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Representation and Asymptotic Formulas for Some 1-dimensional Linearized Eigenvalue Problems

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1 Introduction

In this article we consider the following stationary problem of a scalar reaction diffusion equations

$$\begin{cases} \varepsilon^2 u_{xx}(x) + f(u(x)) = 0, & \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \end{cases} \tag{1.1}$$

and the corresponding linearized eigenvalue problem for each solution $u(x)$ of (1.1)

$$\begin{cases} \varepsilon^2 \varphi_{xx}(x) + f_u(u(x))\varphi(x) + \mu\varphi(x) = 0 & x \in (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{1.2}$$

Here ε is a positive parameter. We are interested in the case that f is a function with *the balanced bistable nonlinearity*: f has exactly three zeros 0, $u_\pm$ such that $u_- < 0 < u_+$ $f_u(0) > 0$, $f_u(u_\pm) < 0$ and $F(u_+) = F(u_-)$, where

$$F(u) := \int_0^u f(s)ds.$$

In other words, we can say that $-F$ is a double well potentials of equal depth. For any $\varepsilon > 0$, (1.1) has two stable trivial solution $u = u_\pm$ and one unstable solution $u = 0$.

Since the pioneer work of Chafee and Infante [4], the solution structure of (1.1) and stability of each stationary solution has been investigated by many authors (See e.g., [4], Henry [5], Smoller-Wasserman [8]). In general, for any $n \in \mathbf{N}$, a curve of n-mode solutions (they have exactly n zeros in $0 \le x \le 1$) bifurcates from $u = 0$ at $\varepsilon = \sqrt{f_u(0)}/(n\pi)$. Furthermore, the Morse index of n- mode solution, the number of negative eigenvalues of (LP), is n (see Brunovský-Fiedler [1]).

In view of dynamical theory for the corresponding reaction diffusion problem, it is important to consider the case when ε is sufficiently small. Then, for each nontrivial stationary solution of (1.1), *transition layers* are formed in neighborhoods of its zeros. Though any nontrivial stationary solution is unstable, precise analysis for (1.2) would be also important to understand the behavior of a certain class of nonstationary solutions. In [3] Carr and Pego have obtained estimates of negative eigenvalues of (1.2), to describe the motion of transition layers for nonstationary solutions.

From the viewpoint of such pattern formation as above, we are interested in profiles of eigenfunctions of (1.2). It is expected that some spike-layer like patterns are formed in eigenfunctions of (1.2) as $\varepsilon \to 0$, although precise analysis has not been done yet.

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The purpose of this article is getting precise information on eigenvalues and eigenfunctions of (1.2) in the following special cases of f:

$$f(u) = \sin u \quad \text{and} \quad f(u) = u(1-u^2).$$

In the first case it is assumed that the solution u of (1.2) satisfies $-\pi \le u(x) \le \pi$ because of periodicity of $\sin u$. These functions are typical examples of balanced bistable nonlinearities. It should be noted that any nontrivial solution of (1.1) can be expressed explicitly with use of Jacobi's elliptic function.

We will introduce a new method to give expressions of all eigenvalues and eigenfunctions of (1.2). Suppose that $u(x)$ is an *n-mode* solution of (1.1). From (1.2) we will introduce an *representation equation*, a linear differential equation whose independent variable is $u = u(x)$, and it does not depend on n and ε. By using a particular solution $h(u;\mu)$ of the equation, we will obtain eigenfunctions of the following two forms:

$$\varphi(x) = \pm\sqrt{h(u(x);\mu)} \quad \text{(special eigenfunction)}$$

($\pm$ should be chosen suitably in several sub-intervals in $0 \le x \le 1$) and

$$\varphi(x) = \sqrt{h(u(x);\mu)}\cos\Big(\frac{1}{\varepsilon}\int_0^x \frac{\sqrt{\rho(\mu)}}{h(u(\xi);\mu)}d\xi\Big) \quad \text{(general eigenfunction)}.$$

Here $\rho(\mu) > 0$ and it is determined by $h(\,\cdot\,;\mu)$ and μ (see Section 3). The boundary condition is reduced to a *characteristic equation*

$$A(\mu) = \frac{j\pi}{2n}$$

for some $j \in \mathbf{N} \cup \{0\}$. For both cases of f, the characteristic function $A(\mu)$ consists of the complete elliptic integral of the third kind. Each eigenvalue is determined by the corresponding characteristic equation. By summarizing these results we will give representation formulas of all eigenvalues and eigenfunctions. We will also derive asymptotic formulas of eigenvalues when ε is sufficiently small. In particular, we will see the following two results (see Corollaries 1 and 2):

(i) If $0 \le j < n$, then

$$\mu_j = -C\cos^2\frac{j\pi}{2n}\cdot e^{-\frac{d}{n\varepsilon}} + o(e^{-\frac{d}{n\varepsilon}}),$$

where $C > 0$ and $d > 0$ are constants.

(ii) There exists $\mu^* > 0$ and $l_0 \in \mathbf{N}$ such that if $j > l_0 n$, then

$$\mu_j = \mu^* + (j - l_0 n)^2\pi^2\varepsilon^2 + o(\varepsilon^2),$$

where μ_j is the $(j+1)$-th eigenvalue of (1.2). These asymptotic formulas enable us to investigate profiles of eigenfunctions of (1.2) when ε is sufficiently small.

The organization of the article is as follows. In Section 2 we first introduce some important functions and lemmas for characteristic equations, and give our main results. In Section 3 we will introduce a method of representation equations to prove representation formulas. In Section 4 we will give a sketch of proof of lemmas for charateristic equations and asymptotic formulas of eigenvalues. As concluding remarks, asymptotic profiles of eigenfunctions will be investigated in Section 5.

2 Main Results

In this section we give our main results. For both cases of f, we will give expressions of nontrivial solutions of (1.1) with use of elliptic integrals and Jacobi's sn-function. Suppose $k \in (0,1)$. We denote by $K(k)$ and $\Pi(\nu, k)$, the complete elliptic integrals of the first kind and the third kind (with a parameter ν), respectively. For complete elliptic integrals, see Section 4. The function $\mathrm{sn}(x,k)$ is defined by

$$x = \int_0^{\mathrm{sn}(x,k)} \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}} ds$$

for $x \in [0, K(k)]$ and it is extended to a periodic function on $\mathbf{R}$ in a standard way. In what follows, $k \in (0,1)$ will be used to parametrize arbitrary pair of $u(x)$ and ε of (1.1).

2.1 Case $f(u) = \sin u$

In this subsection we assume that $f(u) = \sin u$. For $k \in (0,1)$ and $n \in \mathbf{N}$, define

$$\varepsilon_n(k) := \frac{1}{2nK(k)} \quad \text{and} \quad u_n(x;k) := 2\sin^{-1}\Big[k \cdot \mathrm{sn}\big(K(k)(1+2nx); k\big)\Big].$$

Then any nontrivial solution of (1.1) satisfying $-\pi \le u(x) \le \pi$ is given by $(\varepsilon_n(k), u_n(x;k))$ or $(\varepsilon_n(k), -u_n(x;k))$ with some $n \in \mathbf{N}$ and $k \in (0,1)$. It should be noted that

$$u_n(0;k) = \max_{0\le x\le 1} u_n(x;k) = 2\sin^{-1} k, \quad \min_{0\le x\le 1} u_n(x;k) = -2\sin^{-1} k$$

and

$$\frac{\varepsilon_n(k)^2}{2}\Big(\frac{du_n}{dx}(x;k)\Big)^2 = 2\Big(k^2 - \sin^2\frac{u_n(x;k)}{2}\Big).$$

For each $n \in \mathbf{N}$, $\varepsilon_n(k) \to 0$ as $k \to 1$ and then u_n has n layers in the neighborhoods of $z_i = (2i-1)/(2n)$ $(i = 1, \ldots, n)$.

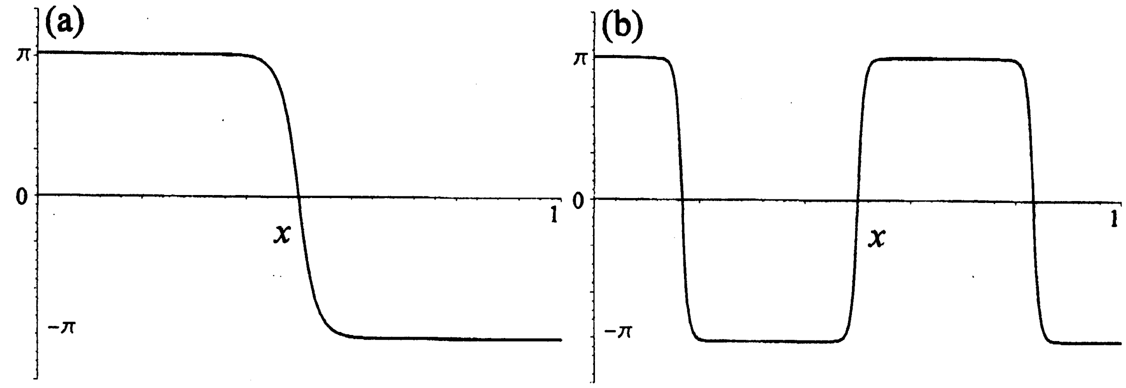

Figure 1: Graphs of $u_n(x;k)$ $(k = 1 - 10^{-20})$: (a) $u_1(x;k)$, (b) $u_3(x;k)$.

The linearized problems associated $(\varepsilon_n(k), u_n(x;k))$ is rewritten as

$$\begin{cases} \varepsilon_n(k)^2\varphi_{xx}(x) + \cos u_n(x;k)\varphi(x) + \mu\varphi(x) = 0, & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{2.1}$$

Hereafter, let arbitrary $n \in \mathbf{N}$ be fixed. By $\mu_j^n(k)$ and $\varphi_j^n(x;k)$ ($j \in \mathbf{N} \cup \{0\}$), we denote the $(j+1)$-th eigenvalue of (2.1) and its corresponding eigenfunction, respectively.

Set

$$h_1(u;\mu,k) := \left(k^2 - \sin^2\frac{u}{2}\right) - \mu, \quad \text{for } u \in [-2\sin^{-1}k, 2\sin^{-1}k],$$
$$\rho_1(\mu,k) := \mu(\mu-k^2)(\mu-k^2+1),$$

and define *the characteristic function* by

$$A_1(\mu,k) := \frac{1}{2}\int_0^{2\sin^{-1}k} \frac{\sqrt{\rho_1(\mu;k)}}{\sqrt{k^2-\sin^2\frac{u}{2}}\,|h_1(u;\mu,k)|}du = \sqrt{\frac{\mu(\mu-k^2+1)}{\mu-k^2}}\Pi\left(\frac{k^2}{\mu-k^2},k\right).$$

It is easy to see that $h_1(u;\mu,k) \geq -\mu > 0$ for $\mu \in (k^2-1,0)$, $h_1(u;\mu,k) \leq k^2-\mu < 0$ for $\mu \in (k^2,+\infty)$ and $\rho_1 > 0$ for $\mu \in (k^2-1,0) \cup (k^2,+\infty)$. Moreover, we can show the following lemma for the characteristic equation.

Lemma 2.1. *Let $p \in (0,+\infty) \setminus \{\pi/2\}$. Then there exists a smooth function $\mu(\,\cdot\,;p) : [0,1) \to \mathbf{R}$ such that*

$$A_1(\mu(k;p);k) = p \quad \textit{for all} \quad k \in [0,1),$$

and for each $k \in [0,1)$, $\mu(k;p)$ is an unique solution of $A_1(\mu,k) = p$. If $0 < p < \pi/2$, then $\mu(k;p) \in (k^2-1,0)$ and $\lim_{k\to1}\mu(k;p) = 0$. If $p > \pi/2$, then $\mu(k;p) \in (k^2,+\infty)$ and $\lim_{k\to1}\mu(k;p) = 1$.

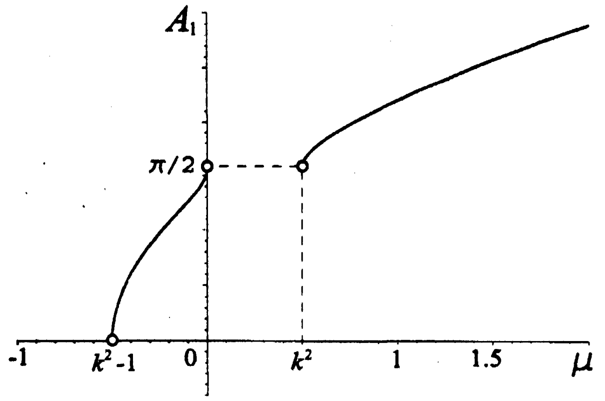

Figure 2: Graph of $A_1(\mu,k)$ ($k = 1/\sqrt{2}$).

Our representation formulas are given by the following two theorems.

Theorem 1 (Special Eigenfunctions). *Problem* (2.1) *has the following pairs of eigenvalues and eigenfunctions.*

(i) $\mu_0^n(k) = -(1-k^2), \quad \varphi_0^n(x;k) = \cos\frac{u_n(x;k)}{2} \quad \left(= \sqrt{|h_1(u_n(x;k);\mu_0^n(k),k)|}\right).$

(ii) $\mu_n^n(k) = k^2, \quad \varphi_n^n(x;k) = \frac{1}{k}\sin\frac{u_n(x;k)}{2} \quad \left(= \frac{1}{k}(-1)^{[2nx]}\sqrt{|h_1(u_n(x;k);\mu_n^n(k),k)|}\right).$

Theorem 2 (General Eigenfunctions). *Suppose $j \neq 0, n$. Then, $\mu_j^n(k) = \mu(k; (j\pi)/(2n))$ and the corresponding eigenfunction is given by*

$$\varphi_j^n(x; k) = \sqrt{|h_1(u_n(x), \mu_j^n(k); k))|} \cos\Big(\frac{1}{\varepsilon_n(k)} \int_0^x \frac{\sqrt{\rho_1(\mu; k)}}{|h_1(u_n(\xi; k); \mu, k)|} d\xi\Big).$$

Theorems 1 and 2 show us expressions of all eigenvalues and eigenfunctions. Furthermore, we give asymptotic formulas of eigenvalues $\mu_j^n(k)$ as $k \to 1$. Fix $n \in \mathbf{N}$ and $j \in \mathbf{N} \cup \{0\}$ arbitrarily. From Lemma 2.1 we see that if $0 < j < n$, then $\lim_{k\to 1} \mu_j^n(k) = 0$ and if $j > n$, then $\lim_{k\to 1} \mu_j^n(k) = 1$. More precisely, we can obtain the following theorem by analyzing the characteristic equation $A_1(\mu, k) = (j\pi)/(2n)$.

Theorem 3 (Asymptotic Formula of Eigenvalue). *Fix $n \in \mathbf{N}$ and $j \in \mathbf{N} \cup \{0\}$. Then, the following* (i) *and* (ii) *hold:*

(i) *If $0 < j < n$, then* $\mu_j^n(k) = -\cos^2 \dfrac{j\pi}{2n} \cdot (1 - k^2) + o(1 - k^2)$ *as* $k \to 1$.

(ii) *If $j > n$, then* $\mu_j^n(k) = 1 + \dfrac{(j-n)^2\pi^2}{4n^2} K(k)^{-2} + o(K(k)^{-2})$ *as* $k \to 1$.

Assertions of Theorem 3 can be expressed in terms of parameter ε $(= \varepsilon_n(k))$. Fix $n \in \mathbf{N}$ arbitrarily. It follows from $\varepsilon = 1/(2nK(k))$ and

$$\lim_{k\to 1} \Big(K(k) - \log \frac{1}{\sqrt{1-k^2}} - 2\log 2\Big) = 0 \tag{2.2}$$

(see Section 4) that

$$1 - k^2 = 16e^{-\frac{1}{n\varepsilon}} + o(e^{-\frac{1}{n\varepsilon}}) \quad \text{as } \varepsilon \to 0.$$

Hence Theorem 3 implies the following corollary.

Corollary 1. *Fix $n \in \mathbf{N}$ and $j \in \mathbf{N} \cup \{0\}$. Then, the following* (i) *and* (ii) *hold:*

(i) *If $0 < j < n$, then* $\mu_j^n = -16\cos^2 \dfrac{j\pi}{2n} \cdot e^{-\frac{1}{n\varepsilon}} + o(e^{-\frac{1}{n\varepsilon}})$ *as* $\varepsilon \to 0$.

(ii) *If $j > n$, then* $\mu_j^n = 1 + (j-n)^2\pi^2\varepsilon^2 + o(\varepsilon^2)$ *as* $\varepsilon \to 0$.

2.2 Case $f(u) = u(1 - u^2)$

In this subsection we assume that $f(u) = u(1 - u^2)$. Set

$$\varepsilon_n(k) = \frac{1}{2n\sqrt{1+k^2}K(k)} \quad \text{and} \quad u_n(x; k) = \sqrt{\frac{2k^2}{1+k^2}} \operatorname{sn}\big(K(k)(1 + 2nx), k\big)$$

for $n \in \mathbf{N}$ and $k \in [0, 1)$. Similarly as in the case $f(u) = \sin u$, it follows that any nontrivial solution of (1.1) is given by $(\varepsilon_n(k), u_n(x; k))$ or $(\varepsilon_n(k), -u_n(x; k))$ with some $n \in \mathbf{N}$ and $k \in (0, 1)$,

$$u_n(0; k) = \max_{0\le x\le 1} u_n(x; k) = \sqrt{\frac{2k^2}{1+k^2}}, \quad \min_{0\le x\le 1} u_n(x; k) = -\sqrt{\frac{2k^2}{1+k^2}}$$

and

$$\frac{\varepsilon_n(k)^2}{2}\left(\frac{du_n}{dx}(x;k)\right)^2 = \frac{k^2}{(1+k^2)^2} - \frac{1}{2}u_n(x;k)^2 + \frac{1}{4}u_n(x;k)^4$$

for all $x \in [0,1]$, $n \in \mathbf{N}$ and $k \in (0,1)$.

The linearized problems associated $(\varepsilon_n(k), \pm u_n(x;k))$ is given by

$$\begin{cases} \varepsilon_n(k)^2\varphi_{xx}(x) + (1 - 3u_n(x;k)^2)\varphi(x) + \mu\varphi(x) = 0, & \text{in } (0,1), \\ \varphi'(0) = \varphi'(1) = 0. \end{cases} \tag{2.3}$$

In what follows, we fix $n \in \mathbf{N}$ arbitrarily and we denote the $(j+1)$-th eigenvalue of (2.1) and its corresponding eigenfunction by $\mu_j^n(k)$ and $\varphi_j^n(x;k)$ $(j \in \mathbf{N} \cup \{0\})$.

For the sake of convenience, we introduce the rescaled parameter

$$\hat{\mu} := (1+k^2)\mu \quad \text{and} \quad w := \sqrt{\frac{1+k^2}{2k^2}}u \in [-1,1],$$

and set

$$\begin{aligned} \hat{h}_2(w;\hat{\mu},k) &:= 9(1+k^2)^2\left[\left(\frac{k^2}{(1+k^2)^2} - \frac{1}{2}u^2 + \frac{1}{4}u^4\right) + \frac{\mu}{6}(u^2-2) + \frac{1}{9}\mu^2\right] \\ &= (\hat{\mu}-3)(\hat{\mu}-3k^2) + 3k^2(\hat{\mu} - 3(1+k^2))w^2 + 9k^4w^4 \\ &= (\hat{\mu}-3)(\hat{\mu}-3k^2)(1+\nu_+(\hat{\mu},k)w^2)(1+\nu_-(\hat{\mu},k)w^2) \\ &= \frac{3}{4}(\hat{\mu}-\hat{\mu}_+(k))(\hat{\mu}-\hat{\mu}_-(k)) + 9k^4\left[\frac{3(1+k^2)-\hat{\mu}}{6k^2} - w^2\right]^2 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \hat{\rho}_2(\hat{\mu},k) &:= \hat{\mu}(\hat{\mu}-3)(\hat{\mu}-3k^2)\left[\hat{\mu}^2 - 2(1+k^2)\mu - 3(1-k^2)^2\right] \\ &= \hat{\mu}(\hat{\mu}-3)(\hat{\mu}-3k^2)(\hat{\mu}-\hat{\mu}_+(k))(\hat{\mu}-\hat{\mu}_-(k)), \end{aligned} \tag{2.5}$$

where

$$\hat{\mu}_\pm(k) := 1 + k^2 \pm 2\sqrt{1-k^2+k^4} \quad (\hat{\mu}_-(k) < 0,\ 3 < \hat{\mu}_+(k)) \tag{2.6}$$

and

$$\begin{aligned} \nu_\pm(\hat{\mu},k) &:= \frac{3k^2\cdot\left[\hat{\mu} - 3(1+k^2) \pm \sqrt{-3\mu^2 + 6(1+k^2)\mu + 9(1-k^2)^2)}\right]}{2(\hat{\mu}-3)(\hat{\mu}-3k^2)} \\ &= \frac{6k^2}{\hat{\mu} - 3(1+k^2) \mp \sqrt{-3\mu^2 + 6(1+k^2)\mu + 9(1-k^2)^2}}. \end{aligned} \tag{2.7}$$

One can show the following properties on $\hat{h}_2$:

(i) if $\hat{\mu} \in (\hat{\mu}_-(k), 0)$, then $\hat{h}_2 > 0$ for $w \in (-1,1)$ and $-1 < \nu_-(\hat{\mu},k) < \nu_+(\hat{\nu},k) < 0$,

(ii) if $\hat{\mu} \in (3k^2, 3)$, then $\hat{h}_2 < 0$ for $w \in (-1,1)$ and $-1 < \nu_+(\hat{\mu},k) < 0 < \nu_-(\hat{\mu},k)$,

(iii) if $\hat{\mu} \in (\hat{\mu}_+(k), +\infty)$, then $\hat{h}_2 > 0$ for $w \in (-1,1)$ and $\nu_\pm(\hat{\mu},k) \in \mathbf{C}$.

Moreover, we define the characteristic function by

$$
\begin{aligned}
A_2(\hat{\mu}, k) &:= \int_0^1 \frac{\sqrt{\hat{\rho}_2(\hat{\mu}, k)}}{\sqrt{(1-w^2)(1-k^2w^2)}|\hat{h}_2(w;\hat{\mu};k)|}dw \\
&= \begin{cases} \dfrac{\sqrt{P}}{3\sqrt{3}k^2}\Big(\nu_+\Pi(\nu_+, k) - \nu_-\Pi(\nu_-, k)\Big) & \text{if } \hat{\mu} \in (\hat{\mu}_-(k), 0), \\ \dfrac{\sqrt{P}}{3\sqrt{3}k^2}\Big(\nu_-\Pi(\nu_-, k) - \nu_+\Pi(\nu_+, k)\Big) & \text{if } \hat{\mu} \in (3k^2, 3), \\ \dfrac{\sqrt{-P}}{3\sqrt{-3}k^2}\Big(\nu_+\Pi(\nu_+, k) - \nu_-\Pi(\nu_-, k)\Big) & \text{if } \hat{\mu} \in (\hat{\mu}_+(k), +\infty), \end{cases}
\end{aligned} \tag{2.8}
$$

where $\nu_\pm = \nu_\pm(\hat{\mu}, k)$ and

$$
P = P(\hat{\mu}, k) := -\hat{\mu}(\hat{\mu}-3)(\hat{\mu}-3k^2). \tag{2.9}
$$

The following lemmas will be proved in Section 4.

Lemma 2.2. *Let $k \in [0, 1)$ be fixed and $\hat{\mu} \in (\hat{\mu}_-(k), 0) \cup (3k^2, 3) \cup (\mu_+(k), +\infty)$. Then the following* (i)-(iii) *hold true:*

(i) $\lim_{\hat{\mu}\to\hat{\mu}_-(k)} A_2(\hat{\mu}, k) = 0, \quad \lim_{\hat{\mu}\to 0} A_2(\hat{\mu}, k) = \dfrac{\pi}{2}.$

(ii) $\lim_{\hat{\mu}\to 3k^2} A_2(\hat{\mu}, k) = \dfrac{\pi}{2}, \quad \lim_{\hat{\mu}\to 3} A_2(\hat{\mu}, k) = \pi.$

(iii) $\lim_{\hat{\mu}\to\hat{\mu}_+(k)} A_2(\hat{\mu}, k) = \pi.$

Lemma 2.3. *Let $k \in [0, 1)$ be fixed and $\hat{\mu} \in (\hat{\mu}_-(k), 0) \cup (3k^2, 3) \cup (\mu_+(k), +\infty)$. Then*

$$
\frac{\partial A_2}{\partial \hat{\mu}}(\hat{\mu}, k) > 0.
$$

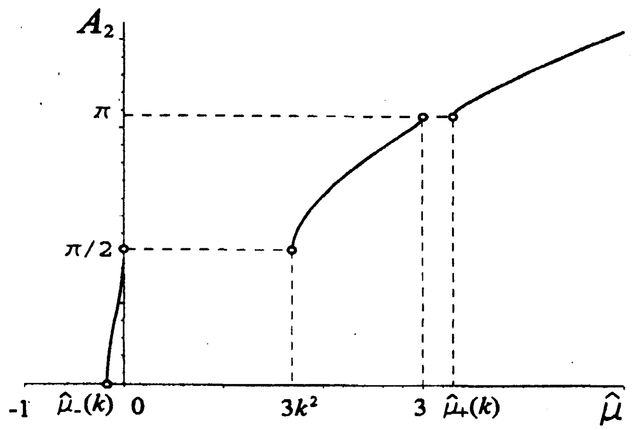

Figure 3: Graph of $A_2(\mu, k)$ $(k = 3/4)$.

By Lemmas 2.2 and 2.3, it follows from the implicit function theorem that for any $p > 0$ with $p \neq \pi/2, \pi$, there exists a smooth function $\hat{\mu}(\,\cdot\,; p) : [0, 1) \to \mathbf{R}$ such that

$$A_2(\hat{\mu}(k; p), k) = p \quad \text{for all} \quad k \in [0, 1),$$

and for each $k \in [0, 1)$, $\hat{\mu}(k; p)$ is an unique solution of $A_1(\mu, k) = p$.

The representation formulas of eigenvalues and eigenfunctions are given by the following two theorems.

Theorem 4 (Special Eigenfunction). *The Linearized problem* (2.3) *has the following pairs of eigenvalues and eigenfunctions:*

(i) $\mu_0^n(k) = \dfrac{\hat{\mu}_-(k)}{1+k^2}$, $\quad \varphi_0^n(x; k) = 1 - \dfrac{(1 + k^2 - \sqrt{1 - k^2 + k^4})(1 + k^2)}{2k^2} \cdot u_n(x; k)^2$,

(ii) $\mu_n^n(k) = \dfrac{3k^2}{1+k^2}$, $\quad \varphi_n^n(x; k) = \sqrt{\dfrac{1+k^2}{2k^2}}u_n(x; k)\sqrt{1 - k^2 \cdot \dfrac{1+k^2}{2k^2}u_n(x; k)^2}$,

(iii) $\mu_{2n}^n(k) = \dfrac{\hat{\mu}_+(k)}{1+k^2}$, $\quad \varphi_{2n}^n(x; k) = -1 + \dfrac{(1 + k^2 + \sqrt{1 - k^2 + k^4})(1 + k^2)}{2k^2} \cdot u_n(x; k)^2$.

Theorem 5 (General Eigenfunction). *Suppose $j \neq 0, n, 2n$. Then,*

$$\mu_j^n(k) = \frac{\hat{\mu}(k; (j\pi)/(2n))}{1+k^2}$$

and the corresponding eigenfunction is given by

$$\varphi_j^n(x; k) = \sqrt{|h_2(u_n(x; k), \mu_j^n(k); k))|}\cos\left(\frac{1}{\varepsilon_n(k)}\int_0^x \frac{\sqrt{\rho_2(\mu_j^n(k), k)}}{|h_2(u_n(\xi; k), \mu_j^n(k), k)|}d\xi\right),$$

where

$$h_2(u, \mu, k) = \frac{1}{9(1+k^2)^2}\hat{h}_2\left(\sqrt{\frac{1+k^2}{2k^2}}u, (1+k^2)\mu, k\right)$$

and

$$\rho_2(\mu, k) = \frac{1}{81(1+k^2)^5}\hat{\rho}_2((1+k^2)\mu, k)$$

Combining Theorems 4 and 5 we have all eigenvalues and eigenfunctions to (2.3). Similarly in the case $f(u) = \sin u$, all eigenvalues are determined by the characteristic equation $A_2(\hat{\mu}, k) = j\pi/(2n)$.

Now we give asymptotic formulas of eigenvalues. We can show that

$$\lim_{k \to 1}\mu_j^n(k) = \begin{cases} 0 & \text{if } 0 \leq j < n, \\ \dfrac{3}{2} & \text{if } n \leq j < 2n \\ 2 & \text{if } j \geq 2n. \end{cases}$$

In the case of $j \geq 2n$, the assertion follows from the following inequality:

$$\frac{j\pi}{2n} = A_2(\hat{\mu}, k) \geq \frac{\sqrt{\hat{\rho}_2(\hat{\mu}, k)}}{\hat{h}_2(1; \hat{\mu}, k)}K(k) = \sqrt{\frac{(\hat{\mu} - 3k^2)(\hat{\mu} - \hat{\mu}_-(k))(\hat{\mu} - \hat{\mu}_+(k))}{\hat{\mu}(\hat{\mu} - 3)}}K(k),$$

where $\hat{\mu} = \hat{\mu}(k; j\pi/(2n))$. Moreover, we have the following theorem.

Theorem 6 **(Asymptotic Formula of Eigenvalue).** *Fix $n \in \mathbf{N}$ and $j \in \mathbf{N} \cup \{0\}$. Then, the following* (i)-(iii) *hold:*

(i) *If $0 \leq j < n$, then* $\mu_j^n(k) = -\frac{3}{8}\cos^2\frac{j\pi}{2n} \cdot (1-k^2)^2 + o((1-k^2)^2)$ *as* $k \to 1$,

(ii) *If $n \leq j < 2n$, then* $\mu_j^n(k) = \frac{3}{2} - \frac{3}{2}\cos^2\frac{(j-n)\pi}{2n} \cdot (1-k^2) + o(1-k^2)$ *as* $k \to 1$,

(iii) *If $j > 2n$, then* $\mu_j^n(k) = 2 + \frac{1}{2}\left(\frac{j-2n}{2n}\right)^2 \pi^2 K(k)^{-2} + o(K(k)^{-2})$ *as* $k \to 1$.

Fix $n \in \mathbf{N}$ arbitrarily. Then, it follows from (2.2) that

$$1 - k^2 = 16e^{-\frac{1}{\sqrt{2}n\varepsilon}} + o\left(e^{-\frac{1}{\sqrt{2}n\varepsilon}}\right) \quad \text{as } \varepsilon \to 0.$$

Therefore, the following corollary comes from Theorem 6.

Corollary 2. *Fix $n \in \mathbf{N}$ and $j \in \mathbf{N} \cup \{0\}$. Then, the following* (i)-(iii) *hold:*

(i) *If $0 \leq j < n$, then* $\mu_j^n = -96\cos^2\frac{j\pi}{2n} \cdot e^{-\frac{\sqrt{2}}{n\varepsilon}} + o\left(e^{-\frac{\sqrt{2}}{n\varepsilon}}\right)$ *as* $\varepsilon \to 0$,

(ii) *If $n < j < 2n$, then* $\mu_j^n(k) = \frac{3}{2} - 24\cos^2\frac{(j-n)\pi}{2n} \cdot e^{-\frac{1}{\sqrt{2}n\varepsilon}} + o\left(e^{-\frac{1}{\sqrt{2}n\varepsilon}}\right)$ *as* $\varepsilon \to 0$,

(iii) *If $j > 2n$, then* $\mu_j^n(k) = 2 + (j-2n)^2\pi^2\varepsilon^2 + o(\varepsilon^2)$ *as* $\varepsilon \to 0$.

3 Proof for Representation Formulas

In this section we introduce a method of representation equation and prove representation formulas. For both cases of f, proofs are done by similar argument. So we only prove Theorems 4 and 5 for the case of $f(u) = u(1-u^2)$ (for the case $f(u) = \sin u$, see [12]).

We begin with the following initial value problem

$$\begin{cases} \varepsilon^2\varphi_{xx}(x) + f_u(u(x))\varphi(x) + \mu\varphi(x) = 0, & \text{in } (0,1), \\ \varphi(0) = 1, \quad \varphi_x(0) = 0, \end{cases} \tag{3.1}$$

for general $f \in C^2(\mathbf{R})$. From (3.1) we will derive the representation equation. By using a particular solution of the equation, we will give expressions of the solution of (3.1). Except for some special cases, the boundary condition $\varphi_x(1) = 0$ is reduced to the characteristic equation. We will show that each eigenvalue is given by a solution of characteristic equation. These results with Lemmas 2.2 and 2.3 give proofs of Theorems 4 and 5.

3.1 Method of Representation Equation

Let $f \in C^2(\mathbf{R})$ and let $(\varepsilon, u(x))$ be a nontrivial solution of (1.1). In what follows, we use notations $\alpha := u(0)$, $\alpha_M := \max_{0\leq x\leq 1} u(x)$ and $\alpha_m := \min_{0\leq x\leq 1} u(x)$. Note that $f(\alpha) \neq 0$, $\alpha \in \{\alpha_m, \alpha_M\}$ and that $u(x)$ satisfies

$$\frac{\varepsilon^2}{2}(u_x(x))^2 + F(u(x)) = F(\alpha) \quad \text{for } x \in [0.1]. \tag{3.2}$$

By (3.2) α_M and α_m are characterized as follows: $F(\alpha_M) = F(\alpha_m) = F(\alpha)$ and

$$F(\alpha) - F(u) > 0 \quad \text{for } u \in (\alpha_m, \alpha_M).$$

The essential idea of representation equation is found in [10]. By the change of variable $\varphi(x) = \Phi(u(x))$, the first equation in (3.1) is led to the representation equation of the second order:

$$2(F(\alpha) - F(u))\Phi_{uu}(u) - f(u)\Phi_u(u) + (f_u(u) + \mu)\Phi(u) = 0 \tag{3.3}$$

for $u \in (\alpha_m, \alpha_M)$. Special eigenfunctions of (1.2) can be obtained by using solutions of (3.3). The above change of independent variable is a useful idea for solving (3.1).

Now we introduce the represenation equation of the third order. Set $R(x) := \varphi(x)^2$, where φ is a solution of the first equation of (3.1). Then it satisfies the following linear differential equation of the third order

$$\varepsilon^2 R_{xxx}(x) + 4(f_u(u(x)) + \mu)R_x(x) + 2f_{uu}(u(x))u_x(x)R(x) = 0. \tag{3.4}$$

Here we apply the change of variable to (3.4). Assume that (3.4) has a solution $R(x) = h(u(x)) = h(u(x);\mu)$. From (1.1) and (3.2), we have

$$\begin{aligned}
\frac{d}{dx}(h(u(x))) &= h_u(u(x))u_x(x), \\
\frac{d^2}{dx^2}(h(u(x))) &= h_{uu}(u(x))(u_x(x))^2 + h_u(u(x))u_{xx}(x) \\
&= \frac{1}{\varepsilon^2}\Big[2(F(\alpha) - F(u(x)))h_{uu}(u(x)) - f(u(x))h_u(u(x))\Big]
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^3}{dx^3}(h(u(x))) = \frac{u_x(x)}{\varepsilon^2}\Big[&2\big(F(\alpha) - F(u(x))\big)h_{uuu}(u(x)) - 3f(u(x))h_{uu}(u(x)) \\
&- f_u(u(x))h_u(u(x))\Big].
\end{aligned}$$

The corresponding equations for h to (3.4) is given by

$$2(F(\alpha) - F(u))h_{uuu}(u) - 3f(u)h_{uu}(u) + (3f_u(u) + 4\mu)h_u(u) + 2f_{uu}(u)h(u) = 0 \tag{3.5}$$

for $u \in (\alpha_m, \alpha_M)$. By multiplying $h(u)$ with respect to (3.5) and integrating, we see that h satisfies the following equation

$$\big(F(\alpha) - F(u)\big)\big(2h_{uu}(u)h(u) - h_u(u)^2\big) - f(u)h_u(u)h(u) + 2(f_u(u) + \mu)h(u)^2 = \rho \tag{3.6}$$

for $u \in (\alpha_m, \alpha_M)$, where

$$\rho(\mu) := \frac{-f(\alpha)h_u(\alpha;\mu)h(\alpha;\mu) + 2(f_u(\alpha) + \mu)h(\alpha;\mu)^2}{2}. \tag{3.7}$$

The following proposition shows us that solution of (3.1) can be expressed by use of $h(u)$, $\rho(\mu)$ (and $u(x)$) as long as $\rho(\mu) \geq 0$.

Proposition 3.1. *Let $\varphi(x)$ be a solution of* (3.1). *Assume that* (3.5) *has a solution $h \in C^3[\alpha_m, \alpha_M]$ such that $h(\alpha) \neq 0$. Then, the following* (i) *and* (ii) *hold:*

(i) *If $\rho(\mu) = 0$, then there exists $\delta > 0$ such that $\varphi(x) = \sqrt{\dfrac{h(u(x))}{h(\alpha)}}$ for $x \in [0, \delta]$.*

(ii) *If $\rho(\mu) > 0$, then*

$$\varphi(x) = \sqrt{\frac{h(u(x))}{h(\alpha)}} \cos\left(\frac{1}{\varepsilon}\int_0^x \frac{\sqrt{\rho(\mu)}}{h(u(\xi))}d\xi\right).$$

Remark 3.1. By (3.6), we can assume that h is positive if $\rho(\mu) > 0$ and that h is non negative if $\rho(\mu) = 0$ without loss of generarity.

Now we consider the boundary condition $\varphi_x(1) = 0$ for $\varphi(x)$ as in Proposition 3.1. In the case $\rho(\mu) = 0$, $\varphi(x) = \sqrt{h(u(x))/h(\alpha)}$ depends on x only through $u_n(x)$; it is called a special solution of (3.1). For special solutions, the boundary condition will be checked in a standard way. In the case $\rho(\mu) > 0$, the condition $\varphi_x(1) = 0$ is reduced to a *characteristic equation* as follows. A direct calculation shows us that $\varphi_x(1) = 0$ if and only if

$$\frac{1}{\varepsilon}\int_0^1 \frac{\sqrt{\rho(\mu)}}{h(u_n(\xi);\mu)}d\xi = j\pi$$

for some $j \in \mathbf{N}$. In what follows we assume that $u(x)$ is n-mode solution; n is the number of the sign-change of $u'(x)$ in $(0,1)$. We introduce a *characteristic function*

$$A(\mu) := \frac{1}{2}\int_{\alpha_m}^{\alpha_M} \frac{\sqrt{\rho(\mu)}}{\sqrt{2(F(\alpha) - F(s))}h(s;\mu)}ds. \tag{3.8}$$

It follows from (3.2) and symmetry of $u(x)$ that

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^1 \frac{\sqrt{\rho(\mu)}}{h(u_n(\xi);\mu)}d\xi &= \sum_{i=1}^{n}\frac{1}{\varepsilon}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \frac{\sqrt{\rho(\mu)}}{h(u_n(\xi);\mu)}d\xi \\ &= n\cdot\frac{1}{\varepsilon}\int_{\alpha_m}^{\alpha_M}\frac{\sqrt{\rho(\mu)}}{h(s;\mu)}\cdot\frac{\varepsilon}{\sqrt{2(F(\alpha)-F(s))}}ds \\ &= 2nA(\mu).\end{aligned}$$

Then, we can show the following proposition.

Proposition 3.2. *Assume Proposition 3.1 and $\rho(\mu) > 0$. Let $u(x)$ be an n-mode solution of* (1.1) *and let φ be a function in* (ii) *of Proposition 3.1. If μ satisfies the characteristic equation*

$$A(\mu) = \frac{j\pi}{2n}$$

for some $j \in \mathbf{N}$, then φ become the $(j+1)$-th eigenfunction of (1.2).

For proofs of Propositions 3.1 and 3.2, see [11] and [12].

3.2 Proof of Theorems 4 and 5

Now we give a sketch of proofs for Theorems 4 and 5. In the case $f(u) = u(1-u^2)$, the represenation equation of the third order is given by

$$2\left(\frac{k^2}{(1+k^2)^2} - \frac{1}{2}u^2 + \frac{1}{4}u^4\right) h_{uuu} - 3u(1-u^2)h_{uu} + (3 - 9u^2 + 4\mu)h_u - 12uh = 0, \quad (3.9)$$

for $u \in (-\sqrt{2k^2/(1+k^2)}, \sqrt{2k^2/(1+k^2)})$. A standard degree argument for polynomials shows us that any polynomial solution h must satisfy $\deg h = 4$. By putting the following polynomial of the form

$$h(u) = \left(\frac{k^2}{(1+k^2)^2} - \frac{1}{2}u^2 + \frac{1}{4}u^4\right) + \mu\bar{h}(u,\mu;k), \quad \deg \bar{h} < 4$$

into (3.9), we have a particular solution

$$h_2(u;\mu,k) = \left(\frac{k^2}{(1+k^2)^2} - \frac{1}{2}u^2 + \frac{1}{4}u^4\right) + \frac{\mu}{6}(u^2-2) + \frac{1}{9}\mu^2.$$

Moreover, the corresponding $\rho(\mu) = \rho_2(\mu,k)$ defined by (3.7) is given by

$$\rho_2(\mu,k) = \frac{1}{81}\mu\left(\mu - \frac{3}{1+k^2}\right)\left(\mu - \frac{3k^2}{1+k^2}\right)\left[\mu^2 - 2\mu - \frac{3(1-k^2)^2}{(1+k^2)^2}\right].$$

Recall that

$$h_2(u;\mu,k) = \frac{1}{9(1+k^2)^2}\hat{h}_2\left(\sqrt{\frac{1+k^2}{2k^2}}u; (1+k^2)\mu, k\right)$$

and

$$\rho_2(\mu,k) = \frac{1}{81(1+k^2)^5}\hat{\rho}_2\left((1+k^2)\mu, k\right),$$

where $\hat{h}_2(w;\hat{\mu},k)$ and $\hat{\rho}_2(\hat{\mu},k)$ are given by (2.4) and (2.5).

Proof of Theorem 4. We see that $\rho_2(\mu,k) = 0$ if and only if

$$\mu = 0, \ \frac{3k^2}{1+k^2}, \ \frac{3}{1+k^2}, \ \frac{\hat{\mu}_\pm(k)}{1+k^2},$$

where $\hat{\mu}_\pm(k)$ is given by (2.6) and see that

$$h_2\left(\sqrt{\frac{2k^2}{1+k^2}};\mu,k\right) = \frac{1}{9}\mu\left(\mu - \frac{3}{1+k^2}\right) \neq 0$$

when $\mu = 3k^2/(1+k^2)$, $\hat{\mu}_\pm/(1+k^2)$. Moreover,

$$h_2\left(u;\frac{\hat{\mu}_\pm}{1+k^2},k\right) = C_1\left[1 - \frac{(1+k^2 \pm \sqrt{1-k^2+k^4})(1+k^2)}{2k^2}u^2\right]^2$$

and

$$h_2\left(u;\frac{3k^2}{1+k^2},k\right) = C_2u^2\left(\frac{2k^2}{1+k^2} - u^2\right).$$

where C_1 and C_2 are positive constants. Suppose that $\mu = 3k^2/(1+k^2)$ and $\mu = \hat{\mu}_\pm(k)$. For each case of μ, it follows from (i) of Proposition 3.1 that

$$\varphi(x) = \sqrt{|h_2(u_n(x;k),\mu,k)|}$$

is a unique local solution of the corresponding initial value problem to (2.3). Moreover, it can be extended to a global solution on $0 \le x \le 1$ and it satisfies (2.3). Hence we have three special eigenfunctions as in Theorem 4. Finally, the standard Sturm-Liouville theory completes the proof of the theorem. $\square$

Remark 3.2. $\sqrt{|h_2(u_n(x;k);\mu,k)|}$ is not an eigenfunction of (1.2) when $\mu = 0, 3/(1+k^2)$.

Proof of Theorem 5. Suppose $\mu \in \left(\frac{\hat{\mu}_-(k)}{1+k^2}, 0\right) \cup \left(\frac{3k^2}{1+k^2}, \frac{3}{1+k^2}\right) \cup \left(\frac{\hat{\mu}_+(k)}{1+k^2}, +\infty\right)$. Then we see $\rho_2(\mu,k) > 0$ and it follows from (ii) of Proposition 3.1 that

$$\varphi(x) = \sqrt{|h_2(u_n(x;k);\mu,k)|}\cos\left(\frac{1}{\varepsilon_n(k)}\int_0^x \frac{\sqrt{\rho_2(\mu,k)}}{h_2(u_n(\xi;k);\mu,k)}d\xi\right)$$

satisfies the first equation of (2.3) and $\varphi_x(0) = 0$. Proposition 3.2 implies that $\varphi(x)$ is the (j+1)-th eigenfunction if $A(\mu) = j\pi/(2n)$, where A is the characteristic function defined by (3.8). By the change of variable $w = \sqrt{(1+k^2)/(2k^2)}$, we have

$$\begin{aligned}
A(\mu) &= \int_0^{\sqrt{\frac{2k^2}{1+k^2}}} \frac{\sqrt{\rho_2(\mu,k)}}{\sqrt{\frac{2k^2}{(1+k^2)^2} - u^2 + \frac{1}{2}u^4 h_2(u;\mu,k)}}du \\
&= \int_0^1 \frac{\sqrt{\rho_2(\mu,k)}}{\sqrt{\frac{2k^2}{(1+k^2)^2} - \left(\sqrt{\frac{2k^2}{1+k^2}}w\right)^2 + \frac{1}{2}\left(\sqrt{\frac{2k^2}{1+k^2}}w\right)^4 h_2\left(\sqrt{\frac{2k^2}{1+k^2}}w;\mu,k\right)}}\cdot\sqrt{\frac{1+k^2}{2k^2}}dw \\
&= \int_0^1 \frac{\sqrt{1+k^2}\cdot\sqrt{\rho_2(\mu,k)}}{\sqrt{(1-w^2)(1-k^2w^2)h_2\left(\sqrt{\frac{2k^2}{1+k^2}}w;\mu,k\right)}}dw \\
&= \hat{A}_2(\hat{\mu},k),
\end{aligned}$$

where $\hat{\mu} := (1+k^2)\mu$. Therefore, the proof is completed by combining Lemmas 2.2 and 2.3. $\square$

4 Analysis for Characterisitic Equation

In this section we will prove key Lemmas 2.1, 2.2 and 2.3 for characteristic equations, and will prove Theorems 4 and 6. We only will prove Lemmas 2.2, 2.3 and Theorem 6 in the case of $f(u) = u(1-u^2)$ (for Lemma 2.1, see [12]). Our analysis is based on the method which is introduced by Kosugi-Morita-Yotsutani [7].

4.1 Elliptic Integrals

Let $k \in (0,1)$ and $\nu \in \mathbf{C}$. The complete elliptic integrals of the first, second and third kind are defined by

$$K(k) := \int_0^1 \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}}ds, \quad E(k) := \int_0^1 \sqrt{\frac{1-k^2s^2}{1-s^2}}ds$$

and

$$\Pi(\nu,k) := \int_0^1 \frac{1}{(1+\nu s^2)\sqrt{(1-s^2)(1-k^2s^2)}}ds, \tag{4.1}$$

respectively. For $K(k)$ (and $E(k)$), many results have been already known; for example, K is monotone increasing in k, $\lim_{k\to 0} K(k) = \pi/2$ and $\lim_{k\to 1} K(k) = +\infty$. Moreover, the change of variable $\tau = s/\sqrt{1-s^2}$ leads us to

$$K(k) = \int_0^{+\infty} \frac{1}{\sqrt{1+\tau^2}\sqrt{1+(1-k^2)\tau^2}}d\tau,$$

and (2.2) comes from the above expression of K.

On the other hand, we need to analyze $\Pi(\,\cdot\, k)$ because the characteristic functions A_1 and A_2 consist of $\Pi(\,\cdot\,,k)$.

Now we introduce some lemmas for $\Pi(\nu,k)$.

Lemma 4.1. *Let $k \in (0,1)$ and $\nu \neq 0,-1,-k^2$. Then*

$$\frac{\partial \Pi}{\partial \nu}(\nu,k) = -\frac{K(k)}{2\nu(1+\nu)} + \frac{E(k)}{2(1+\nu)(k^2+\nu)} + \frac{(k^2-\nu^2)\Pi(\nu,k)}{2\nu(1+\nu)(k^2+\nu)}.$$

Lemma 4.2. *Let $k \in (0,1)$ and $\nu > -1$. Then*

$$\lim_{\nu\to+\infty} \sqrt{1+\nu}\,\Pi(\nu,k) = \frac{\pi}{2} \quad \textit{and} \quad \lim_{\nu\to-1} \sqrt{1+\nu}\,\Pi(\nu,k) = \frac{\pi}{2\sqrt{1-k^2}}.$$

The above formulas are seen in the handbook by Byrd and Friedman [2]. We does not prove these two lemmas here (for proofs, see [11] and [12]). The following expressions of Π will be useful to derive some asymptotic formulas in Lemma 4.2, and Lemmas 4.3 and 4.4 below:

$$\Pi(\nu,k) = \int_0^{+\infty} \frac{\sqrt{1+\tau^2}}{[1+(1+\nu)\tau^2]\sqrt{1+(1-k^2)\tau^2}}d\tau \tag{4.2}$$

and

$$\begin{aligned}\Pi(\nu,k) &= \frac{1}{\sqrt{1+\nu}} \int_0^{+\infty} \frac{\sqrt{1+\nu+t^2}}{(1+t^2)\sqrt{1+\nu+(1-k^2)t^2}}dt \\ &= \frac{1}{\sqrt{1+\nu}\sqrt{1-k^2}} \int_0^{+\infty} \frac{\sqrt{1+\nu+t^2}}{(1+t^2)\sqrt{\frac{1+\nu}{1-k^2}+t^2}}dt.\end{aligned} \tag{4.3}$$

These expressions are derived from (4.1) by changing of variable $\tau = s/\sqrt{1-s^2}$ and $t = \sqrt{1+\nu}\tau$.

Lemma 4.3. *Assume that $1+\nu(k) = q(k)(1-k^2)$ as $k \to 1$, where q is a continuous function in k satisfying $q(k) \in (0,1)$ for $k \in (0,1)$ and $q(k) \to q^* \in [0,1]$ as $k \to 1$. Then, for each $q^* \in [0,1]$,*

$$\lim_{k\to 1} \sqrt{q(k)(1-q(k))}(1-k^2)\cdot\Pi(\nu(k),k) = \frac{\pi}{2} - \tan^{-1}\sqrt{\frac{q^*}{1-q^*}}.$$

Lemma 4.4. *Set* $J(\nu,k) := \sqrt{1+\nu}\,\Pi(\nu,k) - \frac{1}{\sqrt{1+\nu}}K(k)$. *Then,*

$$J(\nu,k) = \frac{\nu}{1+\nu}\int_0^{+\infty} \frac{1}{(1+t^2)\sqrt{1+\frac{1}{1+\nu}t^2}\sqrt{1+\frac{1-k^2}{1+\nu}t^2}}dt$$

and in particular,

$$\lim_{\nu\to+\infty,k\to1} J(\nu,k) = \frac{\pi}{2}.$$

The above two lemmas will be used to show Theorem 6.

Finally, consider an elliptic integral of the form:

$$\begin{aligned}\tilde{\Pi}(a,b,k) &:= \int_0^1 \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}\,[a+(b-s^2)^2]}ds \\ &= \int_0^b \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}\,[a+(b-s^2)^2]}ds \\ &\quad + \int_b^1 \frac{1}{\sqrt{(1-s^2)(1-k^2s^2)}\,[a+(b-s^2)^2]}ds \\ &= \tilde{\Pi}_1(a,b,k) + \tilde{\Pi}_2(a,b,k),\end{aligned} \tag{4.4}$$

where $a > 0$ and $b \in (0,1)$. It is easy to see that $\tilde{\Pi}$ can be expressed with $\Pi((b+\sqrt{-a})/(a+b^2),k)$ and $\Pi((b-\sqrt{-a})/(a+b^2),k)$. To avoid a treatment of elliptic integral with complex parameters, we derive some asymptotic formulas for $\tilde{\Pi}$ as follows.

Lemma 4.5. *Suppose* $a > 0$ *and* $b, b_0 \in (0,1)$. *Then for all* $k \in (0,1)$,

$$\lim_{a\to0,b\to b_0} \sqrt{a}\tilde{\Pi}(a,b,k) = \frac{\pi}{2\sqrt{b_0(1-b_0)(1-k^2b_0)}}.$$

Lemma 4.6. *Set* $\tilde{J}(a,b,k) := \sqrt{a}\tilde{\Pi}(a,b,k) - \frac{\sqrt{a}}{a+(1-b)^2}K(k)$. *Then, under assumptions in* Lemma 4.5,

$$\lim_{a\to0,b\to b_0,k\to1} \tilde{J}(a,b,k) = \frac{\pi}{2\sqrt{b_0}(1-b_0)}.$$

4.2 Proofs of Lemmas 2.2 and 2.3

Now we will give sketches of proofs of Lemmas 2.2 and 2.3. For more details, see [11].

Recall (2.6)-(2.9). In the case of $\hat{\mu} \in (\hat{\mu}_+(k), +\infty)$, $A_2(\hat{\mu},k)$ has another expression

$$A_2(\hat{\mu},k) = \frac{2\sqrt{-P(\hat{\mu},k)}}{3\sqrt{3}k^2} \cdot \sqrt{a(\hat{\mu},k)}\tilde{\Pi}(a(\hat{\mu},k), b(\hat{\mu},k), k), \tag{4.5}$$

where $\tilde{\Pi}$ is given in (4.4) and

$$a(\hat{\mu},k) = \frac{(\hat{\mu}-\hat{\mu}_+(k))(\hat{\mu}-\hat{\mu}_-(k))}{12k^4}, \quad b(\hat{\mu},k) = \frac{3(1+k^2)-\hat{\mu}}{6k^2}.$$

Sketch of Proof of Lemma 2.2. Fix $k \in (0,1)$ arbitrarily. Note that assertions in (i) and (ii) of the lemma are proved in the same way. For simplicity, we only prove $\lim_{\hat{\mu}\to 3} A_2(\hat{\mu},k) = \pi$. In the case of $\hat{\mu} \in (3k^2, 3)$ we have $-1 < \nu_+ < 0 < \nu_-$, and in particular

$$\lim_{\hat{\mu}\to\hat{3}} \nu_+(\hat{\mu},k) = -1, \quad \lim_{\hat{\mu}\to\hat{3}} \nu_-(\hat{\mu},k) = +\infty, \quad \lim_{\hat{\mu}\to 3} \frac{\partial\nu_+}{\partial\hat{\mu}}(\hat{\mu},k) = -\frac{1}{3k^4}.$$

Then, by Lemma 4.2 and the standard l'Hospital's formula, we have

$$\begin{aligned}
&\lim_{\hat{\mu}\to 3} A_2(\hat{\mu},k)\\
&= \lim_{\hat{\mu}\to 3} \frac{\sqrt{\hat{\mu}(\hat{\mu}-3k^2)}}{3\sqrt{3}k^2} \cdot \lim_{\hat{\mu}\to 3} \sqrt{\nu_-(\hat{\mu},k)(3-\hat{\mu})} \cdot \lim_{\nu_-\to+\infty} \sqrt{\frac{\nu_-}{1+\nu_-}} \sqrt{1+\nu_-}\Pi(\nu_-,k)\\
&\quad - \lim_{\hat{\mu}\to 3} \frac{\sqrt{\hat{\mu}(\hat{\mu}-3k^2)}}{3\sqrt{3}k^2} \cdot \lim_{\hat{\mu}\to 3} \sqrt{\frac{3-\hat{\mu}}{1+\nu_+(\hat{\mu},k)}} \cdot \lim_{\nu_-\to -1} \nu_+\sqrt{1+\nu_+}\Pi(\nu_+,k)\\
&= \frac{\sqrt{1-k^2}}{\sqrt{3}k^2} \cdot \frac{\sqrt{3}k^2}{\sqrt{1-k^2}} \cdot \frac{\pi}{2} + \frac{\sqrt{1-k^2}}{\sqrt{3}k^2} \cdot \sqrt{3}k^2 \cdot \frac{\pi}{2\sqrt{1-k^2}}\\
&= \pi.
\end{aligned}$$

Finally, we show the assertion in (iii) of the lemma. We can apply Lemma 4.5 to (4.5); so that

$$\begin{aligned}
\lim_{\hat{\mu}\to\hat{\mu}_+(k)} A_2(\hat{\mu},k) &= \lim_{\hat{\mu}\to\mu_+(k)} \frac{2\sqrt{-P(\hat{\mu},k)}}{3\sqrt{3}k^2} \lim_{a\to 0,\ b\to b_0} \sqrt{a}\tilde{\Pi}(a,b,k)\\
&= \frac{\sqrt{-P(\hat{\mu}_+(k),k)}}{3\sqrt{3}k^2} \cdot \frac{\pi}{\sqrt{b_0(1-b_0)(1-k^2b_0)}},
\end{aligned} \tag{4.6}$$

where

$$b_0 = \frac{3(1+k^2)-\hat{\mu}_+(k)}{6k^2} = \frac{1+k^2-\sqrt{1-k^2+k^4}}{3k^2},$$

After some calculations, we are led to that

$$b_0(1-b_0)(1-k^2b_0) = \frac{1}{27k^4}\left[(1+k^2)(2k^2-1)(2-k^2) + 2(1-k^2+k^4)^{\frac{3}{2}}\right] \tag{4.7}$$

and that

$$-P(\hat{\mu}_+(k),k) = \left[(1+k^2)(2k^2-1)(2-k^2) + 2(1-k^2+k^4)^{\frac{3}{2}}\right]. \tag{4.8}$$

By (4.6)-(4.8), we obtain $\lim_{\hat{\mu}\to\hat{\mu}_+(k)} A_2(\hat{\mu},k) = \pi$. Thus, it completes the proof. □

Sketch of Proof of Lemma 2.3. In the case of $\hat{\mu} \in (\hat{\mu}_-(k), 0)$, it follows from by Lemma 4.1 that

$$\begin{aligned}
\frac{\partial A_2}{\partial\hat{\mu}}(\hat{\mu},k) = &-\frac{\sqrt{P}}{3\sqrt{3}k^2}\left[\frac{1}{2(1+\nu_+)} - \frac{1}{2(1+\nu_-)}\right] K(k)\\
&+ \frac{\sqrt{P}}{3\sqrt{3}k^2}\left[\frac{\nu_+}{2(1+\nu_+)(k^2+\nu_+)} - \frac{\nu_-}{2(1+\nu_-)(k^2+\nu_-)}\right] E(k)\\
&+ \frac{1}{2\sqrt{P}}\left[P_{\hat{\mu}}\nu_+ + \left\{2P + \frac{P(k^2-\nu_+^2)}{(1+\nu_+)(k^2+\nu_+)}\right\}\frac{\partial\nu_+}{\partial\hat{\mu}}\right]\Pi(\nu_+,k)\\
&+ \frac{1}{2\sqrt{P}}\left[P_{\hat{\mu}}\nu_- + \left\{2P + \frac{P(k^2-\nu_-^2)}{(1+\nu_-)(k^2+\nu_-)}\right\}\frac{\partial\nu_-}{\partial\hat{\mu}}\right]\Pi(\nu_-,k),
\end{aligned}$$

where $P = P(\hat{\mu}, k)$ and $\nu_\pm = \nu_\pm(\hat{\mu}, k)$. After some calculations, we conclude that

$$\frac{\partial A_2}{\partial \hat{\mu}}(\hat{\mu}, k) = \frac{(\hat{\mu}^2 - 3k^2\hat{\mu} + 3k^2 - 3)K(k) - 3(\hat{\mu} - k^2 - 1)E(k)}{2\sqrt{P}\sqrt{-(\hat{\mu} - \hat{\mu}_+(k))(\hat{\mu} - \hat{\mu}_-(k))}}. \tag{4.9}$$

For the other two cases of $\hat{\mu}$, $\partial A_2/\partial\hat{\mu}$ are given similarly as in (4.9).

Set

$$G(\hat{\mu}; k) := (\hat{\mu}^2 - 3k^2\hat{\mu} + 3k^2 - 3)K(k) - 3(\hat{\mu} - k^2 - 1)E(k)$$

for $\hat{\mu} \in \mathbf{R}$. Then, we can show the following facts (see [11]):

(i) $G_1(k) := G(0; k) = 3(k^2 - 1)K(k) + 3(k^2 + 1)E(k) \geq 0$,

(ii) $G_2(k) := G(3k^2; k) = 3(k^2 - 1)K(k) - 3(2k^2 - 1)E(k) \leq 0$,

(iii) $G_3(k) := G(3; k) = 6(1 - k^2)K(k) - 3(2 - k^2)E(k) \leq 0$,

(iv) $G_4(k) := \dfrac{G(\hat{\mu}_+(k); k)}{2\sqrt{1 - k^2 + k^4}} = (2 - k^2 + \sqrt{1 - k^2 + k^4})K(k) - 3E(k) \geq 0$

for $k \in (0, 1)$. These facts implies that $\partial A_2/\partial\hat{\mu}$ does not vanishes in

$$\{(\hat{\mu}, k) \mid k \in (0, 1),\ \mu \in (\hat{\mu}_-(k), 0) \cup (3k^2, 3) \cup (\hat{\mu}_+(k), +\infty)\}.$$

Thus it completes the proof.

□

4.3 Sketch of Proof of Theorem 6

Sketch of Proof of Theorem 6. Throughout the proof, we denote $\hat{\mu}(k) := (1 + k^2)\mu_j^n(k)$ for simplicity.

(i) Set $\hat{\mu}(k) = r(k)\hat{\mu}_-(k)$. We see from $\hat{\mu}_-(k) < \hat{\mu}(k) < 0$ that $r(k) \in (0, 1)$. Note that

$$\hat{\mu}_-(k) = -\frac{3(1 - k^2)^2}{1 + k^2 + 2\sqrt{1 - k^2 + k^4}} = -\frac{3}{4}(1 - k^2)^2 + o((1 - k^2)^2).$$

Then,

$$1 + \nu_\pm(k) = \frac{1 \pm \sqrt{1 - r(k)}}{2}(1 - k^2) + o(1 - k^2)$$

and

$$\frac{\sqrt{P(\hat{\mu}(k), k)}}{3\sqrt{3k^2}} = \frac{\sqrt{r(k)}}{2}(1 - k^2) + o(1 - k^2).$$

Now we choose a monotone increasing sequence $\{k_m\}_{m=1}^\infty$ such that k_m and $r(k_m)$ converge to 1 and r^* as $m \to \infty$, respectively. From Lemma 4.3,

$$\begin{aligned}
\lim_{m\to\infty} A_2(\hat{\mu}(k_m), k_m) &= -\lim_{m\to\infty} \frac{\sqrt{r(k_m)}}{2}(1 - k_m^2)\Pi(\nu_+(k_m), k) \\
&\quad + \lim_{m\to\infty} \frac{\sqrt{r(k_m)}}{2}(1 - k_m^2)\Pi(\nu_-(k_m), k) \\
&= -\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{1 + \sqrt{1 - r^*}}{1 - \sqrt{1 - r^*}}}\right) + \left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{1 - \sqrt{1 - r^*}}{1 + \sqrt{1 - r^*}}}\right) \\
&= 2\tan^{-1}\sqrt{\frac{1 + \sqrt{1 - r^*}}{1 - \sqrt{1 - r^*}}} - \frac{\pi}{2}.
\end{aligned}$$

Therefore, by solving the following equation

$$2\tan^{-1}\sqrt{\frac{1+\sqrt{1-r^*}}{1-\sqrt{1-r^*}}}-\frac{\pi}{2}=\frac{j\pi}{2n}$$

with respect to r^*, we obtain $r^* = \cos^2 j\pi/(2n)$. It also implies th at $\lim_{k\to 1} r(k) = \cos^2 j\pi/(2n)$ and it concludes the assertion of (i).

(ii) In the same way as (i), set $\hat{\mu}(k) = 3 - 3r(k)(1-k^2)$. We see that $0 < r(k) < 1$,

$$1+\nu_+(k) = r(k)(1-k^2) + o(1-k^2)$$

and

$$\frac{\sqrt{P(\hat{\mu}(k),k)}}{3\sqrt{3}k^2} = \sqrt{r(k)(1-r(k))}(1-k^2) + o(1-k^2).$$

From Lemma 4.4, we have

$$\begin{aligned}
A_2(\hat{\mu}(k),k) &= \frac{\sqrt{P(\hat{\mu}(k),k)}}{3\sqrt{3}k^2}\nu_-(k)\Pi(\nu_-(k),k) - \frac{\sqrt{P(\hat{\mu}(k),k)}}{3\sqrt{3}k^2}\nu_+(k)\Pi(\nu_+(k),k)\\
&= \left[\frac{\nu_-}{1+\nu_-}\frac{\sqrt{P(\hat{\mu}(k),k)}}{3\sqrt{3}k^2}K(k) + \frac{\sqrt{P(\hat{\mu}(k),k)\nu_-(k)}}{3\sqrt{3}k^2}\sqrt{\frac{\nu_-(k)}{1+\nu_-(k)}}J(\nu_-(k),k)\right]\\
&\quad - \frac{\sqrt{P(\hat{\mu}(k),k)}}{3\sqrt{3}k^2}\nu_+(k)\Pi(\nu_+(k),k).
\end{aligned}$$

In the same way as in (i), we take $\{k_m\}$ and $r(k_m)$; so that

$$\lim_{m\to\infty} A_2(\hat{\mu}(k_m),k_m) = \frac{\pi}{2} + \left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{r^*}{1-r^*}}\right) = \pi - \tan^{-1}\sqrt{\frac{r^*}{1-r^*}}.$$

By solving the equation

$$\pi - \tan^{-1}\sqrt{\frac{r^*}{1-r^*}} = \frac{j\pi}{2n},$$

we have $r^* = \cos^2(j-n)\pi/(2n)$. It concludes the proof of (ii).

(iii) Note that $\hat{\mu}(k) \to 4$ as $k \to 1$. By Lemma 4.6,

$$\begin{aligned}
A_2(\hat{\mu}(k),k) &= \frac{2\sqrt{-P(\hat{\mu}(k),k)}}{3\sqrt{3}k^2}\frac{\sqrt{a(\hat{\mu}(k),k)}}{a(\hat{\mu}(k),k)+(1-b(\hat{\mu}(k),k))^2}K(k)\\
&\quad + \frac{2\sqrt{-P(\hat{\mu}(k),k)}}{3\sqrt{3}k^2}\tilde{J}(a(\hat{\mu}(k),k),b(\hat{\mu}(k),k),k).
\end{aligned} \tag{4.10}$$

and it leads us to that

$$\begin{aligned}
\sqrt{\hat{\mu}(k)-\hat{\mu}_+(k)}K(k) &= \left(-J(a(\hat{\mu}(k),b(\hat{\mu}(k),k) + \frac{3\sqrt{3}k^2}{2\sqrt{-P(\hat{\mu}(k),k)}}\cdot\frac{j\pi}{2n}\right)\\
&\quad \cdot\frac{2\sqrt{3}k^2[a(\hat{\mu}(k),k)+(1-b(\hat{\mu}(k),k))^2]}{\sqrt{\hat{\mu}(k)-\hat{\mu}_-(k)}}.
\end{aligned}$$

By letting $k \to 1$, we have

$$\lim_{k\to 1}\sqrt{\hat{\mu}(k)-\hat{\mu}_+(k)}K(k) = \frac{(j-2n)\pi}{2n}.$$

Thus, it completes the proof. $\square$

5 Concluding Remarks

At the end, we give some remarks on profiles of eigenfunctions φ_j^n when k is sufficiently close to 1 (ε is sufficiently small). Fix $n \in \mathbf{N}$ arbitrarily, and denote zeros of n-mode solution u_n by

$$z_i := \frac{2i-1}{2n} \quad (i = 1, 2, \cdots, n).$$

In the case of $f(u) = \sin u$, we give a conjecture on profiles of eigenfunctions below.

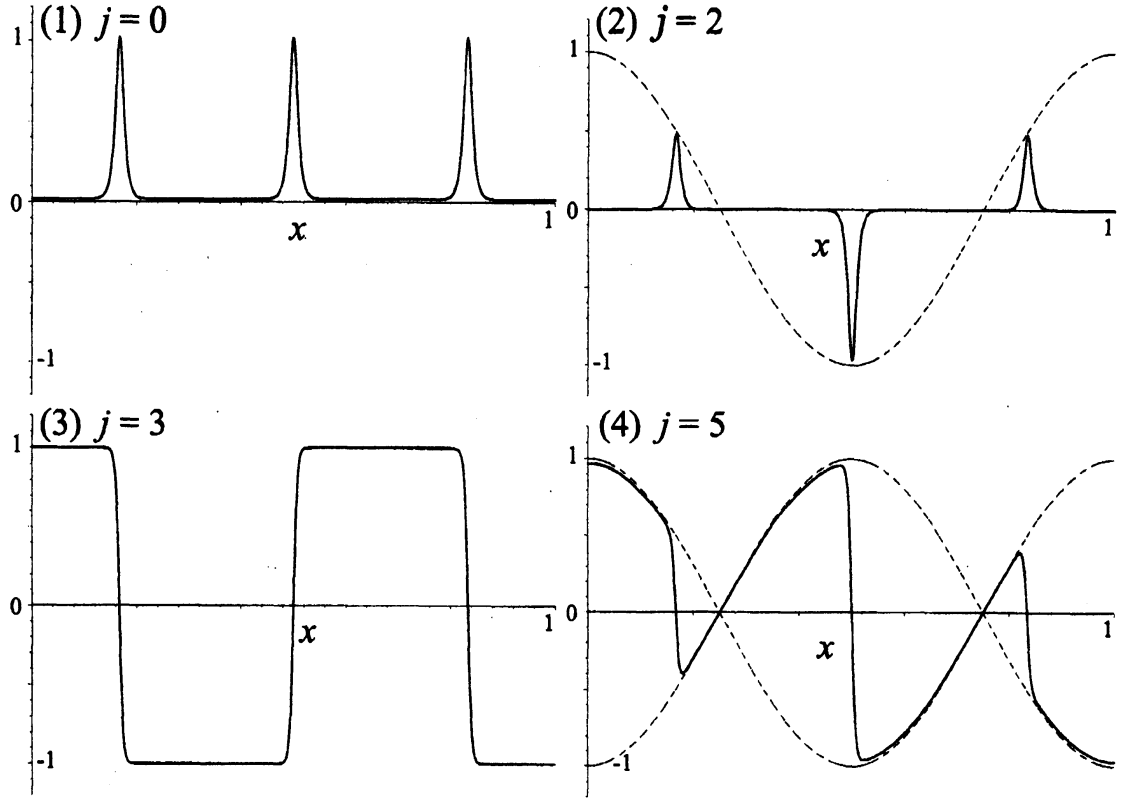

Figure 4: Graphs of eigenfunctions for case $f(u) = \sin u$ ($n = 3$ and $k = 1 - 10^{-20}$): (1) $\varphi_0^3(x;k)$, (2) $\varphi_2^3(x;k)$, (3) $\varphi_2^3(x;k)$, (4) $\varphi_5^3(x;k)$.

Profiles of two special eigenfunctions φ_0^n and φ_n^n are given in (1) and (3) of Fig. 4; they have spikes and transition layers in neighborhoods of $\{z_i\}$, respectively. An observation on Theorems 2, 3 and numerical simulations shows us formally the following asymptotic formulas:

(i) if $0 < j < n$, then $\varphi_j^n(x;k) \sim \varphi_0^n(x;k) \cos j\pi x$ (Fig.4 (2)),

(ii) if $j > n$, then $\varphi_j^n(x;k) \sim \varphi_n^n(x;k) \cos(j-n)\pi x$ (Fig.4 (4)),

where $A \sim B$ means that A is close to B in a certain sense. In the case of $0 < j < n$, E. Yanagida shows us a conjecture for general $f(u)$ that φ_j^n has spikes in neighborhoods of z_i ($i = 1, 2, \cdots, n$); the height of each spike is proportional to $\cos j\pi z_i$. The above formal result implies that Yanagida's conjecture holds for $f(u) = \sin u$, since $\varphi_0^n(x;k) = \cos(u_n(x;k)/2)$ has spikes with the same height at z_i ($i = 1, 2, \cdots, n$).

We also show a conjectures on profiles of eigenfunctions in the case of $f(u) = u(1-u^2)$:

(i) if $0 < j < n$, then $\varphi_j^n(x;k) \sim \varphi_0^n(x;k)\cos j\pi x$,

(ii) if $n < j < 2n$, then $\varphi_j^n(x;k) \sim \varphi_n^n(x;k)\cos(j-n)\pi x$,

(iii) if $j > 2n$, then $\varphi_j^n(x;k) \sim \varphi_{2n}^n(x;k)\cos(j-2n)\pi x$.

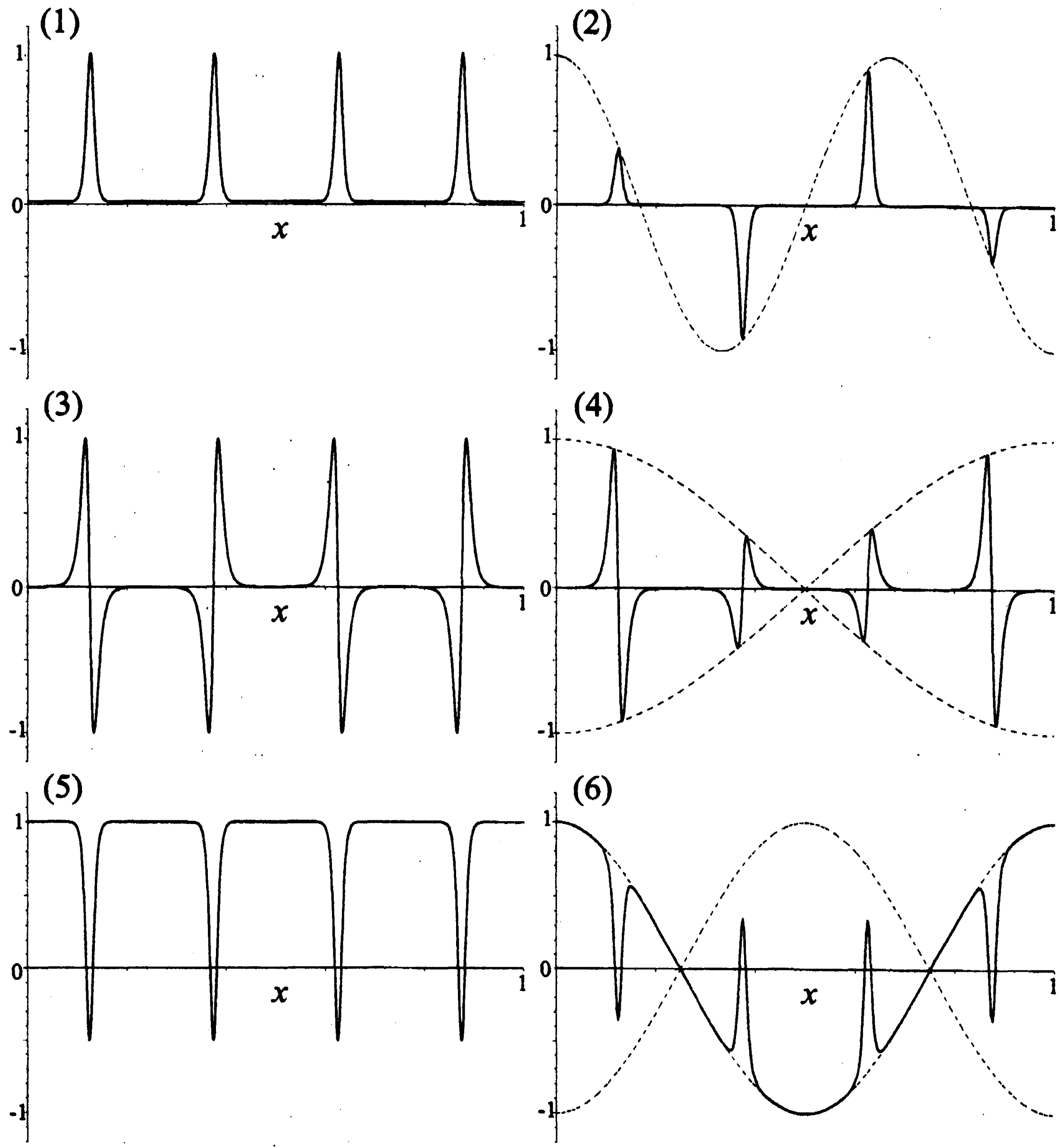

Figure 5: Graphs of eigenfunctions for case $f(u) = u(1-u^2)$ ($n = 4$ and $k = 1 - 10^{-10}$): (1) $\varphi_0^4(x;k)$, (2) $\varphi_3^4(x;k)$, (3) $2\varphi_4^4(x;k)$, (4) $2\varphi_5^4(x;k)$, (5) $\varphi_8^4(x;k)/2$, (6) $\varphi_{10}^4(x;k)/2$.

For a certain class of $f(u)$, we suspect that similar properties as in two cases hold for eigenfunctions arising from linearized eigenvalue problems. In the forthcoming papers, we will show asymptotic formulas of eigenfunctions rigorously and consider the above general conjecture.

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