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Recent Topics from Competitive Game Theory

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Some of the recent works by the present author are given.
A lot of interesting open problems are mentioned

1A. Two-player One-sided Games of Deception. Two numbers x_1 and x_2 are chosen from $[0,1]$ by means of independent bivariate uniform distribution on $[0,1]^2$. Player I now looks at the numbers privately and chooses one of the two and opens it to Player II, and the other number is covered. Player II then accepts either one of the opened number or the covered number, and receives from player I the number he accepted. Player I (II) aims to minimize (maximize) the expected payoff to II.

In Baston and Bostock (Ref.[1]) it is proven that the strategies;

σ^* : Choose the nearest number to $\frac{1}{2}$ among x_1 and x_2, and open it. The other number is covered.

for I and

τ^* : Accept the opened (covered) number if it is $>$ $(<)\frac{1}{2}$,

for II, constitute an optimal strategy-pair, and the value of the game is 7/12.

By Sakaguchi (Ref.[5]) it is proven that, if x_1 and x_2 are independent bivariate standard normal distribution in $(-\infty,\infty)^2$, then the above strategy-pair σ^* and τ^* with $\frac{1}{2}$ replaced by 0, is optimal, and the value of the game is $\frac{2-\sqrt{2}}{\sqrt{2\pi}} \approx 0.2337$.

1B. Two-player Two-sided Games of Deception. Let X_1, X_2, Y_1, Y_2 are *i.i.d.* r.v.s with an identical p.d.f. Player I observes (X_1, X_2) and chooses his decision number $\theta_1 \in [0,1]$. Player II observes (Y_1, Y_2) and chooses his decision number $\theta_2 \in [0,1]$. Each player's choice of his decision number is made independently of the opponent's choice.

Player I chooses the nearest number to θ_1 among x_1 and x_2 and open it and the other number is covered. Player II chooses the nearest number to θ_2 among y_1 and y_2 and opens it and the other number is covered. If II's opened number is $>$ $(<)\theta_1$, then I gets II's opened (covered) number. If I's opened number is $>$ $(<)\theta_2$, then II gets I's opened (covered) number. For the sake of symmetry it should be $\theta_1 = \theta_2 (= \theta$, say). Both players want to choose the optimal θ which maximizes the common expected payoff, they can get.

The result is:

pdf of X	Opt. choice of θ	Common opt reward
1 in $[0,1]$	$1/2$	$7/12$
$2x$ in $[0,1]$	$2/3$	$296/405 \doteqdot 0.7309$
e^{-x} in $[0,\infty)$	1	$\frac{1}{2}+2e^{-1} \doteqdot 1.2358$
xe^{-x} in $(0,\infty)$	2	$2(1+4e^{-2}) \doteqdot 3.0827$
$\varphi(x) \equiv \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ in $(-\infty,\infty)$	0	$(2-\sqrt{2})/\sqrt{2\pi} \doteqdot 0.2337$

Now let (X_1, X_2) be a bivariate r.v. obeying a perfectly symmetric distribution. It is shown that for (1) bivariate uniform and (2) bivariate normal distrib.

the less correlated is between the two component variables, the more effective player I's deception becomes, at least, when the correlation is oppositely directed. It is also shown that, for (2) if independence is assumed, the value of the game is $(2-\sqrt{2})/\sqrt{2\pi} \doteqdot 0.2337$.

$$1+\gamma(1-2x_1)(1-2x_2),\quad (x_1,x_2)\in[0,1]^2,\quad |\gamma|\le 1$$

$$\frac{1}{2\pi\sqrt{1-\rho^2}}\exp. -\frac{1}{2(1-\rho^2)}(x_1^2-2\rho x_1 x_2+x_2^2),\quad (x_1,x_2)\in(-\infty,\infty)^2,\quad |\rho|<1.$$

の game value は それぞれ $7/12-(1/20)\gamma$,

$$\{1-\rho-\sqrt{2(1-\rho)}+\sqrt{1-\rho^2}\}\phi(0)-2\rho\int_0^\infty \Phi(x_2)\Phi\left(\frac{\rho x_2}{\sqrt{1-\rho^2}}\right)dx_2.$$ ただし $\Phi(x)\equiv\int_x^\infty \phi(t)dt$

である. See Ref. [3], [4]. Now, let

$$(1)\qquad f(x_1,x_2)=e^{-(x_1+x_2)}\{1+\gamma(2e^{-x_1}-1)(2e^{-x_2}-1)\},(x_1,x_2)\in(0,\infty)^2$$

where $\gamma, |\gamma|\le 1$, be a given constant. It is easily seen that $f(x_1,x_2)$ is symmetric with identical marginal pdf $g(x)=e^{-x}, 0\le x<\infty$, and that the correlation coefficient is equal to $(1/4)\gamma$. This bivariate pdf is one of the simplest one that has identical exponential marginals and correlated component variables. For (1) and

the bivariate normal

$$(2)\qquad f(x_1,x_2)=\frac{1}{\sigma_1}\phi\left(\frac{x_1-\mu_1}{\sigma_1}\right)\cdot\frac{1}{\sigma_2}\phi\left(\frac{x_2-\mu_2}{\sigma_2}\right),(x_1,x_2)\in(-\infty,\infty)^2,$$

(i.e., asymmetric but independent), an optimal strategy-pair would be difficult to find.

1C. Sequential Deception Game. [Ref. 5]

The classical full-information best-choice problem first discussed by Gilbert and Mosteller in 1966 is extended to a zero-sum bilateral sequential game by introducing an opponent player (I) who looks at a sequence of r.v.'s privately and decides whether to "open" or "cover" each r.v. He is allowed to cover at most m times. If the statistician (player II) accepts a r.v. he receives from I his accepted variable, and the game terminates. If II rejects a r.v., the game is continued to the next round starting with the next r.v. privately observed by I. Player I (II) wants to minimise (maximise) the expected payoff to II. For this m-opportunity n-round sequential game, an optimal strategy-pair of the players and the value of the game are found for general distribution of the *iid* sequence of r.v.'s. It is shown that under the optimal play player I deceives his opponent, *i.e.* covers the r.v. if it is very small as well as if it is considerably large. Two examples are given to illustrate the procedure to derive the solution of the game. It is shown, for example, that when the distribution is uniform in [0,1], and $n=12$, II can get 0.8126, 0.7561, 0.7067, if $m = 1, 2, 3$, respectively, although he can get 0.8791 if his opponent is not allowed to cover (*i.e.* $m = 0$). Finally the Poisson-arrival version of the continuous-time sequential game with a given final time is also formulated and solved.

The solution for $m=3$ $n=12$ is:

$$a_{12}^{(3)} = 0.0467,\ v_{11}^{(3)} = 0.6897,\ v_{11}^{(2)} = 0.7410, \text{ and } \beta_{12}^{(3)} = 0.0739$$

Game value is $v_{12}^{(3)} = 0.7067$,

where the horizontal line segment represents the abcissa of $0 \leqq x \leqq 1$, and 0-r, 0-a, and c-m mean the decision-pair open-reject, open-accept and cover-mix with $< \beta, \bar{\beta} >$, respectively.

ON TWO-PERSON "REAL" POKER BY NEWMAN Players I and II each ante 1 unit and are each dealt a "hand", namely, a randomly chosen real number in [0,1]. Each sees his, but not other's hand. First I bets any amount (≥ 0) he chooses. Next II decides whether he sees the bet or folds. The payoff is as usual. Hence the rule of the game is described by the diagram (in which sgn $z \equiv 1$ ($z > 0$), $0 (z = 0)$, or $-1 (z < 0)$):

Player	Hand	1st Move	2nd Move	Payoff to I
I	x	bet $\beta(x)$		
II	y		fold	1
			see	$(1 + \beta(x))sgn(x - y)$

The solution ingeneously suggested by Richard Bellman and proved by Newman is as follows: An optimal strategy for I is, when his hand is x, to bet

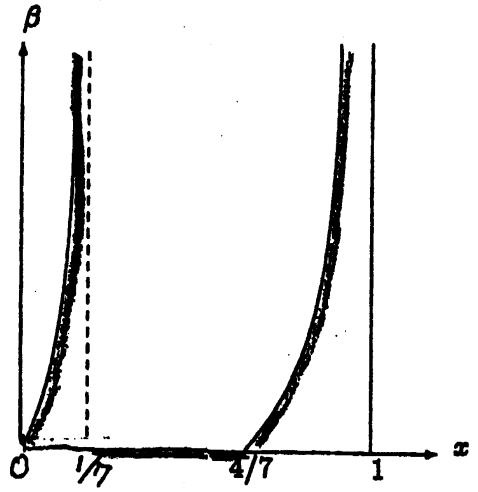
$$\beta^*(x) = \begin{cases} \text{unique root } \beta \text{ in } [0, \infty) \text{ of the equation} & \\ 3(2/(\beta+2))^2 - 2(2/(\beta+2))^3 = 1 - 7x, & \text{if } 0 \leqq x < 1/7, \\ 0, & \text{if } 1/7 \leqq x \leqq 4/7, \\ \{(12/7)(1-x)^{-1}\}^{1/2} - 2, & \text{if } 4/7 < x \leqq 1. \end{cases}$$

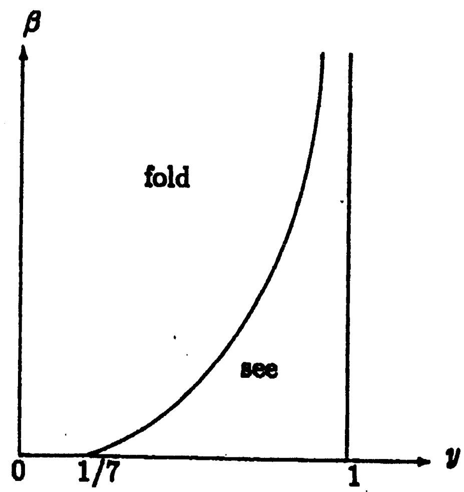
The optimal strategy for II is to see the I's bet, β, if and only if his hand y exceeds $1 - (12/7)(\beta + 2)^{-1}$. The value of the game is 1/7. (Ref. [2])

Figure 1 describes this optimal strategy-pair. Newman pointed out the following two interesting points:

(1) II's strategy is featureless. He merely sees the bet if his hand is good enough. I's strategy, on the other hand, is very slick. He very systematically and boldly bluffs on $0 < x < 1/7$. For example he bets β^* (0.142815) = 200(cf.$1/7 = 0.\dot{1}4285\dot{7}$), which would otherwise correspond to the superb hand $x = 0.99996$.

(2) The number 7 is present in an essential way in the solution. (The explanation of this mystical appearance is not given.)





The appearance of the number 7 comes from the three reasons; (a) x and y are dealt by $U[0,1]$, (b) payoff is defined by $sgn\,(x-y)$ and (c) player I is allowed to choose an arbitrary real amount of bet. Once either of the three is removed, the presence of the number 7 in the solution disappears.

For example, let (c) be replaced by (c′) ($\beta(x)$ is restricted to the positive integers). Then π appears instead of 7. The value of the game is $1-\frac{1}{2}l-\frac{1}{12}l^2 \doteqdot -0.4465$, where $l = 9/(6\pi^2-55) \doteqdot 2.1338$

Another example is: Assume that (d) Umpire tells I which is true $x<y$ or $x>y$. Then under (a)~(d), the number 4 becomes a mysterious number. We can prove;

An optimal strategy for I, when his hand is x, is to bet

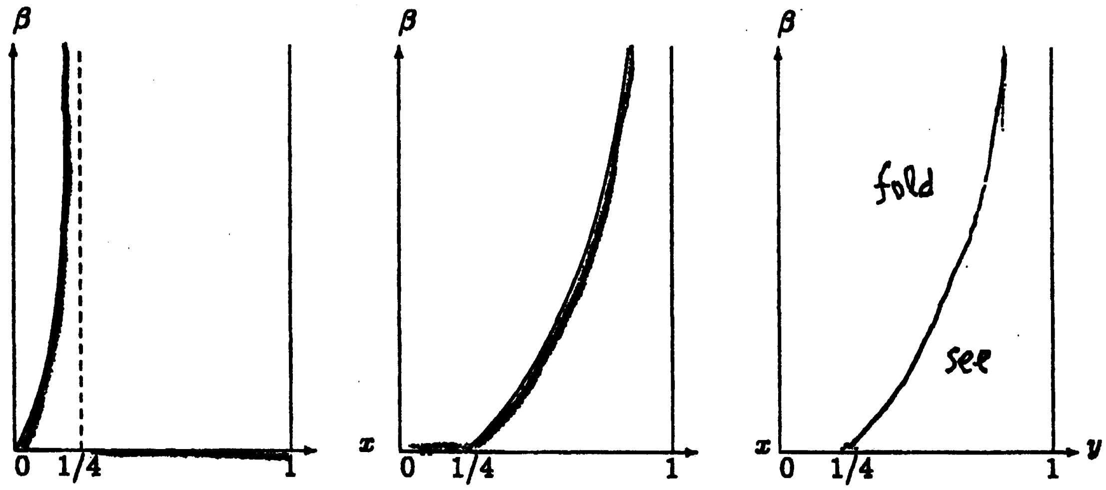
$$\beta_1^*(x) = \begin{cases} \text{a unique root } \beta \text{ in } (0,\infty) \text{ of the equation} & \\ 3\xi^2 - 2\xi^3 = 1-4x, \text{ where } \xi = 2/(2+\beta), & \text{if } 0 \leqq x < 1/4, \\ 0, & \text{if } 1/4 < x \leqq 1, \end{cases}$$

if $x < y$ becomes known;

$$\beta_2^*(x) = \begin{cases} 0, & \text{if } 0 \leqq x < 1/4 \\ \sqrt{3/(1-x)} - 2, & \text{if } 1/4 < x \leqq 1 \end{cases}$$

if $x > y$ becomes known.

The optimal strategy for II is to see the I's bet β, if and only if his hand y exceeds $y_0^*(\beta) = 1 - (3/2)/(2+\beta)$. The value of the game is 1/4.



$\beta_1^*(x)$ $\beta_2^*(x)$ $y_0(\beta)$

It is interesting to observe how (1) and (2), mentioned in the previous page, will make changes in these two examples.

An open problem: If x has pdf $2x$, his high bet is risky. Is there I's hand distrib. under which he wouldn't make a featureless bet? (Ref.[6],[7])

On Three-Member Committee

A 3-player(=member) committee has players I, II and III. The committee wants to employ one specialist among n applicants. It interviews applicants sequentially one-by-one. Facing each applicant player I(II, III) evaluates the management ability at $X_1(Y_1, Z_1)$ and computer ability at $X_2(Y_2, Z_2)$. Evaluation by the players are made independently and each player chooses, based on his evaluation, either one of R and A. The committee's choice is made by simple majority. If the committee rejects the first $n-1$ applicants, then it should accept the n-th applicant. Denote

$$\xi = x_1 \wedge x_2, \eta = y_1 \wedge y_2, \zeta = z_1 \wedge z_2. \tag{1.1}$$

If the committee accepts an applicant with talents evaluated at x, y, z by I, II, III, resp., then the game stops and each player is paid ξ, η, ζ to I, II, III, resp.. If the committee rejects an applicant, then the next applicant is interviewed and the game continues. Each player of the committee aims to maximize the expected payoff he can get.

The two different kinds of talents (management and computer abilities) for each applicant, are bivariate r.v.s, *i.i.d.* with pdf

$$h(x_1,x_2)=1+\gamma(1-2x_1)(1-2x_2),\ \ \forall(x_1,x_2)\in[0,1]^2, |\gamma|\le 1 \tag{1.2}$$

for player I. For II and III, pdfs are $h(y_1,y_2)$ and $h(z_1,z_2)$ respectively, with the same γ. If $X_1(X_2)$ for I is the evaluation of ability of management (foreign language), then γ will be $0\le\gamma\le 1$. If X_2 is the evaluation of the computer ability, then γ may be $-1\le\gamma\le 0$.

The bivariate pdf (1.2) is one of the simplest pdf that has the identical uniform margins and correlated component variables. The correlation coefficient is equal to $\gamma/3$.

Denote the state $(j,\mathbf{x},\mathbf{y},\mathbf{z})$ where $\mathbf{x}=(x_1,x_2)$, *etc.*, to mean that ① the first $j-1$ applicants were rejected by the committee, ② the j-th applicants is currently evaluated at $\mathbf{x},\mathbf{y},\mathbf{z}$, by I, II, III resp. and ③ $n-j$ applicants remain un-interviewed if the j-th is rejected by the committee. The state is illustrated by the Figure 1.

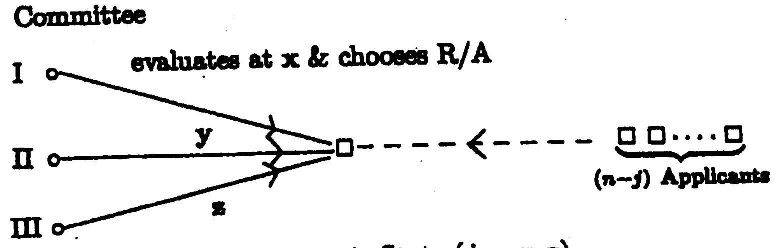


Figure 1. State $(j,\mathbf{x},\mathbf{y},\mathbf{z})$

We define u_j = Expected payoff, player I can get, if I is in state $(j,\mathbf{x},\mathbf{y},\mathbf{z})$ and all players play optimally hereafter.

Define v_j, for II, and w_j, for III, similarly. Moreover we introduce a number

$$c\equiv E_{\mathbf{x}}(\xi)=2\int_0^1 dx_1\int_0^{x_1}x_2h(x_1,x_2)dx_2=\frac{1}{3}+\frac{1}{30}\gamma$$

where is in $[3/10, 11/30]$ for $\forall\gamma\in[-1,1]$.

The Optimality Equation of our 3-player 2-choice n-stage game is

$$(u_j,v_j,w_j)=E_{\mathbf{x},\mathbf{y},\mathbf{z}}[\text{Expected payoffs facing } M_j(x,y,z)] \tag{1.3}$$
$$(j\in[1,n], u_n=v_n=w_n==E_{\mathbf{x}}(\xi)=c),$$

where

$$\begin{tikzcd} & \mathbf{M}_{j,R}(\mathbf{x},\mathbf{y},\mathbf{z}) \\ \mathbf{M}_j(\mathbf{x},\mathbf{y},\mathbf{z}) \arrow[ur, "\text{R by I}"] \arrow[dr, "\text{A by I}"'] & \\ & \mathbf{M}_{j,A}(\mathbf{x},\mathbf{y},\mathbf{z}) \end{tikzcd} \tag{1.4}$$

(1.5) $\mathbf{M}_{jR}(\mathbf{x},\mathbf{y},\mathbf{z})=$

	R by III	A by III
R by II	$u,\ v,\ w$	$u,\ v,\ w$
A by II	$u,\ v,\ w$	$\xi,\ \eta,\ \zeta$

(1.6) $\mathbf{M}_{jA}(\mathbf{x},\mathbf{y},\mathbf{z})=$

$u,\ v,\ w$	$\xi,\ \eta,\ \zeta$
$\xi,\ \eta,\ \zeta$	$\xi,\ \eta,\ \zeta$

because of the simple majority rule.

(In each cell, the subscript $j+1$ of $u_{j+1}, v_{j+1}, w_{j+1}$ is omitted. We use this convention hereafter too, when needed.)

It is clear that, for I *in state* (j, x, y, z), R (A) *dominates* A (R), *if* $u_{j+1} > (<) \xi$. By symmetry *for* II (III), u_{j+1} *and* ξ *are replaced by* v_{j+1} *and* η (w_{j+1} *and* ζ).

Let

$$f(u) \equiv E_x I(\xi > u) = (\bar{u})^2(1+\gamma u^2),$$

$$g(u) \equiv E_x[\xi I(\xi > u)] = c - u^2 + (2/3)u^3 + \gamma u^3\left(\frac{2}{3} - \frac{3}{2}u + \frac{4}{5}u^2\right).$$

We prove : *Optimal expected payoff to* 1 *satisfies the recursion*

$$u_j = Q(u_{j+1}), \quad \forall j \in [1, n-1], u_n = c,$$

where

$$Q(u) = u\,[1 - 3(f(u))^2 + 2(f(u))^3] + (f(u))^2(c - 2g(u)) + 2f(u)g(u).$$

and

$$u_1 > u_2 > \cdots > u_{n-1} > u_n = c.$$

In real world n may be 20 or so, So making the table

	u_n	u_{n-1}	u_{n-2}	$\cdots$ u_2	u_1
$\gamma = -1$	3/10	0.3420	0.3685	$\cdots$	
0	1/3	0.3821	0.4121	$\cdots$	
1	11/30	0.4284	0.4620	$\cdots$	

is usefull. (Ref.[9])

Another kind of 3-member committee is: It wants to employ one spezialist among n applicants.

The committee interviews applicants sequentially one-by-one. Facing each applicant each member chooses either A(=accept) or R(=reject). If choices are different, odd-man's judgement is not neglected and he can make some arbitration for deciding the committee's A or R. Let (X_j, Y_j, Z_j) be the evaluations of the j-th applicant's ability by the committee members, where X_j, Y_j, Z_j are *i.i.d.* with $U_{[0,1]}$ distribution. Each member of the committee wants to maximize the expected value u_n of the applicant accepted by the committee. This three-player two-choice multistage game is formulated and is given a solution, as a function of $p \in [0, \frac{1}{2}]$ *i.e.*, odd-man's power of arbitration. It is shown that $u_n \uparrow u_\infty(p)$ and $u_\infty(p)$ decreases as $p \in [0, \frac{1}{2}]$ increases.

Define the state (n, x, y, z) to mean that the committee evaluates the present applicant at $x(y, z)$ by I (II, III) and $n-1$ uninterviewed applicants remain if the present applicant is rejected by the committee.

Let EQV(=eq. value) for the n-stage game be (u_n, v_n, w_n). Then the Optimality Equation is

(2.1) $$(u_n, v_n, w_n) = E_{x,y,z}[\text{EQV of } \mathbf{M}_n(x,y,z)], \quad \left(n \geq 1, u_1 = v_1 = w_1 = \frac{1}{2}\right),$$

where the payoff matrix $\mathbf{M}_n(x,y,z)$ in state (n,x,y,z) is represented by

(2.2) $$\mathbf{M}_n(x,y,z) \begin{cases} \xrightarrow{\text{R by I}} \mathbf{M}_{n,R}(x,y,z) \\ \xrightarrow{\text{A by I}} \mathbf{M}_{n,A}(x,y,z) \end{cases}$$

(2.3) $$\mathbf{M}_{n,R}(x,y,z) = \begin{array}{c} \\ \text{II's R} \\ \text{II's A} \end{array} \begin{array}{cc} \text{III's R} & \text{III's A} \\ \hline u,\ v,\ w & p(x,y,z)+\bar{p}(u,v,w) \\ p(x,y,z)+\bar{p}(u,v,w) & p(u,v,w)+\bar{p}(x,y,z) \end{array}$$

(2.4) $$\mathbf{M}_{n,A}(x,y,z) = \begin{array}{|c|c|} \hline p(x,y,z)+\bar{p}(u,v,w) & p(u,v,w)+\bar{p}(x,y,z) \\ \hline p(u,v,w)+\bar{p}(x,y,z) & x,\ y,\ z \\ \hline \end{array}$$

(In each cell, the subscript $n-1$ of $u_{n-1}, v_{n-1}, w_{n-1}$ is omitted. We use this convention hereafter too, if needed.)

It is shown that $u_n \uparrow u$, where u is a unique root in $(\frac{1}{2}, 1)$ of
$$(3p-1)(u^3-u^2)+(p+\tfrac{1}{2})u-\tfrac{1}{2}=0, \quad \text{if } p \neq \tfrac{1}{3}.$$
Computation gives

$p=0$	0.1	0.2	0.3	1/3	0.35	0.4	1/2
$u_\infty(p) = \frac{1}{\sqrt{2}} \approx 0.7071$	0.6605	0.6304	0.6069	3/5	0.5967	0.5872	0.5698

If the odd-man appears, and has some power of arbitration the committee stands at disadvantage, in the sense that its gain $u_\infty(p) - \frac{1}{2}$ decreases as $p \in [0, \frac{1}{2}]$ increases. The committee gets less, as odd-man's power of arbitration becomes stronger. See Ref[8].

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