数理解析研究所講究録
第 1590 巻 2008 年 146-148

Some comments about notes A, B and C.

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March 11, 2008

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Abstract

We provide some complements to the notes A, B and C by M. Yor, and we use freely his main references.

About note A : the Keating-Snaith philosophy.

- For every $k\in\mathbb{N}$, the Keating-Snaith conjecture gives the leading order of the $2k^{th}$ moment of the Riemann zeta function on the critical axis. One may ask for the next terms in this expansion. In [3], it is conjectured that there exists a polynomial P_k with degree k^2 such that for any $\epsilon>0$ and any weight function g (for example $g=\mathbb{1}_{[0,T]}$)

$$\int_{\mathbb{R}}\left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k}g(t)\mathrm{d}t=\int_{\mathbb{R}}P_k\left(\log\frac{t}{2\pi}\right)\left(1+O\left(t^{-1/2+\epsilon}\right)\right)g(t)\mathrm{d}t.$$

  This generalizes the Keating-Snaith conjecture and gives another heuristics for the appearance of the constant $H_{Mat}(k)$: this constant arises naturally from a general recipe enabling to conjecture moments of shifted L-functions (see subsection 2.1 in [3]).

- Selberg (resp Keating and Snaith) obtained a central limit theorem for $\log\zeta(1/2+\mathrm{i}t)$ ($0\leq t\leq T$ and $T\to\infty$) (resp $\det(\mathrm{Id}-u)$, u being Haar-distributed on $U(n)$, and $n\to\infty$).

  There is no hope to get convergence in law to a non-zero random variable with finite moments, after normalization, for $\zeta(1/2+\mathrm{i}t)$

itself (resp: $\det(\mathrm{Id}-u)$) because its k^{th} moment is conjectured (resp: proved) to have order $(\log T)^{k^2}$ (resp: n^{k^2}). More precisely, let $X_n = \det(\mathrm{Id}_n - u)$ and suppose that there exists a sequence of constants $(c(n), n \geq 0)$ and a random variable X, not identically 0, with finite moments, such that $c(n)X_n \xrightarrow{\text{law}} X$. Necessarily $\mathbb{E}(c(n)^k|X_n|^k) \sim c(n)^k a(k) n^{k^2}$ as $n \to \infty$, and the latter sequence must converge to $0 < \mathbb{E}(|X|^k) < \infty$. The case $k = 1$ implies $c(n) \sim c/n$ for a constant $c > 0$, while the case $k = 2$ implies $c(n) \sim d/n^2$ for a constant $d > 0$, leading to a contradiction.

- The proof of

$$\det(\mathrm{Id}_n - A_n) \stackrel{\text{(law)}}{=} (1 - M_{11}) \det(\mathrm{Id} - A_{n-1})$$

(with the notations of the note A) is a little delicate, as it relies on the following suitable choice of the transformations $(M_n, n \geq 1)$: they need to be reflections, i.e.: $\mathrm{Id}_n - M_n$ must have rank 0 or 1: for this choice

$$\det\left(\mathrm{Id}_n - M_n \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}\right) = (1 - M_{11}) \det(\mathrm{Id}_{n-1} - A_{n-1}),$$

as proved in [1].

About note B : Selberg's integrals and the derivatives of the characteristic polynomial.

- The Selberg's distributions (as they are called in Section 2) are often referred to in the literature as the Jacobi distributions.
- From a decomposition as a product of independent random variables, the density of the first derivative of the characteristic polynomial is obtained, for the Haar measure on $SO(2n+1)$ conditioned to have one eigenvalue at 1. This agrees with Nina Snaith's results. In the same manner, the asymptotics of the density of the first non-zero derivative can be obtained, for any Jacobi and circular Jacobi ensembles. Moreover, this can be determined jointly at points 1 and -1 (:Jacobi ensemble) and at point 1 jointly for the real and imaginary parts (:Jacobi circular ensemble). See [2].

About note C : from Selberg's central limit theorem to total disorder. Consider the Euler product of the zeta function. Taking the logarithm gives $-\log(1-p^s) \approx p^{-s}$ as $p \to \infty$. If $X(T) \to \infty$ and $X(T) = O(T)$, then the following convergence in law holds :

$$\text{(1)} \qquad \frac{\sum_{p \leq X(T)} p^{-\frac{1}{2}+\mathrm{i}UT}}{\sqrt{\log\log X(T)}} \underset{T\to\infty}{\longrightarrow} \frac{1}{\sqrt{2}}(\mathcal{N}_1 + \mathrm{i}\,\mathcal{N}_2),$$

with U uniform on $(0,1)$ and $\mathcal{N}_1, \mathcal{N}_2$ independent standard normal variables. The above result relies on basic ergodic theory, and hinges on the fact that the $\log p$'s are linearly independent over $\mathbb{Q}$. A result by Selberg shows that partial sums like the LHS of (1) give a good approximation (in L^2 norm) for the zeta function, on the critical line.

Hence the above paragraph makes Selberg's central limit theorem intuitive. Thus, its multidimensional generalization which is the subject of note C may be thought of as intuitive because a multidimensional analogue of (1) holds.

References

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[3] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, N. C. Snaith, Integral moments of L-functions, Proc. London Math. Soc. 91: 33-104, 2005.