in L^1_{loc}(\mathbb{R}^N)$ is a nonnegative function. It is known that for

$$N \geq 3, \qquad p > p_{sing} := \frac{N}{N-2},$$

(1.1) has an explicit singular steady state $\varphi(|x|) \in C^\infty(\mathbb{R}^N \setminus \{0\})$ with a singular point 0;

$$\varphi(|x|) = L|x|^{-m}, \quad m = \frac{2}{p-1}, \ L^{p-1} = m(N-m-2).$$

Then $\varphi(|x|)$ satisfies (1.1) in the distribution sense, and

$$\varphi_{rr} + \frac{N-1}{r}\varphi_r + \varphi^p = 0, \qquad r = |x| > 0. \tag{1.2}$$

Clearly, the spatial singularity of $u = \varphi(|x|)$ persists for all $t > 0$, but the singular point does not move in time.

Our aim of this paper is to discuss the existence of a solution of (1.1) whose spatial singularity moves in time. More precisely, we define a solution with a moving singularity as follows.

Definition 1. The function $u(x,t)$ is said to be a solution of (1.1) with a moving singularity $\xi(t) \in \mathbb{R}^N$ for $t \in (0,T)$, where $0 < T \le \infty$, if the following conditions hold:

(i) u, $u^p \in C([0,T); L^1_{loc}(\mathbb{R}^N))$ satisfy (1.1) in the distribution sense.

(ii) $u(x,t)$ is defined on $\{(x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t \in (0,T)\}$, and is twice continuously differentiable with respect to x and continuously differentiable with respect to t.

(iii) $u(x,t) \to \infty$ as $x \to \xi(t)$ for every $t \in [0,T)$.

In this paper, we study the time-local existence for a solution with a moving singularity of the Cauchy problem (1.1). In order to state our result, we first introduce a critical exponent given by

$$p_* := \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}},$$

which appeared in the papers of Véron [8] and Chen-Lin [3]. It was shown in [8] that p_* is related to the linearized stability of the singular steady state, while it was shown in [3] that p_* plays a crucial role for the existence of solutions with a prescribed singular set of the Dirichlet problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$. In fact, in [3], they proved that if $N \ge 3$, $p_{sing} < p < p_*$, then for any closed set $K \subset \Omega$, there exists a singular solution having K as a singular set. We note that p_* is larger than p_{sing} and is smaller than the Sobolev critical exponent $p_S := (N+2)/(N-2)$. We also introduce the important numbers

$$\lambda_1 := \frac{N - 2 - \sqrt{(N-2)^2 - 4pL^{p-1}}}{2},$$

$$\lambda_2 := \frac{N - 2 + \sqrt{(N-2)^2 - 4pL^{p-1}}}{2}.$$

We note that for $N \geq 3$, $p_{sing} < p < p_*$, the constants $\lambda_1 < \lambda_2$ are positive roots of

$$\lambda^2 - (N-2)\lambda + pL^{p-1} = 0.$$

Finally, for $a \in \mathbb{R}$, $[a]$ denotes the largest integer not greater than a.

Our result is concerning the time-local existence of a solution of (1.1) with a moving singularity.

Theorem 1. *Let $N \geq 3$ and $p_{sing} < p < p_*$. Assume the following conditions:*

(A1) $\xi(t) \in C^{i+\alpha}([0,\infty);\mathbb{R}^N)$ $(\alpha > 0)$ *with* $i = [\frac{|m-\lambda_2|+1}{2}] + 1$.

(A2) u_0 *is nonnegative and continuous in* $x \in \mathbb{R}^N \setminus \xi(0)$, *and is uniformly bounded for* $|x - \xi(0)| \geq 1$.

(A3) *If $m - \lambda_2$ is not an integer, then*

$$u_0(x) = L|x-\xi(0)|^{-m}\Big\{1 + \sum_{i=1}^{[m-\lambda_2]} b_i\Big(\frac{x-\xi(0)}{|x-\xi(0)|}, 0\Big)|x-\xi(0)|^i + O(|x-\xi(0)|^{m-\lambda_2+\varepsilon})\Big\}$$

as $x \to \xi(0)$ for some $\varepsilon > 0$, where $b_i(\omega,t)$ are functions on S^{N-1} defined later by (2.3)-(2.5). If $m - \lambda_2$ is an integer, then

$$u_0(x) = L|x-\xi(0)|^{-m}\Big\{1 + \sum_{i=1}^{m-\lambda_2} b_i\Big(\frac{x-\xi(0)}{|x-\xi(0)|}, 0\Big)|x-\xi(0)|^i + c(0)|x-\xi(0)|^{m-\lambda_2}\log|x-\xi(0)| + O(|x-\xi(0)|^{m-\lambda_2+\varepsilon})\Big\}$$

as $x \to \xi(0)$ for some $\varepsilon > 0$, where $b_i(\omega,t)$ are functions on S^{N-1} defined later by (2.3)-(2.5) and $b_{m-\lambda_2}(\omega,t)$ and $c(t)$ satisfy (3.1)

Then for some $T > 0$, there exists a solution of (1.1) *with a moving singularity $\xi(t)$.*

Remark 1. *If $N \geq 3$ and*

$$p_{sing} < p < \min\Big\{p_*, \frac{3N+5}{3N-3}\Big\},$$

then $0 \leq m - \lambda_2 < 1$ so that $[m-\lambda_2] = 0$. In this case, (A1) *implies $\xi(t) \in C^{1+\alpha}([0,\infty);\mathbb{R}^N)$ $(\alpha > 0)$, and* (A3) *is simplified as*

$$u_0(x) = L|x-\xi(0)|^{-m} + O(|x-\xi(0)|^{-\lambda_2+\varepsilon}) \quad \text{as } x \to \xi(0). \tag{1.3}$$

In this paper, we consider only the time-local existence of the Cauchy problem with a moving singularity. Needless to say, the existence of time-global solutions are important questions. Also, when the solution with a moving singularity is not time-global, it is interesting to ask what happens at the maximal existence time. These questions will be future works.

This paper is organized as follows: In Section 2 we carry out formal analysis for a solution of (1.1) as a perturbation of the singular steady state. In Section 3 we state the outline of proof of the time-local existence.

2 Formal expansion at a singular point

In this section, we consider the formal expansion of a solution $u(x,t)$ of (1.1) with a moving singularity $\xi(t)$. Assuming that the solution resembles the singular steady state around $\xi(t)$, we may naturally expand $u(x,t)$ as

$$u(x,t) = Lr^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega,t)r^i + v(y,t)r^m\Big\}, \tag{2.1}$$

where

$$y = x - \xi(t)\ ;\ r = |x - \xi(t)|\ ;\ \omega = \frac{1}{r}(x-\xi) \in S^{N-1}\ ,\ k = [m],$$

and the remainder term v satisfies

$$v(y,t) = o(|y|^{-m}) \quad \text{as} \quad |y| \to 0. \tag{2.2}$$

Substituting (2.1) into (1.1), and using

$$r_t = -\frac{(x-\xi)\cdot\xi_t}{r}, \qquad \omega_t = -\frac{1}{r}\xi_t + \frac{\omega\cdot\xi_t}{r}\omega,$$

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}$$

and the Taylor expansion, we compare the coefficients of r^{-m+i-2} for $i = 0, 1, \ldots, k$. Then we obtain

$$r^{-m-2};\ (Lr^{-m})_{rr} + \frac{N-1}{r}(Lr^{-m})_r + (Lr^{-m})^p = 0,$$

$$\begin{aligned} r^{-m-1};\ \Delta_{S^{N-1}} b_1 + \{(-m+1)(N-m-1) + pm(N-m-2)\}b_1 \\ = m\omega\cdot\xi_t, \end{aligned} \tag{2.3}$$

$$r^{-m};\ \Delta_{S^{N-1}}b_2+\{(-m+2)(N-m)+pm(N-m-2)\}b_2$$
$$=(m-1)b_1\omega\cdot\xi_t-(\xi_t-(\omega\cdot\xi_t)\omega)\cdot\nabla b_1+\frac{p(p-1)}{2}m(N-m-2)b_1^2, \quad (2.4)$$

$$r^{-m+i-2};\ \Delta_{S^{N-1}}b_i+\{(-m+i)(N-m+i-2)+pm(N-m-2)\}b_i$$
$$=G_i(\omega;b_1,b_2,\ldots,b_{i-1},\xi) \qquad (i=3,4,\ldots,k). \quad (2.5)$$

where $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1} and the function $G_i(\omega;b_1,b_2,\ldots,b_{i-1},\xi)$ on $S^{N-1}\times[0,\infty)$ is determined by $(b_1,b_2,\ldots,b_{i-1},\xi)$.

The equality for r^{-m-2} always holds by (1.2). From other equations, we have the above system of inhomogeneous elliptic equations for b_i on S^{N-1}: By these equations, $b_1,b_2,\ldots$ are determined sequentially.

Let us consider the solvability of (2.3), (2.4) and (2.5). It is well known (see, e.g. [2]) that for every $j=0,1,2,\ldots$, the eigenvalues of $-\Delta_{S^{N-1}}$ are given by

$$\mu_j=j(N+j-2), \quad j=0,1,2,\ldots.$$

and the eigenspace E_j associated with μ_j is given by

$$E_j=\{\,f|_{S^{N-1}}:\ f \text{ is a harmonic homogeneous polynomial of degree } j\,\}.$$

Therefore, unless

$$(-m+i)(N-m+i-2)+pm(N-m-2)=j(N+j-2), \qquad (2.6)$$

the operators in the left-hand side of (2.3), (2.4) and (2.5) are invertible. We define a set Λ by

$$\Lambda:=\left\{p>1:\ (2.6)\text{ holds for some } i\in\{1,2,\ldots,[\frac{2}{p-1}]\},\ j\in\{0,1,2,\ldots,i\}\right\}.$$

Moreover, we consider $G_i(\omega;b_1,b_2,\ldots,b_{i-1},\xi)$ in detail and obtain next lemma.

Lemma 1. *Suppose that $\xi(t)$ satisfies* (A1). *If $p\notin\Lambda$, then there exist $b_1(\omega,t),b_2(\omega,t),\ldots,b_k(\omega,t)\in C^{\infty,1}(S^{N-1}\times[0,\infty))$ such that* (2.3), (2.4) *and* (2.5) *hold.*

By this lemma, in order to consider the existence of the solution of (1.1) with a moving singularity, it suffices to consider $v(y,t)$. By taking $b_i(\omega,t)$ as Lemma 1, (1.1) is satisfied if $v(y,t)$ satisfies

$$v_t=\Delta v+\xi_t\cdot\nabla v+F(v,y,t) \quad \text{in } \mathbb{R}^N\times(0,\infty). \qquad (2.7)$$

where $F(v,y,t)$ is determined by $b_1, b_2, \dots, b_k$ and ξ. After tedious computations, we notice that

$$F(v,y,t) = \frac{pL^{p-1}}{r^2} v + o(r^{-2}) \quad \text{as } r \to 0.$$

In order to consider the existence of solutions of (2.7), we first consider

$$v_t = \Delta v + \frac{pL^{p-1}}{r^2} v \ \text{ in } \ \mathbb{R}^N \times (0,\infty). \tag{2.8}$$

This equation has been investigated in [1, 7, 6], and it was shown that (2.8) is well-posed when

$$0 < pL^{p-1} < \frac{(N-2)^2}{4}, \tag{2.9}$$

and

$$|v(y,0)| \le Cr^{-\lambda} \ \text{ for some } \lambda_1 < \lambda < \lambda_2, \ C > 0.$$

The inequalities (2.9) hold if and only if p satisfies

$$p_{sing} < p < p_* \ \text{ for } \ N \ge 3, \ \text{or} \ \ p > p_{JL} := \frac{N - 2\sqrt{N-1}}{N-4-2\sqrt{N-1}} \ \text{ for } \ N > 10.$$

Here the exponent p_{JL} was first introduced by Joseph-Lundgren [4] and is known to play an important role for the dynamics of solutions of (1.1).

Since the gradient term in (2.7) and the higher order term of F do not affect the well-posedness, we must assume (2.9) for the solvability of (2.7). If $p > p_{JL}$, then $\lambda_1 < m$ does not hold so that (2.2) may not be true. Hence we exclude the case $p_{JL} < p$. Based on the above formal analysis, we will focus on the case $p_{sing} < p < p_*$.

3 Time-local existence

Taking into account of the formal analysis in the previous section, we will show the existence of a time-local solution with a moving singularity. To this end, we develop the idea of Marchi [6] for the well-posedness of the linear equation (2.8).

The outline of the proof is divided into three steps. Roughly speaking, we construct a suitable supersolution and subsolution with a moving singularity in Subsection 3.1. In Subsection 3.2, we construct a sequence of approximate solutions and find a convergent subsequence. In Subsection 3.3, we show that the limiting function is indeed a solution of (1.1) with a moving singularity.

3.1 Construction of a supersolution and a subsolution

In this subsection, we construct a supersolution and a subsolution of (1.1) that are suitable for our purpose.

First we note that if $m - \lambda_2$ is not an integer, then (2.6) does not hold for all $i = 1, 2, \dots, [m - \lambda_2]$, $j = 0, 1, \dots, i$. Indeed, if (2.6) does not hold for some $1 \le i \le m - \lambda_2, j = 1, \dots, i$, then $i = -\lambda_2, j = 0$, contradicting that $m - \lambda_2$ is not an integer. Therefore, if $m - \lambda_2$ is not an integer, then by Lemma 1 and (A1), we can determine $b_1(\omega, t)$, $b_2(\omega, t), \dots$, $b_{[m-\lambda_2]}(\omega, t) \in C^{2,1}(S^{N-1} \times [0, \infty))$ by (2.3), (2.4) and (2.5).

On the other hand, if $m - \lambda_2$ is an integer, (2.6) holds for $i = m - \lambda_2$, $j = 0$. However, we carry out similar argument by replacing $b_{[m-\lambda_2]}(\omega, t) r^{[m-\lambda_2]}$ with $(b_{m-\lambda_2}(\omega, t) + c(t) \log r) r^{m-\lambda_2}$ that satisfies

$$\Delta_{S^{N-1}} b_{m-\lambda_2} = (I - P_0) G(\omega, t), \quad c(t) = (N - 2\lambda_2 - 2)^{-1} P_0 G(\omega, t), \quad (3.1)$$

where P_0 is define the projection on E_0 and $G(\omega, t)$ is the right-hand side of (2.5) with $i = m - \lambda_2$.

Now we fix $\lambda = \lambda_2 - \epsilon$ satisfying

$$\min\{\lambda_1, m - [m - \lambda_2] - 1\} < \lambda < \lambda_2$$

and replace k defined in Section 2 with $k := [m - \lambda_2]$. From (A2) and (A3), it follows that $u_0 \in C(\mathbb{R}^N \setminus \xi(0)) \bigcap L^\infty(\mathbb{R}^N \setminus B(\xi(0), 1))$, $u_0 \ge 0$, and

$$u_0(x) = L|x - \xi(0)|^{-m} \Big\{ 1 + \sum_{i=1}^{k} b_i \Big(\frac{x - \xi(0)}{|x - \xi(0)|}, 0 \Big) |x - \xi(0)|^i + O(|x - \xi(0)|^{m-\lambda}) \Big\} \quad \text{as} \quad x \to \xi(0).$$

Then there exist constants $C > 0$ and $R > 0$ such that

$$\Big| u_0(x) - L|x - \xi(0)|^{-m} \Big\{ 1 + \sum_{i=1}^{k} b_i(\omega, 0) \Big(\frac{x - \xi(0)}{|x - \xi(0)|} \Big) |x - \xi(0)|^i \Big\} \Big| < CL|x - \xi(0)|^{-\lambda} \text{ in } B(\xi(0), R).$$

Fix any $T_1 > 0$.

First we construct a supersolution and a subsolution of (1.1) in a neighborhood of $\xi(t)$ by using (2.7). By (2.1), we have

$$u_t - \Delta u - u^p = L\{v_t - \Delta v - \xi_t \cdot \nabla v - F(v, y, t)\}.$$

Hence

$$\overline{u}(x,t) = Lr^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega,t)r^i + v^+(y,t)r^m\Big\}$$

is a supersolution of (1.1) if and only if v^+ is a supersolution of (2.7). Since it follows from tedious calculation that $\overline{v} := Cr^{-\lambda}$ is a supersolution of (2.7) on $B_R \times (0,T_1)$ if $R > 0$ is sufficiently small,

$$\overline{u} := L|x-\xi(t)|^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega,t)|x-\xi(t)|^i + C|x-\xi(t)|^{m-\lambda}\Big\}$$

is a supersolution of (1.1) on $\bigcup_{0\le t\le T_1} B_R(\xi(t)) \times \{t\}$ for small $R > 0$. Similarly, we can show that

$$\underline{u} := L|x-\xi(t)|^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega,t)|x-\xi(t)|^i - C|x-\xi(t)|^{m-\lambda}\Big\}$$

is a subsolution of (1.1) on $\bigcup_{0\le t\le T_1} B_R(\xi(t)) \times \{t\}$ for small $R > 0$.

Next, we construct a supersolution and a subsolution near infinity. By direct calculation, it is shown that

$$\overline{u} := C_1\Big(1 - \frac{t}{2T_2}\Big)^{-\frac{1}{2(p-1)}}$$

is a supersolution of (1.1) on $\mathbb{R}^N \setminus B(\xi(t),1) \times (0,T_2)$, provided that

$$C_1 > \|u_0\|_{L^\infty(\mathbb{R}^N\setminus B(\xi(0),1))}, \qquad T_2 < 2\sqrt{2}(p-1)C_1^{p-1}.$$

Clearly $u \equiv 0$ is a subsolution (1.1).

Finally, connecting these supersolutions and subsolutions in the intermediate region, we obtain a supersolution $\overline{u}$ and a subsolution $\underline{u}$ such that $\overline{u}$, $\overline{u}^p$, $\underline{u}$, $\underline{u}^p \in L^1_{loc}(\mathbb{R}^N \times [0,T])$ and the following properties hold:

(i) $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are defined on $\{(x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\},\ t \in [0,T]\}$ and are twice continuously differentiable with respect to x and continuously differentiable with respect to t.

(ii) For every $t \in [0,T]$, $\overline{u}(x,t)$, $\underline{u}(x,t) \to \infty$ as $x \to \xi(t)$. In particular,

$$\overline{u}(x,t) = L|x-\xi(t)|^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega,t)|x-\xi(t)|^i + C|x-\xi(t)|^{m-\lambda}\Big\},$$

$$\underline{u}(x,t) = L|x-\xi(t)|^{-m}\Big\{1 + \sum_{i=1}^{k} b_i(\omega,t)|x-\xi(t)|^i - C|x-\xi(t)|^{m-\lambda}\Big\}$$

for $|x-\xi(t)| \le R_0$ and $0 \le t \le T$.

(iii) The inequalities

$$\overline{u}(x,0) > u_0(x) > \underline{u}(x,0) \quad \text{in} \quad \mathbb{R}^N \setminus \{\xi(0)\},$$
$$\overline{u}(x,t) > \underline{u}(x,t) \quad \text{in} \quad \mathbb{R}^N \times [0,T] \setminus \bigcup_{0 \le t \le T} (\xi(t),t)$$

hold.

(iv) The inequalities

$$\overline{u}_t \ge \Delta \overline{u} + \overline{u}^p \quad \text{in } \mathbb{R}^N \times [0,T] \setminus \bigcup_{0 \le t \le T} (\xi(t),t),$$

$$\underline{u}_t \le \Delta \underline{u} + \underline{u}^p \quad \text{in } \mathbb{R}^N \times [0,T] \setminus \bigcup_{0 \le t \le T} (\xi(t),t)$$

hold.

for some small R_0 and T.

3.2 Construction of approximate solutions

In this subsection, by using the supersolution and subsolution given in the previous subsection, we construct a series of approximate solutions that is convergent in an appropriate function space.

Define a sequence of bounded domains

$$A_n(t) := \{x \in \mathbb{R}^N : \ |x - \xi(t)| \le n, \ |x - \xi(t)| \ge \frac{1}{n} \} \qquad (n = 1, 2, \ldots).$$

For each n, let $u_n(x,t)$ be a classical solution of

$$\begin{cases} u_{n,t} = \Delta u_n + u_n^p & \text{in} \quad \bigcup_{0 \le t \le T} A_n(t) \times \{t\}, \\ u_n = \underline{u} & \text{on} \quad \bigcup_{0 \le t \le T} \partial A_n(t) \times \{t\}, \\ u_n(x,0) = u_{0,n}(x) & \text{in} \quad A_n(0), \end{cases}$$

where the initial value is assumed to satisfy

$$\underline{u}(x,0) \le u_{0,n}(x) \le u_{0,n+1}(x) \le \overline{u}(x,0) \quad \text{in} \quad A_n(0),$$
$$u_{0,n}(x) = \underline{u}(x,0) \quad \text{on} \quad \partial A_n(0), u_{0,n} \nearrow u_0 \quad \text{as} \quad n \to \infty.$$

It is easily seen that $\underline{u} \le u_n \le \overline{u}$ in $\bigcup_{0\le t\le T} A_n(t) \times \{t\}$ by the comparison principle. Furthermore, by the standard parabolic theory [5] and the Ascoli-Arzelà theorem, from $\{u_n\}$, we can obtain a subsequence $\{u_{n(j)}\}_j$ and some function $u(x,t)$ such that

$$u_{n(j)} \to u \text{ locally uniformly in } R^N \times (0,T) \setminus \bigcup_{0<t<T} (\xi(t),t) \text{ as } n(j) \to \infty$$

Hence the limiting function $u(x,t)$ satisfies

$$u \in C(R^N \times (0,T) \setminus \textstyle\bigcup_{0<t<T}(\xi(t),t)),$$
$$\underline{u} \le u \le \overline{u} \text{ in } \mathbb{R}^N \times (0,T) \setminus \textstyle\bigcup_{0<t<T}(\xi(t),t).$$

3.3 Completion of the proof

In this subsection, we show that the limiting function $u(x,t)$ obtained in Subsection 3.2 is indeed a solution of (1.1) with a moving singularity $\xi(t)$ for $t \in (0,T)$.

First, by $\underline{u} \le u \le \overline{u}$ and the Lebesgue convergence theorem, we can show that the function u satisfies (1.1) in the distribution sense. Next, by $\underline{u} \le u \le \overline{u}$ and the standard parabolic theory [5], the function u has the desired properties as stated in Definition 1. Consequently, it is shown that the function u is a solution of (1.1) with a moving singularity $\xi(t)$ for $t \in (0,T)$. ∎

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