数理解析研究所講究録
第 1591 巻 2008 年 159-163

A remark on the mean field equation for equilibrium vortices with arbitrary sign

Tonia Ricciardi

Dipartimento di Matematica e Applicazioni "R. Caccioppoli"

Università di Napoli Federico II

Via Cintia, 80126 Napoli, Italy

tonia.ricciardi@unina.it

Abstract

We consider the problem:

$$-\Delta u=\lambda\left(\frac{e^{u}}{\int_{\Omega}e^{u}\,\mathrm{d}x}-\frac{e^{-u}}{\int_{\Omega}e^{-u}\,\mathrm{d}x}\right)\qquad \text{in } \Omega,\qquad u=0\qquad \text{on } \partial\Omega$$

where Ω is a bounded domain in $\mathbb{R}^2$ and $\lambda>0$. We show that uniqueness of the trivial solution $u\equiv 0$ may *not* be expected when $\lambda>j_0^2\pi\approx 5.76\pi$, where $j_0\approx 2.40$ denotes the first zero of the Bessel function of the first kind of order zero. This result is related to recent studies by Sawada, Suzuki and Takahashi.

In the recent article [5], Sawada, Suzuki and Takahashi considered the following problem:

$$(1)\qquad -\Delta u=\lambda\left(\frac{e^{u}}{\int_{\Omega}e^{u}\,\mathrm{d}x}-\frac{e^{-u}}{\int_{\Omega}e^{-u}\,\mathrm{d}x}\right)\qquad \text{in } \Omega,\qquad u=0\qquad \text{on } \partial\Omega,$$

where Ω is a bounded open subset of $\mathbb{R}^2$ and $\lambda>0$ is a constant. Equation (1) was derived by Joyce and Montgomery [2] and Pointin and Lundgren [4] in the context of the statistical mechanics description of two-dimensional turbulence.

We note that (1) always admits the trivial solution $u=0$. Thus, uniqueness of the trivial solution for (1) is a natural question. In this direction, the following results were obtained in [5]:

Theorem 1 ([5]). *If Ω is simply connected and $0 < \lambda \leq 4\pi$, then* (1) *does not admit any nontrivial solution.*

On the other hand, the following uniform estimate holds:

Theorem 2 ([5]). *For every $\varepsilon > 0$ there exists $C > 0$ such that any solution to* (1) *with $0 < \lambda \leq 8\pi - \varepsilon$ satisfies:*

$$\|u\|_\infty \leq C.$$

It is clear that the value 8π is related to the blow-up of solutions. For example, Bartolucci and Pistoia [1] recently constructed a family of blow-up solutions u_ρ to the problem

$$-\Delta u = \rho(e^u - e^{-u}) \qquad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega,$$

such that $\rho \int_\Omega e^{u_\rho}\, dx \to 8\pi$, $\rho \int_\Omega e^{-u_\rho}\, dx \to 8\pi$ as $\rho \to 0$, thus providing evidence that blow-up solutions to (1) should exist near $\lambda = 8\pi$. We also recall that in view of the well-known uniqueness result of Suzuki [7], the related problem

$$(2) \qquad -\Delta u = \lambda \frac{e^u}{\int_\Omega e^u\, dx} \qquad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega$$

admits a unique solution when $\lambda \in (0, 8\pi]$ and Ω is simply connected. Thus, the following question is natural:

Question 1. *What happens for $\lambda \in (4\pi, 8\pi]$?*

It is shown in [5] that in general uniqueness may *not* be expected in the whole interval $(0, 8\pi)$, unlike what happens for problem (2). Indeed, when Ω is the unit disk, a branch of nontrivial solutions bifurcates at

$$\lambda^* = \frac{j_1^2\pi}{2} \approx 7.34\pi < 8\pi.$$

Here, $j_1 \approx 3.83$ denotes the first positive zero of J_1, the Bessel function of the first kind of order one. Thus, as a first step towards answering Question 1, we may ask:

Question 2. *Is $u \equiv 0$ the unique solution for problem* (1) *when $\lambda \in (0, \lambda^*]$ and Ω is simply connected?*

In what follows, we will show that the answer to Question 2 is *negative*. Indeed, we will show that if Ω consists of two equal disjoint disks joined by a "thin corridor" K_ε, such that $|K_\varepsilon| = o(1)$, then bifurcation of a branch of nontrivial solutions occurs at $\lambda_\varepsilon = j_0^2\pi + o(1) < j_1^2\pi/2$. Here, $j_0 \approx 2.40$ denotes the first zero of J_0, the Bessel function of the first kind of order zero, and $o(1)$ is a quantity which vanishes as $\varepsilon \longrightarrow 0$. We set

$$\lambda^{**} = j_0^2\pi \approx 5.76\pi < \lambda^* = \frac{j_1^2\pi}{2} \approx 7.34\pi.$$

We have the following.

Theorem 3. *For every $\varepsilon > 0$ let $\Omega_\varepsilon = B_1 \cup B_2 \cup K_\varepsilon$, where $B_1 = B(p_1, \sqrt{2}/2)$, $B_2 = B(p_2, \sqrt{2}/2)$ are disjoint disks centered at $p_1 = (-1,0)$, $p_2 = (1,0)$ and $K_\varepsilon = [-1,1] \times [-\varepsilon, \varepsilon]$. Then, problem (1) with $\Omega = \Omega_\varepsilon$ admits a branch of nontrivial solutions bifurcating from $\lambda_\varepsilon = \lambda^{**} + o(1)$.*

The proof of Theorem 3 relies on the analysis of the linearization of (1) at $u = 0$, which is given by:

$$-\Delta\phi = \mu\left(\phi - \frac{1}{|\Omega|}\int_\Omega \phi\,\mathrm{d}x\right) \qquad \text{in } \Omega, \qquad \phi = 0 \qquad \text{on } \partial\Omega, \tag{3}$$

with $\mu = 2\lambda/|\Omega|$. In fact, problem (3) may also be viewed as the linearization about $u \equiv 0$ of the more general problem

(4)

$$-\Delta u = \lambda_1\left(\frac{e^u}{\int_\Omega e^u\,\mathrm{d}x} - \frac{1}{|\Omega|}\right) - \lambda_2\left(\frac{e^{-u}}{\int_\Omega e^{-u}\,\mathrm{d}x} - \frac{1}{|\Omega|}\right) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

In this case $\mu = (\lambda_1 + \lambda_2)/|\Omega|$. Let $H_c^1(\Omega) \equiv H_0^1(\Omega) \oplus \mathbb{R}$ denote the set of functions in $H^1(\Omega)$ which are constant on $\partial\Omega$ (in the sense of $H_0^1(\Omega)$). Setting $\psi = \phi - |\Omega|^{-1}\int_\Omega \phi\,\mathrm{d}x$, one may check that the first eigenvalue for (3) is given by the minimization problem:

$$\mu_1(\Omega) = \inf\left\{\frac{\|\nabla\psi\|_{2,\Omega}^2}{\|\psi\|_{2,\Omega}^2} : \ \psi \in H_c^1(\Omega) \setminus \{0\}, \ \int_\Omega \psi\,\mathrm{d}x = 0\right\}. \tag{5}$$

See, e.g., [5, 7]. We shall use the following result, which was proved by Lucia [3] by symmetrization techniques.

Theorem 4 ([3]). *The following estimate holds:*

$$\mu_1(\Omega) \geq \frac{2j_0^2\pi}{|\Omega|}.$$

Equality holds if and only if Ω is the disjoint union of two equal disks. Moreover, if $\Omega = B_1 \cup B_2$ with $B_1 = B(p_1, \sqrt{2}/2)$, $B_2 = B(p_2, \sqrt{2}/2)$ with $p_1, p_2 \in \mathbb{R}^2$ such that $B_1 \cap B_2 = \emptyset$, then up to a constant factor the first eigenfunction is given by

$$\psi_1(x) = \begin{cases} J_0(\sqrt{2}j_0|x-p_1|), & \text{if } x \in B_1 \\ -J_0(\sqrt{2}j_0|x-p_2|), & \text{if } x \in B_2 \end{cases}.$$

We note that Theorem 4 was used in [3] in the somewhat different context of proving the existence of mountain pass solutions to problem (4) with $\lambda_2 = 0$. With this motivation, the relevant consequence of Theorem 4 is that $\mu_1(\Omega)|\Omega| \geq 2j_0^2\pi > 8\pi$, for any bounded open set Ω. Hence, the method of Struwe and Tarantello [6] may be applied. On the other hand, the relevant consequence of Theorem 4 in our case is that for $\Omega = B_1 \cup B_2$, we have $\mu_1(\Omega)|\Omega|/2 = j_0^2\pi < j_1^2\pi/2$. Hence, in this case, a branch of nonzero solutions bifurcates from $\lambda^{**} = j_0^2\pi$. This fact proves that uniqueness of the zero solution to problem (1) may not be expected for $\lambda > \lambda^{**}$, if we allow Ω to be disconnected. So, in order to complete the proof of Theorem 3 we are left to show that even if we require Ω to be simply connected, uniqueness of the zero solution may not be expected for $\lambda > \lambda^{**}$.

Proof of Theorem 3 completed. Let $\overline{\psi} \in H_c^1(\Omega_\varepsilon)$ be the function defined by:

$$\overline{\psi}(x) = \begin{cases} \psi_1(x), & \text{if } x \in B_1 \cup B_2 \\ 0, & \text{otherwise} \end{cases}.$$

Then, in view of Theorem 4 and the fact $|\Omega_\varepsilon| = \pi + o(1)$, we have

$$\mu_1(\Omega_\varepsilon) \geq \frac{2j_0^2\pi}{|\Omega_\varepsilon|} = \frac{2j_0^2\pi}{\pi + o(1)}.$$

On the other hand, using $\overline{\psi}$ as a test function in (5), we have

$$\mu_1(\Omega_\varepsilon) \leq \frac{\|\nabla\overline{\psi}\|_{2,\Omega_\varepsilon}^2}{\|\overline{\psi}\|_{2,\Omega_\varepsilon}^2} = \frac{\|\nabla\psi_1\|_{2,B_1\cup B_2}^2}{\|\psi_1\|_{2,B_1\cup B_2}^2} = 2j_0^2.$$

Hence, we conclude that $\mu_1(\Omega_\varepsilon) \to 2j_0^2$ as $\varepsilon \to 0$. □

Acknowledgements

I thank Professors Kimie Nakashima, Takashi Suzuki and Futoshi Takahashi for offering me the opportunity to participate in the Kyoto RIMS Symposium *Variational Problems and Related Topics*, June 19–21, 2007.

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