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Asymptotic nondegeneracy of the least energy solutions to an elliptic problem with the critical Sobolev exponent

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1 Introduction

This is an abbreviated version of the forthcoming paper [12].

In this paper, we consider the problem

$$(P_{\varepsilon,k})\begin{cases} -\Delta u = c_0 u^p + \varepsilon k(x) u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 4)$ is a smooth bounded domain, $c_0 = N(N-2)$, $p = (N+2)/(N-2)$ is the critical Sobolev exponent with respect to the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a small positive parameter. Here, k is a function in $C^2(\overline{\Omega})$.

We are interested in some qualitative property of solutions to $(P_{\varepsilon,k})$ when $\varepsilon > 0$ is sufficiently small. First, recall that a solution u of $(P_{\varepsilon,k})$ is said to be *nondegenerate*, if the linearized operator around u: $L_u := -\Delta - N(N-2)pu^{p-1}I - \varepsilon k(x)I$ with the Dirichlet boundary condition is invertible. Equivalently, the solution u is nondegenerate if the linearized problem

$$(L_{\varepsilon,k})\begin{cases} -\Delta v = N(N-2)pu^{p-1}v + \varepsilon k(x)v & \text{in } \Omega, \\ v|_{\partial\Omega} = 0 \end{cases}$$

admits only the trivial solution $v \equiv 0$.

The problem $(P_{\varepsilon,k})$ lies in the limit case of the Palais-Smale compactness condition, therefore the existence of solutions is not so straightforward. However, when $\varepsilon > 0$ is sufficiently small such that $-\Delta - \varepsilon k(x)I$ is coercive, Brezis and Nirenberg [1] proved that if $k(x) > 0$ somewhere on Ω, there exists a solution u_ε of $(P_{\varepsilon,k})$ with the property that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx - \varepsilon \int_\Omega k(x) u_\varepsilon^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p+1} dx\right)^{\frac{2}{p+1}}} = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \varepsilon \int_\Omega k(x) u^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

We call u_ε *the least energy solution* to $(P_{\varepsilon,k})$. In what follows, we consider only the least energy solutions to $(P_{\varepsilon,k})$.

Since the best constant of the Sobolev embedding theorem

$$S_N = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}$$

cannot be attained on domains other than $\mathbb{R}^N$, it is easily checked that $\|u_\varepsilon\|_{L^\infty(\Omega)} \to \infty$ as $\varepsilon \to 0$ for the least energy solution u_ε. In the following, we denote $\|\cdot\|_{L^\infty(\Omega)}$ by $\|\cdot\|$. Thus if $x_\varepsilon \in \Omega$ is a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$, we call any accumulation point $x_0 \in \overline{\Omega}$ of $\{x_\varepsilon\}$ as $\varepsilon \to 0$ a *blow-up point* of the sequence $\{u_\varepsilon\}$. It is also known that the set of blow-up points of $\{u_\varepsilon\}$ (more generally, of solutions minimizing the Sobolev inequality) consists of one point in $\overline{\Omega}$.

On the location of the blow-up point of the least energy solutions, the following fact has been proved before.

Theorem 1.1 *([11]) Assume $N \geq 4$ and $\Omega_+ := \{x \in \Omega | k(x) > 0\} \neq \phi$. Let $x_0 \in \overline{\Omega}$ be the blow-up point of the least energy solutions $\{u_\varepsilon\}$ to $(P_{\varepsilon,k})$. Then we have $x_0 \in \Omega_+$, in particular x_0 is an interior point of Ω, and x_0 is a maximum point of the function $F : \Omega_+ \to \mathbb{R}_+$, defined by*

$$F(x) = \frac{k(x)}{R(x)^{\frac{2}{N-2}}}, \quad x \in \Omega_+. \tag{1.1}$$

Here $R(x)$ is the (positive) Robin function associated with the Green function $G(x,y)$ of $-\Delta$ with the Dirichlet boundary condition:

$$R(x) = \lim_{y \to x} \left[\frac{1}{(N-2)\sigma_N} |x-y|^{2-N} - G(x,y) \right],$$

where σ_N is the volume of the $(N-1)$ dimensional unit sphere in $\mathbb{R}^N$.

In this paper, we will show the following theorem concerning the qualitative property of the blowing-up solutions.

Theorem 1.2 *(Asymptotic Nondegeneracy) Assume $N \geq 6$ and $\Omega_+ \neq \phi$. Let x_0 be the blow-up point of the least energy solutions $\{u_\varepsilon\}$. If x_0 is a nondegenerate point of the matrix*

$$\left(\frac{k_{x_i,x_j}}{k} - \frac{2}{N-2}\frac{R_{x_i,x_j}}{R}\right)_{1\leq i,j\leq N}(x), \quad x \in \Omega_+, \tag{1.2}$$

then u_ε is nondegenerate for $0 < \varepsilon << 1$ sufficiently small.

Here we note that the matrix in (1.2) is different from the Hessian matrix of $\log F$ where F is in (1.1), since $(\mathrm{Hess}\log F)(x)$ is

$$\left[\left(\frac{k_{x_i,x_j}}{k} - \frac{2}{N-2}\frac{R_{x_i,x_j}}{R}\right) - \left(\frac{k_{x_i}k_{x_j}}{k} - \frac{2}{N-2}\frac{R_{x_i}R_{x_j}}{R}\right)\right]_{1\leq i,j\leq N}(x).$$

To prove Theorem 1.2, we need a precise asymptotics of the L^∞ norm of the solution. This is achieved via the "blow-up analysis" as in Han [9], and the proof of this proposition is omitted. We only note that since our equation in $(P_{\varepsilon,k})$ has a variable coefficient, we cannot use the Gidas-Ni-Nirenberg theory [6] directly to control the blow-up point to be away from the boundary. However, for more restrictive class of solutions, that is, for least energy solutions, we can check that the blow-up point does not approach to the boundary, from the energy comparison argument [11]. The argument of Han works well once the fact that the blow-up point is an interior point of Ω is assured, See also [10] for another possible proof.

Proposition 1.3 *(Asymptotics) Assume $N \geq 4$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. Then after passing to a subsequence, the followings hold true.*

(1) There exists a constant $C > 0$ independent of ε such that

$$u_\varepsilon(x) \leq C\frac{\|u_\varepsilon\|}{\left(1+\|u_\varepsilon\|^{\frac{4}{N-2}}|x-x_\varepsilon|^2\right)^{\frac{N-2}{2}}}, \quad (\forall x \in \Omega). \tag{1.3}$$

(2)

$$\|u_\varepsilon\|u_\varepsilon \to (N-2)\sigma_N G(\cdot, x_0) \text{ in } C^2_{loc}(\overline{\Omega} \setminus \{x_0\}), \tag{1.4}$$

as $\varepsilon \to 0$.

(3)

$$\lim_{\varepsilon \to 0} \varepsilon \|u_\varepsilon\|^{\frac{2(N-4)}{N-2}} = \frac{(N-2)^3}{2a_N} \sigma_N \frac{R(x_0)}{k(x_0)} \qquad (N \geq 5), \tag{1.5}$$

$$\lim_{\varepsilon \to 0} \varepsilon \log \|u_\varepsilon\| = 4\sigma_4 \frac{R(x_0)}{k(x_0)} \qquad (N = 4),$$

where $a_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N-2}}$.

When $N \geq 5$ and $k \equiv 1$, Grossi [8] proved the above nondegeneracy result for solutions satisfying

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx - \varepsilon \int_\Omega u_\varepsilon^2 dx}{\left(\int_\Omega |u_\varepsilon|^{p+1} dx\right)^{\frac{2}{p+1}}} \to S_N \quad (\varepsilon \to 0),$$

under the assumption that the blow-up point x_0 of the solution sequence $\{u_\varepsilon\}$ is a nondegenerate critical point of the Robin function, i.e.

$$\left(\frac{\partial^2 R}{\partial x_i \partial x_j}\right)_{1 \leq i,j \leq N} (x_0) \text{ is an invertible matrix.}$$

Theorem 1.2 can be regarded as an extension of Grossi's theorem for the case $k \not\equiv 1$. However, note that we have to impose more restricted assumption on solutions, that is, we can deal with only the least energy solutions. Also in the course of proof, we need some new arguments which are not in [8].

2 Preliminaries

We recall some facts which are useful in the sequel.

Let $G = G(x, z)$ denote the Green function of $-\Delta$ under the Dirichlet boundary condition:

$$\begin{cases} -\Delta G(\cdot, z) = \delta_z & \text{in } \Omega, \\ G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.1 *(Pohozaev identities for the Green function) The identities*

$$\int_{\partial\Omega}((x-y)\cdot\nu)\left(\frac{\partial G(x,y)}{\partial\nu_x}\right)^2 ds_x=(N-2)R(y) \tag{2.1}$$

and

$$\int_{\partial\Omega}\left(\frac{\partial G}{\partial x_i}\right)\frac{\partial}{\partial\nu_x}\left(\frac{\partial G}{\partial z_j}\right)(x,y)ds_x=\frac{1}{2}\frac{\partial^2 R}{\partial x_i\partial x_j}(y) \tag{2.2}$$

hold true for any $y\in\Omega$.

Proof: See [2]:Theorem 4.3 for (2.1) and [8]:Lemma 3.2 for (2.2). □

Lemma 2.2 *Let u_ε be a solution to $(P_{\varepsilon,k})$ and v_ε be a solution to $(L_{\varepsilon,k})$. Then the following identities hold true:*

$$\int_{\partial\Omega}((x-y)\cdot\nu)\left(\frac{\partial u_\varepsilon}{\partial\nu}\right)\left(\frac{\partial v_\varepsilon}{\partial\nu}\right)ds_x=\varepsilon\int_\Omega u_\varepsilon v_\varepsilon\left(2k(x)+(x-y)\cdot\nabla k(x)\right)dx \tag{2.3}$$

for any $y\in\mathbb{R}^N$ and

$$\int_{\partial\Omega}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial v_\varepsilon}{\partial\nu}\right)ds_x=\varepsilon\int_\Omega u_\varepsilon v_\varepsilon\left(\frac{\partial k}{\partial x_i}\right)dx,\quad i=1,2,\cdots,N. \tag{2.4}$$

Proof: Set $w_\varepsilon(x)=(x-y)\cdot\nabla u_\varepsilon+\frac{N-2}{2}u_\varepsilon$. Direct computation yields that

$$-\Delta w_\varepsilon=N(N+2)u_\varepsilon^{p-1}w_\varepsilon+\varepsilon k w_\varepsilon+2\varepsilon k u_\varepsilon+\varepsilon u_\varepsilon(x-y)\cdot\nabla k(x).$$

Since v_ε satisfies $-\Delta v_\varepsilon=N(N+2)u_\varepsilon^{p-1}v_\varepsilon+\varepsilon k v_\varepsilon$, we have

$$(\Delta v_\varepsilon)w_\varepsilon-(\Delta w_\varepsilon)v_\varepsilon=2\varepsilon k u_\varepsilon v_\varepsilon+\varepsilon u_\varepsilon v_\varepsilon(x-y)\cdot\nabla k(x).$$

Integrating this identity on Ω, using integration by parts and noting $w_\varepsilon=(x-y)\cdot\nu(\frac{\partial u_\varepsilon}{\partial\nu})$ on $\partial\Omega$, we have (2.3).

On the other hand, differentiating the equation in $(P_{\varepsilon,k})$ with respect to x_i, we have

$$-\Delta\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)=N(N+2)u_\varepsilon^{p-1}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)+\varepsilon k(x)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)+\varepsilon\left(\frac{\partial k}{\partial x_i}\right)u_\varepsilon.$$

Multiplying this equation by v_ε, and the equation of v_ε by $\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)$ and subtracting, we obtain

$$(\Delta v_\varepsilon)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) - \left(\Delta\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right) v_\varepsilon = \varepsilon\left(\frac{\partial k}{\partial x_i}\right) u_\varepsilon v_\varepsilon.$$

Finally, integration by parts yields (2.4). □

Now, let us consider the scaled function

$$\tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{\frac{2}{N-2}}(\Omega - x_\varepsilon). \tag{2.5}$$

We see $0 < \tilde{u}_\varepsilon \leq 1, \tilde{u}_\varepsilon(0) = 1$, and $\tilde{u}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{u}_\varepsilon = c_0 \tilde{u}_\varepsilon^p + \frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}} k_\varepsilon(y)\tilde{u}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $k_\varepsilon(y) = k\left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon\right)$. Since $\|u_\varepsilon\| \to \infty$ as $\varepsilon \to 0$, we see $\Omega_\varepsilon \to \mathbb{R}^N$ and $k_\varepsilon \to k(0)$ compact uniformly on $\mathbb{R}^N$ as $\varepsilon \to 0$. By standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_\varepsilon$ that

$$\tilde{u}_\varepsilon \to U \quad \text{compact uniformly in } \mathbb{R}^N \tag{2.6}$$

as $\varepsilon \to 0$ for some function U. Passing to the limit, we obtain that U is a solution of

$$\begin{cases} -\Delta U = c_0 U^p \quad \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \ U(0) = 1, \\ \lim_{|y|\to\infty} U(y) = 0. \end{cases}$$

Then according to the uniqueness theorem by Caffarelli, Gidas and Spruck [4], we obtain

$$U(y) = \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}}.$$

Note that (1.3) in Proposition 1.3 can be written as

$$\tilde{u}_\varepsilon(y) \leq CU(y) \quad \text{for } \forall y \in \Omega_\varepsilon. \tag{2.7}$$

We recall here the classification theorem proved by Bianchi and Egnell [3].

Lemma 2.3 *Let v_0 be a solution to*

$$\begin{cases} -\Delta v_0 = c_0 p U^{p-1} v_0 & \text{in } \mathbb{R}^N, \\ v_0 \in D^{1,2}(\mathbb{R}^N) \end{cases}$$

where $D^{1,2}(\mathbb{R}^N) = \{v \in L^{2N/(N-2)}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} |\nabla v|^2 dy < \infty\}$.

Then there exist constants a_j $(j = 1, 2, \cdots, N)$ and b in $\mathbb{R}$ such that v_0 can be written as

$$v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{N/2}} + b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}. \tag{2.8}$$

Final lemma is a well-known unique solvability result of linear first order PDE's with the initial condition. Proof of this lemma is done by the standard method of characteristics.

Lemma 2.4 *Let $a = (a_1, a_2, \cdots, a_N) \neq 0$ is a constant vector and $f, g \in C^1(\mathbb{R}^N)$. Let $\Gamma_a = \{x \in \mathbb{R}^N | a \cdot x = 0\}$ be the $(N-1)$-plane perpendicular to a through the origin. Then there exists a unique solution of the following initial value problem of the linear first order PDE*

$$a \cdot \nabla u = 0,$$
$$u|_{\Gamma_a} = g.$$

More precisely, this solution is obtained as

$$u(x) = \int_0^{\phi(x)} f(\tau a + \alpha(\psi(x)))d\tau + g(\alpha(\psi(x))), \quad x \in \mathbb{R}^N$$

where

$$\phi(x) = \frac{a \cdot x}{|a|^2}, \quad \psi(x) = (\psi_1(x), \cdots, \psi_{N-1}(x)),$$
$$\psi_j(x) = \frac{|a|^2 x_j - (a \cdot x) a_j}{|a|^2}, (j = 1, \cdots, N-1)$$
$$\alpha(s) = (s, -\frac{1}{a_N}\sum_{j=1}^{N-1} a_j s_j) \in \mathbb{R}^N, \quad s = (s_1, \cdots, s_{N-1}) \in \mathbb{R}^{N-1},$$

if we assume (w.l.o.g) $a_N \neq 0$. Furthermore, if $f(x) = O(|x|^\beta), g(x) = O(|x|^\beta)$ as $|x| \to \infty$, then $u(x) = O(|x|^{\beta+1})$ as $|x| \to \infty$.

3 The asymptotic nondegeneracy result

In this section, we will prove Theorem 1.2. As noticed earlier, we mainly follow the argument by Grossi [8], but some new argument is needed.

We argue by contradiction and assume that there exists a non-trivial solution v_ε to $(L_{\varepsilon,k})$. Since the problem is linear, we may assume $\|v_\varepsilon\| = \|u_\varepsilon\|$ for any $\varepsilon > 0$, where u_ε is the least energy solution to $(P_{\varepsilon,k})$ obtained by Brezis and Nirenberg.

Let us consider the scaled function

$$\tilde{v}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} v_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{2}{N-2}}(\Omega - x_\varepsilon). \tag{3.1}$$

We see $0 < \tilde{v}_\varepsilon \le 1$ and $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{v}_\varepsilon = c_0 p \tilde{u}_\varepsilon^{p-1} \tilde{v}_\varepsilon + \frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}} k_\varepsilon(y) \tilde{v}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \tilde{v}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1 \end{cases} \tag{3.2}$$

where $k_\varepsilon(y) = k\left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon \right)$. By $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$ and the elliptic estimate, we see there exists v_0 such that

$$\tilde{v}_\varepsilon \to v_0 \quad \text{uniformly on compact subsets of } \mathbb{R}^N \tag{3.3}$$

and v_0 satisfies

$$-\Delta v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.$$

Now, we claim that

$$\int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy \le \exists C \tag{3.4}$$

for some $C > 0$ independent of $\varepsilon > 0$. Though the proof of this claim is the same as in the derivation of the inequality (3.8) in [8], or the inequality (10) in [5], we recall it here for the reader's convenience.

Denote $a_\varepsilon(y) = c_0 p \tilde{u}_\varepsilon^{p-1}(y)$. By (3.2), we have

$$\int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2 dy = \int_{\Omega_\varepsilon} (a_\varepsilon(y) + \frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}} k_\varepsilon(y)) \tilde{v}_\varepsilon^2 dx.$$

By the Poincaré inequality and the scaling property of the eigenvalue of Laplacian $\lambda_1(s\Omega) = s^{-2}\lambda_1(\Omega)$, we see that

$$\frac{\varepsilon}{\|u_\varepsilon\|^{4/(N-2)}}\int_{\Omega_\varepsilon} k_\varepsilon(y)\tilde{v}_\varepsilon^2 dx \le \frac{\varepsilon\|k\|}{\lambda_1(\Omega_\varepsilon)\|u_\varepsilon\|^{4/(N-2)}}\int_{\Omega_\varepsilon}|\nabla\tilde{v}_\varepsilon|^2 dx = \frac{\varepsilon\|k\|}{\lambda_1(\Omega)}\int_{\Omega_\varepsilon}|\nabla\tilde{v}_\varepsilon|^2 dx.$$

From these, we have

$$(1+o(1))\int_{\Omega_\varepsilon}|\nabla\tilde{v}_\varepsilon|^2 dx \le \int_{\Omega_\varepsilon} a_\varepsilon(y)\tilde{v}_\varepsilon^2 dx.$$

Let $0<\delta<4/(N-2)$. Then by the Sobolev inequality, we have

$$\begin{aligned}(1+o(1))S_N\left(\int_{\Omega_\varepsilon}|\tilde{v}_\varepsilon|^{p+1}dy\right)^{2/(p+1)} &\le (1+o(1))\int_{\Omega_\varepsilon}|\nabla\tilde{v}_\varepsilon|^2 dy\\ &\le \int_{\Omega_\varepsilon} a_\varepsilon(y)\tilde{v}_\varepsilon^2 dy \le \int_{\Omega_\varepsilon}|a_\varepsilon(y)|\tilde{v}_\varepsilon^{2-\delta}dy,\end{aligned}$$

here, the last inequality comes from the fact that $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le 1$.

Now, by the Hölder inequality and (1.3), we have

$$\begin{aligned}\int_{\Omega_\varepsilon}|a_\varepsilon(y)|\tilde{v}_\varepsilon^{2-\delta}dy &\le \left(\int_{\Omega_\varepsilon}|\tilde{v}_\varepsilon|^{p+1}dy\right)^{(2-\delta)/(p+1)}\left(\int_{\Omega_\varepsilon}|a_\varepsilon(y)|^{(p+1)/(p-1+\delta)}dy\right)^{(p-1+\delta)/(p+1)}\\ &\le C\left(\int_{\Omega_\varepsilon}|\tilde{v}_\varepsilon|^{p+1}dy\right)^{(2-\delta)/(p+1)}\left(\int_{\Omega_\varepsilon}U(y)^{(p-1)(p+1)/(p-1+\delta)}dy\right)^{(p-1+\delta)/(p+1)},\end{aligned}$$

thus we obtain

$$\left(\int_{\Omega_\varepsilon}|\tilde{v}_\varepsilon|^{p+1}dy\right)^{\delta/(p+1)} \le C\left(\int_{\mathbb{R}^N}U(y)^{(p-1)(p+1)/(p-1+\delta)}dy\right)^{(p-1+\delta)/(p+1)}.$$

Note that $((N-2)/2)(p-1)(p+1)/(p-1+\delta) > N/2$ if $\delta < 4/(N-2)$, so the last integral is bounded by a constant. Therefore, we have

$$\int_{\Omega_\varepsilon}|\tilde{v}_\varepsilon|^{p+1}dy \le C. \tag{3.5}$$

Finally, again by the Hölder inequality, (3.5) and (1.3), we have

$$
\begin{aligned}
&(1+o(1))\int_{\Omega_\varepsilon}|\nabla\tilde{v}_\varepsilon|^2dy \le \int_{\Omega_\varepsilon} a_\varepsilon(y)\tilde{v}_\varepsilon^2dy\\
&\le \left(\int_{\Omega_\varepsilon}|\tilde{v}_\varepsilon|^{p+1}dy\right)^{2/(p+1)}\left(\int_{\Omega_\varepsilon}|a_\varepsilon(y)|^{(p+1)/(p-1)}dy\right)^{(p-1)/(p+1)}\\
&\le C\left(\int_{\mathbb{R}^N}U(y)^{p+1}dy\right)^{(p-1)/(p+1)} \le C.
\end{aligned}
$$

This proves (3.4).

By (3.4) and Fatou's lemma, we also have

$$\int_{\mathbb{R}^N}|\nabla v_0|^2dy \le C.$$

Thus by Lemma 2.3, we have (2.8), i.e.

$$v_0=\sum_{j=1}^N a_j\frac{y_j}{(1+|y|^2)^{N/2}}+b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}. \tag{3.6}$$

In the following, we divide the proof into several steps.

Step 1. $b=0$.

Step 2. $a_j=0, j=1,\cdots,N$.

Step 3. $v_0=0$ leads to a contradiction.

We need the following pointwise estimate for the scaled function $\tilde{v}_\varepsilon$.

Lemma 3.1 *Assume $N\ge 5$. Let $\tilde{v}_\varepsilon$ be as in (3.1). Then there exists a constant $C>0$ independent of ε such that*

$$|\tilde{v}_\varepsilon(y)|\le C\left(\frac{1}{1+|y|^2}\right)^{(N-2)/2} \tag{3.7}$$

holds true for all $y\in\Omega_\varepsilon$.

Proof. Since Ω is bounded, we see that there exists $\gamma > 0$ such that $\Omega_\varepsilon \subset B(0, \gamma\|u_\varepsilon\|^{2/N-2})$. We employ the Kelvin transformation of $\tilde{v}_\varepsilon$:

$$w_\varepsilon(z) := |z|^{2-N}\tilde{v}_\varepsilon(\frac{z}{|z|^2}), \quad z \in \Omega_\varepsilon^*,$$

here $\Omega_\varepsilon^* := \{z = \frac{y}{|y|^2} | y \in \Omega_\varepsilon\}$. Note that Ω_ε^* is a domain contained $\mathbb{R}^N \setminus B(0, 1/(\gamma\|u_\varepsilon\|^{2/N-2}))$. Then it is enough to show that

$$\sup_{z\in\Omega_\varepsilon^*\cap B(0,1)} w_\varepsilon(z) \leq C$$

to obtain the result, because by the fact that $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, we only have to bound $\tilde{v}_\varepsilon$ for $|y|$ sufficiently large. By the property of the Kelvin transformation, we have for $z \in \Omega_\varepsilon^*$,

$$\Delta w_\varepsilon(z) = \frac{1}{|z|^{N+2}}(\Delta\tilde{v}_\varepsilon)(\frac{z}{|z|^2}), \quad \int_{\Omega_\varepsilon^*} |w_\varepsilon(z)|^{2N/(N-2)}dz = \int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon(y)|^{2N/(N-2)}dy.$$

Set

$$a_\varepsilon(z) := \frac{1}{|z|^4}\left(c_0 p\tilde{u}_\varepsilon^{p-1}(\frac{z}{|z|^2}) + \frac{\varepsilon}{\|u_\varepsilon\|^{p-1}}k_\varepsilon(\frac{z}{|z|^2})\right)$$

for $z \in \Omega_\varepsilon^*$. Then w_ε satisfies

$$\begin{cases} -\Delta w_\varepsilon = a_\varepsilon(z)w_\varepsilon & \text{in } \Omega_\varepsilon^*, \\ w_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon^*, \end{cases}$$

Then the same reasoning as in [5] p.107 leads to the fact that $a_\varepsilon \in L^\alpha(\Omega_\varepsilon^*)$ for some $\alpha > N/2$ when $N \geq 5$. Thus by the classical elliptic estimate (for example, [7] Lemma 8.17) and (3.5), we confirm that

$$\begin{aligned} \sup_{z\in\Omega_\varepsilon^*\cap B(0,1)} |w_\varepsilon(z)| &\leq C\left(\int_{\Omega_\varepsilon^*\cap B(0,2)} |w_\varepsilon|^{p+1}dz\right)^{1/(p+1)} \leq C\left(\int_{\Omega_\varepsilon^*} |w_\varepsilon|^{p+1}dz\right)^{1/(p+1)} \\ &= C\left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1}dz\right)^{1/(p+1)} \leq C. \end{aligned}$$

□

By this pointwise estimate for $\tilde{v}_\varepsilon$, we obtain the following convergence result: see [8] (3.26).

Lemma 3.2 *Let ω be a neighborhood of $\partial\Omega$ not containing x_0. Then we have*

$$\|u_\varepsilon\| v_\varepsilon \to (2-N)\sigma_N b G(\cdot, x_0) \quad \text{in } C^{1,\alpha}(\omega) \tag{3.8}$$

as $\varepsilon \to 0$ for some $\alpha \in (0,1)$.

Assume for the moment that the proof of Step 1 and 2 is finished. Then the proof of Step 3 is as follows. By Step 1 and Step 2, we deduce that the limit function $\lim_{\varepsilon\to 0} \tilde{v}_\varepsilon = v_0 \equiv 0$. Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $x_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(x_\varepsilon) = 1$ and $|x_\varepsilon| \to \infty$ because the above convergence $\tilde{v}_\varepsilon \to v_0 \equiv 0$ is uniformly on compact sets of $\mathbb{R}^N$. But this is not possible because of Lemma 3.1.

Proof of Step 1.

Putting $y = x_0$ in (2.3) and multiplying $\|u_\varepsilon\|^2$, we have

$$\begin{aligned} &\int_{\partial\Omega} ((x-x_0)\cdot\nu) \left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial\nu}\right)\left(\frac{\partial\|u_\varepsilon\|v_\varepsilon}{\partial\nu}\right) ds_x \\ &= \varepsilon\|u_\varepsilon\|^2 \int_\Omega u_\varepsilon v_\varepsilon \left(2k(x) + (x-x_0)\cdot\nabla k(x)\right) dx \end{aligned} \tag{3.9}$$

First, by Proposition 1.3 (1.4) and (3.8), the LHS of (3.9) tends to

$$-(N-2)^2\sigma_N^2 b \int_{\partial\Omega} ((x-x_0)\cdot\nu) \left(\frac{\partial G(x,x_0)}{\partial\nu}\right)^2 ds_x = -(N-2)^3\sigma_N^2 b R(x_0).$$

Here we have used (2.1) in Lemma 2.1.

On the other hand, set $L(x) := 2k(x) + (x-x_0)\cdot\nabla k(x)$ for $x \in \Omega$. Then L is continuous on Ω and $L(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon) \to L(x_0) = 2k(x_0)$ uniformly on compact sets of $\mathbb{R}^N$. By a change of variable, the limit of the RHS of (3.9) is

$$\begin{aligned} &\varepsilon\|u_\varepsilon\|^{4-\frac{2N}{N-2}} \int_{\Omega_\varepsilon} L\Big(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon\Big)\tilde{u}_\varepsilon\tilde{v}_\varepsilon dy \\ &\to \big(\lim_{\varepsilon\to 0}\varepsilon\|u_\varepsilon\|^{2(N-4)/(N-2)}\big)\cdot L(x_0)\int_{\mathbb{R}^N} U(y)v_0(y)dy \\ &= \frac{(N-2)^3\sigma_N}{2a_N}\,\frac{R(x_0)}{k(x_0)}\cdot 2k(x_0)\times \\ &\int_{\mathbb{R}^N}\left(\frac{1}{1+|y|^2}\right)^{(N-2)/2}\left(\sum_{j=1}^N a_j\frac{y_j}{(1+|y|^2)^{N/2}} + b\frac{1-|y|^2}{(1+|y|^2)^{N/2}}\right)dy. \end{aligned} \tag{3.10}$$

Here we have used Proposition 1.3 (1.5) with the use of the pointwise estimates (1.3), (3.7) and Lebesgue's dominated convergence theorem.

Note that the integral

$$\int_{\mathbb{R}^N} \left(\frac{1}{1+|y|^2}\right)^{(N-2)/2} \frac{y_j}{(1+|y|^2)^{N/2}} dy = 0$$

for any $j = 1, 2, \cdots, N$ by the oddness of the integrand,

$$\int_{\mathbb{R}^N} \left(\frac{1}{1+|y|^2}\right)^{(N-2)/2} \frac{1-|y|^2}{(1+|y|^2)^{N/2}} dy = -\sigma_N \frac{\Gamma(N/2)\Gamma(N/2-2)}{\Gamma(N-1)}$$

and

$$a_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N-2}} dr = \frac{\Gamma(N/2)\Gamma(N/2-2)}{2\Gamma(N-2)}.$$

Here we have used a formula

$$\int_0^\infty \frac{r^\alpha}{(1+r^2)^\beta} dr = \frac{\Gamma((\alpha+1)/2)\Gamma(\beta-(\alpha+1)/2)}{2\Gamma(\beta)}$$

for $\alpha, \beta > 0$ with $\beta - (\alpha+1)/2 > 0$. Thus, we have

$$(3.10) = -2(N-2)^2\sigma_N^2 R(x_0)b.$$

As a result of the above, we obtain

$$-(N-2)^3\sigma_N^2 bR(x_0) = -2(N-2)^2\sigma_N^2 R(x_0)b$$

which leads to an obvious contradiction if $b \neq 0$.

Thus we have proved Step 1.

Proof of Step 2.

In this step, we prove $a_j = 0, j = 1, 2, \cdots, N$ in (3.6). For this purpose, we need a lemma, which is not in [8].

Lemma 3.3 *Assume $b = 0$ and $\vec{a} = (a_1, \cdots, a_N) \neq 0$ in (3.6). Then we have*

$$\|u_\varepsilon\|^{N/(N-2)} v_\varepsilon \to \sigma_N \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial z_j}(x,z)\right)\Big|_{z=x_0}$$

in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$.

Proof. For any $x_0 \in \overline{\Omega} \setminus \{x_0\})$, the Green representation formula for the solution v_ε to $(L_{\varepsilon,k})$ implies that

$$\begin{aligned}v_\varepsilon(x) &= N(N+2)\int_\Omega G(x,z)u_\varepsilon^{p-1}(z)v_\varepsilon(z)dz + \varepsilon\int_\Omega G(x,z)k(z)v_\varepsilon(z)dz\\ &=: I_1(\varepsilon) + I_2(\varepsilon).\end{aligned} \tag{3.11}$$

By a change of variables, we see

$$\begin{aligned}I_1(\varepsilon) &= N(N+2)\int_\Omega G(x,z)u_\varepsilon^{p-1}(z)v_\varepsilon(z)dz\\ &= \frac{N(N+2)}{\|u_\varepsilon\|}\int_{\Omega_\varepsilon} G_\varepsilon(x,y)\tilde{u}_\varepsilon^{p-1}\tilde{v}_\varepsilon(y)dy\end{aligned}$$

where $G_\varepsilon(x,y) = G(x, \frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon)$ for $y \in \Omega_\varepsilon$. By (2.6) and (3.3), we know that

$$\begin{aligned}&\tilde{u}_\varepsilon^{p-1}(y) \to U^{p-1}(y),\\ &\tilde{v}_\varepsilon(y) \to v_0 = \sum_{j=1}^N a_j\frac{y_j}{(1+|y|^2)^{N/2}} = \frac{-1}{(N-2)}\sum_{j=1}^N a_j\frac{\partial U}{\partial y_j}\end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^N$, thus

$$\tilde{u}_\varepsilon^{p-1}\tilde{v}_\varepsilon(y) \to \sum_{j=1}^N a_j\left(\frac{\partial}{\partial y_j}\frac{-1}{(N+2)}U^p(y)\right)$$

uniformly on compact subsets of $\mathbb{R}^N$.

Now, let us consider the following linear first order PDE

$$\sum_{j=1}^N a_j\frac{\partial w}{\partial y_j} = \tilde{u}_\varepsilon^{p-1}\tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N$$

with the initial condition $w|_{\Gamma_a} = \frac{-1}{(N+2)}U^p(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. By Lemma 2.4, we have a solution w_ε of this problem with the estimate $w_\varepsilon(y) = O(|y|^{-(N+1)})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p-1}\tilde{v}_\varepsilon(y) = O(U^p(y)) = O(|y|^{-(N+2)})$ by (2.7) and (3.7). Also we have

$$w_\varepsilon \to \frac{-1}{(N+2)}U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} w_\varepsilon(y)dy \to \frac{-1}{(N+2)} \int_{\mathbb{R}^N} U^p dy = \frac{-1}{N(N+2)}\sigma_N$$

by the dominated convergence theorem.

Using integration by parts, we have

$$\begin{aligned} I_1(\varepsilon) &= \frac{N(N+2)}{\|u_\varepsilon\|} \int_{\Omega_\varepsilon} G_\varepsilon(x,y) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\ &= -\frac{N(N+2)}{\|u_\varepsilon\|} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} G_\varepsilon(x,y) \cdot w_\varepsilon(y)dy \\ &= -\frac{N(N+2)}{\|u_\varepsilon\|^{N/(N-2)}} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \left(\frac{\partial G}{\partial z_j}\right)(x,z)\Big|_{z=\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}}+x_\varepsilon} w_\varepsilon(y)dy. \end{aligned}$$

Thus we obtain

$$\|u_\varepsilon\|^{N/(N-2)} I_1(\varepsilon) \to \sigma_N \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial z_j}(x,z)\right)\Big|_{z=x_0} \tag{3.12}$$

for $x \in \overline{\Omega} \setminus \{x_0\}$.

Next we consider $I_2(\varepsilon)$.

$$\begin{aligned} I_2(\varepsilon) &= \varepsilon \int_\Omega G(x,z)k(z)v_\varepsilon(z)dz \\ &= \frac{\varepsilon}{\|u_\varepsilon\|^{(N+2)/(N-2)}} \int_{\Omega_\varepsilon} G_\varepsilon(x,y)k_\varepsilon(y)\tilde{v}_\varepsilon(y)dy. \end{aligned}$$

As before, consider the following linear first order PDE

$$\sum_{j=1}^N a_j \frac{\partial w}{\partial y_j} = \tilde{v}_\varepsilon(y) \quad (y \in \mathbb{R}^N), \quad w|_{\Gamma_a} = \frac{-1}{(N-2)}U(y).$$

Lemma 2.4 assures the existence of solution w_ε with the property that $w_\varepsilon(y) = O(|y|^{3-N})$ as $|y| \to \infty$, because $\tilde{v}_\varepsilon(y) = O(U(y)) = O(|y|^{2-N})$ by Lemma 3.1. Since

$$\tilde{v}_\varepsilon(y) \to \sum_{j=1}^N a_j \frac{y_j}{(1+|y|^2)^{N/2}} = \sum_{j=1}^N a_j \left(\frac{\partial}{\partial y_j}\frac{-1}{(N-2)}U(y)\right),$$

we have $w_\varepsilon \to \frac{-1}{(N-2)}U(y)$ compact uniformly on $\mathbb{R}^N$.

Now, by integration by parts, we have

$$\begin{aligned}
I_2(\varepsilon) &= \frac{\varepsilon}{\|u_\varepsilon\|^{(N+2)/(N-2)}} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} G_\varepsilon(x,y)k_\varepsilon(y)\frac{\partial w_\varepsilon}{\partial y_j}dy \\
&= -\frac{\varepsilon}{\|u_\varepsilon\|^{(N+2)/(N-2)}} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j}\{G_\varepsilon(x,y)k_\varepsilon(y)\}\, w_\varepsilon(y)dy \\
&= -\frac{\varepsilon}{\|u_\varepsilon\|^{(N+2)/(N-2)}} \times
\end{aligned}$$

$$\sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{1}{\|u_\varepsilon\|^{2/(N-2)}} \left\{\frac{\partial}{\partial z_j}(G(x,z)k(z))\right\}\Bigg|_{z=\left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}}+x_\varepsilon\right)} w_\varepsilon(y)dy.$$

Since $\Omega_\varepsilon \subset B(0,\gamma\|u_\varepsilon\|^{2/(N-2)})$ for some $\gamma > 0$, we have

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} w_\varepsilon(y)dy\right| &\le C + C\int_{B(0,\gamma\|u_\varepsilon\|^{2/(N-2)})\setminus B(0,1)} |y|^{3-N}dy \\
&\le C + C\int_1^{\gamma\|u_\varepsilon\|^{2/(N-2)}} r^{3-N}r^{N-1}dr \\
&\le C\|u_\varepsilon\|^{6/(N-2)}.
\end{aligned}$$

On the other hand, Proposition 1.3 (1.5) implies that $\varepsilon = O(\|u_\varepsilon\|^{-2(N-4)/(N-2)})$ as $\varepsilon \to 0$ for $N \ge 5$. Thus we have

$$\begin{aligned}
\|u_\varepsilon\|^{N/(N-2)}|I_2(\varepsilon)| &\le \|u_\varepsilon\|^{N/(N-2)}\varepsilon\frac{1}{\|u_\varepsilon\|^{(N+4)/(N-2)}}C\left|\int_{\Omega_\varepsilon} w_\varepsilon(y)dy\right| \\
&\le C\|u_\varepsilon\|^{N/(N-2)}\|u_\varepsilon\|^{-2(N-4)/(N-2)}\frac{1}{\|u_\varepsilon\|^{(N+4)/(N-2)}}\|u_\varepsilon\|^{6/(N-2)} \\
&\le C\|u_\varepsilon\|^{2(5-N)/(N-2)} = o(1) \qquad (3.13)
\end{aligned}$$

as $\varepsilon \to 0$ when $N \ge 6$.

From (3.12) and (3.13), we see

$$\begin{aligned}
\|u_\varepsilon\|^{N/(N-2)}v_\varepsilon &= \|u_\varepsilon\|^{N/(N-2)}(I_1 + I_2) \\
&\to \sigma_N \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial z_j}(x,z)\right)\Big|_{z=x_0}
\end{aligned}$$

for any $x \in \overline{\Omega}\setminus\{x_0\}$. Standard elliptic estimate assures that this convergence also holds in $C^1_{loc}(\overline{\Omega}\setminus\{x_0\})$. This proves Lemma. □

Now, we multiply both sides of (2.4) in Lemma 2.2 by $\|u_\varepsilon\|^{N/(N-2)} \times \|u_\varepsilon\|$. Letting $\varepsilon \to 0$, we see

$$\begin{aligned}
&\int_{\partial\Omega}\left(\frac{\partial\|u_\varepsilon\|u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial\|u_\varepsilon\|^{N/(N-2)}v_\varepsilon}{\partial \nu}\right)ds_x\\
&\to (N-2)\sigma_N^2\int_{\partial\Omega}\sum_{j=1}^N a_j\left(\frac{\partial G}{\partial x_i}\right)(x,x_0)\frac{\partial}{\partial\nu}\left(\frac{\partial G}{\partial z_j}\right)(x,x_0)ds_x\\
&= \frac{(N-2)}{2}\sigma_N^2\sum_{j=1}^N a_j\frac{\partial^2 R}{\partial x_i\partial x_j}(x_0) \qquad (3.14)
\end{aligned}$$

for the LHS of the identity, here we have used Proposition 1.3 (1.4), Lemma 3.3 and Lemma 2.1 (2.2).

On the other hand, the RHS can be written as

$$\begin{aligned}
&\varepsilon\|u_\varepsilon\|^{N/(N-2)}\|u_\varepsilon\|\int_\Omega u_\varepsilon v_\varepsilon\left(\frac{\partial k}{\partial x_i}\right)dx\\
&= \varepsilon\|u_\varepsilon\|^{-N/(N-2)}\|u_\varepsilon\|^3\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon(y)\tilde{v}_\varepsilon(y)\left(\frac{\partial k}{\partial x_i}\right)\left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}}+x_\varepsilon\right)dy. \qquad (3.15)
\end{aligned}$$

We know

$$\begin{aligned}
\tilde{u}_\varepsilon\tilde{v}_\varepsilon(y) \to U(y)v_0(y) &= \sum_{j=1}^N a_j\frac{y_j}{(1+|y|^2)^{N-1}}\\
&= \sum_{j=1}^N a_j\frac{\partial}{\partial y_j}\frac{1}{2(2-N)}\left(\frac{1}{1+|y|^2}\right)^{N-2}
\end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^N$. As before, we exploit the solution w_ε of the linear first order PDE

$$\sum_{j=1}^N a_j\frac{\partial w}{\partial y_j} = \tilde{u}_\varepsilon(y)\tilde{v}_\varepsilon(y) \quad (y\in\mathbb{R}^N), \quad w|_{\Gamma_a} = \frac{1}{2(2-N)}\left(\frac{1}{1+|y|^2}\right)^{N-2}$$

with the property that $w_\varepsilon(y) = O(|y|^{5-2N})$ for $|y|$ large and

$$w_\varepsilon \to \frac{1}{2(2-N)}\left(\frac{1}{1+|y|^2}\right)^{N-2}$$

uniformly on compact subsets of $\mathbb{R}^N$. Note that $w_\varepsilon \in L^1(\mathbb{R}^N)$ by our assumption $N > 5$. Thus,

$$
\begin{aligned}
(3.15) &= \varepsilon \|u_\varepsilon\|^{3-\frac{N}{N-2}} \int_{\Omega_\varepsilon} \sum_{j=1}^N a_j \frac{\partial w_\varepsilon(y)}{\partial y_j} \left(\frac{\partial k}{\partial x_i}\right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon\right) dy \\
&= -\varepsilon \|u_\varepsilon\|^{3-\frac{N}{N-2}} \int_{\Omega_\varepsilon} w_\varepsilon(y) \sum_{j=1}^N a_j \frac{\partial}{\partial y_j} \left(\frac{\partial k}{\partial x_i}\right) \left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon\right) dy \\
&= -\varepsilon \|u_\varepsilon\|^{3-\frac{N}{N-2}} \int_{\Omega_\varepsilon} w_\varepsilon(y) \frac{1}{\|u_\varepsilon\|^{\frac{2}{N-2}}} \sum_{j=1}^N a_j \frac{\partial}{\partial x_j} \left(\frac{\partial k}{\partial x_i}(x)\right) \Bigg|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}}+x_\varepsilon} dy \\
&= -\varepsilon \|u_\varepsilon\|^{2(N-4)/(N-2)} \int_{\Omega_\varepsilon} w_\varepsilon(y) \sum_{j=1}^N a_j \frac{\partial^2 k}{\partial x_i \partial x_j} \left(\frac{y}{\|u_\varepsilon\|^{\frac{2}{N-2}}} + x_\varepsilon\right) dy \\
&\to -\frac{(N-2)^3 \sigma_N}{2a_N} \frac{R(x_0)}{k(x_0)} \times -\frac{1}{2(2-N)} \int_{\mathbb{R}^N} U^2(y)dy \times \sum_{j=1}^N a_j \frac{\partial^2 k}{\partial x_i \partial x_j}(x_0) \\
&= \frac{(N-2)^2 \sigma_N^2}{4} \frac{R(x_0)}{k(x_0)} \sum_{j=1}^N a_j \frac{\partial^2 k}{\partial x_i \partial x_j}(x_0). \qquad (3.16)
\end{aligned}
$$

Here again we have used Proposition 1.3 (1.5) and the dominated convergence theorem. Note that $\sigma_N a_N = \int_{\mathbb{R}^N} U^2 dy$.

By (3.14) and (3.16), we have

$$
\sum_{j=1}^N a_j \left\{ \frac{1}{k(x_0)} \left(\frac{\partial^2 k}{\partial x_i \partial x_j}(x_0) \right) - \frac{2}{N-2} \frac{1}{R(x_0)} \left(\frac{\partial^2 R}{\partial x_i \partial x_j}(x_0) \right) \right\} = 0.
$$

Finally we obtain $a_j = 0$ for all $j = 1, \cdots, N$ by our nondegeneracy assumption of the matrix (1.2) at x_0. Thus we have proved Step 2 and this ends the proof of Theorem 1.2. □

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