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A New Approach to the Null Condition for Nonlinear Wave Equations

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1. Introduction

This article is based on the joint work [12] with Prof. Hideo Kubo (Osaka University).

We consider the Cauchy problem for the following system of semilinear wave equations:

$$\square u_i\big(:= (\partial_t^2 - \Delta_x)u_i\big) = F_i(\partial u) \text{ in } (0,\infty)\times\mathbb{R}^3\ (1\le i\le N) \tag{1.1}$$

with small initial data

$$u = \varepsilon f \text{ and } \partial_t u = \varepsilon g \text{ at } t = 0, \tag{1.2}$$

where $u = (u_j)_{1\le j\le N}$, and $\partial u = (\partial_a u_j)_{0\le a\le 3, 1\le j\le N}$. Here we have set $\partial_0 = \partial_t$ and $\partial_k = \partial_{x_k}$ $(k = 1,2,3)$. ε is a small and positive parameter. We suppose

$$F(\partial u) = (F_i(\partial u))_{1\le i\le N} = O(|\partial u|^2) \text{ near } \partial u = 0. \tag{1.3}$$

The contents of this article can be extended to the quasi-linear case with nonlinearity of higher order depending also on u as well as its derivatives, but for the sake of simplicity, we restrict our statement to the above semilinear system (1.1).

We are interested in global existence of classical solutions for small initial data, because some solutions blow up in finite time no matter how small ε is. In fact, John [8] considered

$$\square u = (\partial_t u)^2 \text{ in } (0,\infty)\times\mathbb{R}^3 \tag{1.4}$$

with initial data (1.2), and showed that there exist f and $g \in C_0^\infty(\mathbb{R}^3)$ such that the solution to (1.4)–(1.2) blows up in finite time for any $\varepsilon > 0$. Hence we have to impose some special condition on nonlinearity to get global existence of small solutions. The *null condition*, introduced by Klainerman [15], is one of such conditions:

Definition 1.1 (The null condition). *We say that* $F = (F_i)_{1\le i\le N}$ *satisfies the null condition, if each* F_i $(1 \le i \le N)$ *satisfies*

$$F_i^{(2)}\left((\omega_a \mu_j)_{0\le a\le 3, 1\le j\le N}\right) = 0$$

for any $\mu = (\mu_j)_{1\le j\le N} \in \mathbb{R}^N$, *and any* $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ *with* $\omega_0 = -1$, *where* $F_i^{(2)}$ *denotes the quadratic part of* F_i, *namely we have* $F(\partial u) - F^{(2)}(\partial u) = O(|\partial u|^3)$ *around* $\partial u = 0$.

The null condition is associated with the *null forms* Q_0 and Q_{ab}, which are defined by

$$Q_0(v, w) = (\partial_t v)(\partial_t w) - (\nabla_x v)\cdot(\nabla_x w), \tag{1.5}$$

$$Q_{ab}(v, w) = (\partial_a v)(\partial_b w) - (\partial_b v)(\partial_a w) \quad (0 \le a < b \le 3). \tag{1.6}$$

It is known that F satisfies the null condition, if and only if each F_i $(1 \le i \le N)$ has the form

$$\begin{aligned} F_i(\partial u) = &\sum_{1\le j,k\le N} A_i^{jk} Q_0(u_j, u_k) + \sum_{\substack{1\le j,k\le N\\ 0\le a<b\le 3}} B_i^{jk,ab} Q_{ab}(u_j, u_k) \\ &+ O(|\partial u|^3) \end{aligned} \tag{1.7}$$

with suitable constants A_i^{jk} and $B_i^{jk,ab}$.

Klainerman and Christodoulou proved the following global existence theorem independently by different methods.

Theorem 1.2 (Klainerman [15], Christodoulou [3]). *Suppose that* F *satisfies the null condition. Then, for any* f, $g \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N)$, *there exists a positive constant* ε_0 *such that the Cauchy problem* (1.1) – (1.2) *admits a unique global solution* $u \in C^\infty([0,\infty)\times\mathbb{R}^3; \mathbb{R}^N)$ *for any* $\varepsilon \in (0, \varepsilon_0]$.

Christodoulou's method is called the *conformal mapping method*, where the problem for $[0,\infty)\times\mathbb{R}^3$ is transformed into that for $[0,\pi)\times S^3$. On the other hand, Klainerman's method is called the *vector fields method*, where the following vector fields are used:

$$L_j = x_j\partial_t + t\partial_j \ (1 \le j \le 3),\ \Omega_{jk} = x_j\partial_k - x_k\partial_j \ (1 \le j < k \le 3),$$
$$S = t\partial_t + x\cdot\nabla_x.$$

These vector fields not only play an important role in Klainerman's weighted L^1–L^∞ estimate for wave equations (which was later extended by Hörmander [5]), but reveal the special structure of the null forms.

To explain this structure, first we observe

$$\begin{aligned} Q_0(u_j, u_k) =& \frac{1}{2}\{(D_+u_j)(D_-u_k) + (D_-u_j)(D_+u_k)\} \\ &+ \frac{1}{r} \sum_{1\le \ell, m\le 3; \ell\ne m} \omega_m(\partial_\ell u_j)(\Omega_{\ell m}u_k) \end{aligned} \tag{1.8}$$

for $1 \le j, k \le N$, where $D_\pm = \partial_t \pm \partial_r$, $\partial_r = \sum_{j=1}^3 \omega_j\partial_j$ and $\omega_j = x_j/|x|$ for $j = 1, 2, 3$. Here we have set $\Omega_{\ell m} = -\Omega_{m\ell}$ for $1 \le m < \ell \le 3$.

We define $L_r = \sum_{j=1}^3 \omega_j L_j = r\partial_t + t\partial_r$ with $r = |x|$. Then we have

$$D_+ = \frac{1}{t+r}(S + L_r), \tag{1.9}$$

$$\Omega_{jk} = \frac{1}{t}(x_jL_k - x_kL_j). \tag{1.10}$$

We also define $L = (L_1, L_2, L_3)$, and $\Omega = (\Omega_{12}, \Omega_{13}, \Omega_{23})$.

From (1.8), (1.9) and (1.10), we find

$$|Q_0(u_j, u_k)| \le C(1 + t + r)^{-1}|\Gamma u|\,|\partial u| \tag{1.11}$$

where $\Gamma u = (Su, Lu, \Omega u, \partial u)$. In fact, $|Q_0(u_j, u_k)| \le C|\partial u|^2$ is triviality, since Q_0 is quadratic. (1.8) and (1.9) yield $r|Q_0(u_j, u_k)| \le C|\Gamma u|\,|\partial u|$. If we further use (1.10) to replace $\Omega_{\ell m}$ in (1.8), we also find $t|Q_0(v, w)| \le C|\Gamma u|\,|\partial u|$.

(1.11) holds also for other null forms. To make use of (1.11), we need some estimate for Γu, as well as ∂u. It is easy to obtain

$$[S, \Box] = -2\Box, \ [L_j, \Box] = [\Omega_{jk}, \Box] = [\partial_a, \Box] = 0,$$

where $[A, B] = AB - BA$ for operators A and B. Hence, decay estimates for the solution u to the wave equation immediately imply decay estimates for Γu. For the solution v to the homogeneous wave equation $\Box v = 0$ in $(0, \infty) \times \mathbb{R}^3$, we have

$$|v(t, x)| \le C(1 + t + r)^{-1}(1 + |t - r|)^{-\rho}, \tag{1.12}$$

$$|\partial v(t, x)| \le C(1 + t + r)^{-1}(1 + |t - r|)^{-\rho-1} \tag{1.13}$$

with some $\rho \geq 0$ depending on decay rate of the initial data at the spatial infinity (if the initial data are compactly supported, ρ can be as large as we want, because of the strong Huygens principle which is true for odd space dimensions).

Since we are considering small solutions to nonlinear wave equations, we may expect that the solutions to the nonlinear equations behave like those to the homogeneous equations (of course this expectation is *not always* true, as is shown by the blow-up result mentioned above). When this expectation is true, we find that the null forms behave like $C(1+t+r)^{-3}(1+|t-r|)^{-1-2\rho}$ in view of (1.11), while general quadratic terms of ∂u is expected to behave like $C(1+t+r)^{-2}(1+|t-r|)^{-2-2\rho}$. In other words, the null forms gain extra decay $(1+t+r)^{-1}(1+|t-r|)$ compared to general quadratic forms. Observe that, away from the light cone $t = r$, this gain is nothing. More precisely, if $|t-r| \geq \delta t$ with some small $\delta (> 0)$, then we have $1+t+r \leq C(1+|t-r|)$, and $(1+t+r)^{-1}(1+|t-r|)$ is bounded from above and below. Hence we can say that (1.11) plays its essential role only in a neighborhood of the light cone. This observation is important when we consider systems of wave equations with *multiple* propagations speeds.

Set $\Box_c = \partial_t^2 - c^2\Delta_x$ for $c > 0$, and consider the following system of wave equations with multiple propagation speeds:

$$\Box_{c_i} u_i = F_i(\partial u) \text{ in } (0,\infty) \times \mathbb{R}^3 \tag{1.14}$$

for $i = 1, \ldots, N$, where c_i's are given positive constants. We still have

$$[S, \Box_{c_i}] = -2\Box_{c_i}, \ [\Omega_{jk}, \Box_{c_i}] = [\partial_a, \Box_{c_i}] = 0.$$

Hence these vector fields are favorable also for the multiple speeds case. However, we have $[L_j, \Box_c] = 0$ if and only if $c = 1$. Of course, we can modify the definition of L_j's to get $[L_j, \Box_c] = 0$ for given c, but this modification depends on the propagation speed c, and does not work for systems with multiple propagation speeds. Thus we need the vector fields method without L_j's.

This kind of vector fields method was studied by many authors (see Kovalyov [18, 19], Klainerman – Sideris [17], Yokoyama [26], Kubota – Yokoyama [20], Sideris – Tu [24], Sogge [25], Hidano [4], the author [9–11], Katayama – Yokoyama [14] for example). In [4, 9, 10, 14, 19, 20, 24–26], global existence results for the multiple speeds case are treated. If we restrict our attention to (1.14), the null condition for the multiple speeds case is satisfied if and only if each F_i has the form

$$F_i(\partial u) = N_i(\partial u) + R_i^I(\partial u) + R_i^{II}(\partial u) + O(|\partial u|^3), \tag{1.15}$$

$$N_i(\partial u) = \sum_{j,k;\, c_j=c_k=c_i} \left(A_i^{jk} Q_0(u_j, u_k; c_i) + B_i^{jk,ab} Q_{ab}(u_j, u_k)\right) \tag{1.16}$$

$$R_i^I(\partial u) = \sum_{j,k;\, c_j=c_k\neq c_i} C_i^{jk,ab}(\partial_a u_j)(\partial_b u_k), \tag{1.17}$$

$$R_i^{II}(\partial u) = \sum_{j,k;\, c_j\neq c_k} D_i^{jk,ab}(\partial_a u_j)(\partial_b u_k) \tag{1.18}$$

with some constants A_i^{jk}, $B_i^{jk,ab}$, $C_i^{jk,ab}$ and $D_i^{jk,ab}$, where

$$Q_0(v, w; c) = (\partial_t v)(\partial_t w) - c^2(\nabla_x v)\cdot(\nabla_x w) \tag{1.19}$$

(among the above works, the case

$$F_i = N_i(\partial u) + R_i^I(\partial u) + R_i^{II}(\partial u) + O(|u|^3 + |\partial u|^3)$$

is treated in [9,20]; even some quadratic nonlinearities depending both on u and ∂u are handled in [10, 14], but we do not go into further details here).

The new features for the multiple speeds case are the *nonresonant terms* R_i^I and R_i^{II}. The reason why these terms behave better can be explained by the decay property for ∂u. Similarly to the single speed case, we can expect

$$|\partial u_j(t,x)| \leq C(1+t+r)^{-1}(1+|c_j t - r|)^{-1-\rho} \tag{1.20}$$

with some $\rho \geq 0$, which yields

$$|R_i^{II}(\partial u)| \leq (1+t+r)^{-3-\rho}\Big(\min_{1\leq \ell\leq N}(1+|c_\ell t - r|)\Big)^{-1} \tag{1.21}$$

because we have

$$\begin{aligned}&(1+|c_j t - r|)^{-1}(1+|c_k t - r|)^{-1}\\ &\qquad \leq C(1+t+r)^{-1}\left(\min\{1+|c_j t - r|, 1+|c_k t - r|\}\right)^{-1}\end{aligned}$$

for $c_j \neq c_k$. Hence there is some gain in the decay rate for R_i^{II}, compared to general quadratic terms. It is rather difficult to explain the advantage of R_i^I; there is no gain in its universal decay rate, but R_i^I behaves like

$$|R_i^I(\partial u)| \leq C(1+t+r)^{-4-2\rho} \tag{1.22}$$

near the light cone $c_i t = r$, because R_i^I consists of ∂u_j whose propagation speed c_j is not equal to c_i. This property is helpful, because the principal influence of nonlinear terms to the element u_i comes from some neighborhood of the light cone $c_i t = r$. We do not go into further details here, but the point of these observations is that the vector fields in Γ do not play any essential role in the treatment of the nonresonant terms. Hence we concentrate on the null forms.

Fix $i \in \{1, \dots, N\}$. Observing that the null forms which are included in the nonlinearity for u_i consist of elements u_j and u_k which have the same propagation speed as u_i, we may assume $c_i = 1$ for each fixed i without loss of generality. Thus we shall discuss $Q_0(u_j, u_k)$ instead of $Q_0(u_j, u_k; c_i)$. In the above works, various kinds of decay estimates without L_j's are adopted instead of Klainerman's L^1–L^∞ estimate which contains L_j's. However, the estimates for null forms used in those works are essentially the same as

$$|Q_0(u_j, u_k)| \leq C(1+r)^{-1}(|\Gamma_* u|\,|\partial u| + (1+|t-r|)|\partial u|^2), \tag{1.23}$$

where $\Gamma_* u = (Su, \Omega u, \partial u)$. Other null forms also enjoy the same estimate. A variant of (1.23) was first introduced by Hoshiga–Kubo [6] for the two space dimensional systems. To prove (1.23), we use

$$D_+ = \frac{1}{r}(S - (t-r)\partial_t) \tag{1.24}$$

instead of (1.9). (1.8) and (1.24) lead to (1.23) immediately.

(1.23) may seem to be quite weaker than (1.11), but this is not true, because this kind of estimate for null forms has its meaning only in a neighborhood of the light cone $t = r$, as we have pointed out in the above. When $|t-r| < \delta t$ with some small $\delta > 0$, we have $1+t+r \leq C(1+r)$, and in view of (1.12) and (1.13), we expect from (1.23) that Q_0 behaves like $C(1+t+r)^{-3}(1+|t-r|)^{-1-2\rho}$, which is the same expectation as we have got from (1.11). In this way, we can exclude L_j's from estimates for the null forms.

Note that weighted L^∞–L^∞ decay estimates, where we need only Ω and ∂ (see Lemmas 3.2 and 3.3 below), are used in [9,20], and (1.23) is the only reason why S was adopted in these works.

Our aim in this article is to get rid of not only L_j's, but also S from the argument. In other words, we would like to prove Theorem 1.2 using only ∂ and Ω. Instead of using (1.9) or (1.24), we will directly obtain enhanced decay of D_+u for the solution u to the wave equation, using only ∂ and Ω. Once we get such estimates, we are able to observe extra decay of null forms by going back to (1.8).

2. Enhanced Decay of a Tangential Derivative to the Light Cone

In this section, we introduce the main result of [12], which gives us enhanced decay of D_+u. To state the result, we introduce some notations.

We put $Z = (Z_j)_{1\le j\le 7} = (\partial, \Omega)$, and $Z^\alpha = Z_1^{\alpha_1}\cdots Z_7^{\alpha_7}$ with a multi–index $\alpha = (\alpha_1, \dots, \alpha_7)$.

For a function $v = v(t,x)$ and a nonnegative integer s, we define

$$|v(t,x)|_s = \sum_{|\alpha|\le s} |Z^\alpha v(t,x)|, \text{ and } \|v(t,\cdot)\|_s = \big\| |v(t,\cdot)|_s \big\|_{L^2(\mathbb{R}^3)}. \tag{2.1}$$

We introduce

$$A_{\rho,k}[f,g] = \sup_{x\in\mathbb{R}^3} \langle x\rangle^\rho \left(|f(x)|_k + |\nabla_x f(x)|_k + |g(x)|_k \right) \tag{2.2}$$

for $\rho \in \mathbb{R}$, a nonnegative integer k, and $f,\ g \in C_0^\infty(\mathbb{R}^3)$, where $\langle x\rangle = \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^3$. We shall also write $\langle a\rangle = \sqrt{1+|a|^2}$ for $a \in \mathbb{R}$.

Let $c_1, \dots c_N$ be given positive constants. We set $c_0 = 0$, and define

$$w(t,r) = \min_{0\le j\le N} \langle c_j t + r\rangle\,. \tag{2.3}$$

We introduce

$$N_{\rho,\kappa,k}[G](T) = \sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \langle x\rangle \, \langle t+|x|\rangle^\rho \, w(t,|x|)^\kappa |G(t,x)|_k \tag{2.4}$$

for $\rho, \kappa \in \mathbb{R}$, a nonnegative integer k, and a smooth function $G = G(t,x)$ decaying sufficiently fast as $|x| \to \infty$.

Consider the Cauchy problem for the linear wave equation

$$\Box_c v(t,x) = G(t,x) \qquad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}^3, \tag{2.5}$$

$$v(0,x) = \phi(x), \ (\partial_t v)(0,x) = \psi(x) \qquad \text{for } x \in \mathbb{R}^3, \tag{2.6}$$

where c is a given positive constant, and $\Box_c = \partial_t^2 - c^2\Delta_x$. We write $U_c[G]$ for the solution u to the problem (2.5)–(2.6) with $\phi = \psi \equiv 0$, and $U_c^*[\phi,\psi]$ for the solution u to the problem (2.5)–(2.6) with $G \equiv 0$.

We define

$$D_{\pm,c} = \partial_t \pm c\partial_r \tag{2.7}$$

for $c > 0$, where $\partial_r = \sum_{j=1}^3 \omega_j \partial_j$, and $\omega_j = x_j/|x|$ for $j = 1,2,3$ as before. Now we are in a position to state our result.

Theorem 2.1 (Katayama-Kubo [12]). (i) *Let* $1 < \rho < 2$, $\kappa > 1$, $\mu \geq 0$ *and* $\delta > 0$. *Then there exists a positive constant C depending only on* ρ, κ, δ *and* c *such that we have*

$$\langle t+|x|\rangle^{2-\mu} \langle ct-|x|\rangle^{\rho-1} |D_{+,c}U_c[G](t,x)| \leq CN_{\rho-\mu,\kappa,2}[G](t) \tag{2.8}$$

for any (t,x) *satisfying* $t \geq 0$ *and* $|x| \geq \delta t$.

(ii) *Let* $1 < \rho < 2$, $\kappa > 1$, $\mu \geq 0$ *and* $\delta > 0$. *Then there exists a positive constant C depending only on* ρ, κ *and* c *such that we have*

$$\langle t+|x|\rangle^{2-\mu} \langle ct-|x|\rangle^{\rho-1} |D_{+,c}U_c^*[\phi,\psi](t,x)| \leq CA_{\rho-\mu+\kappa,2}[\phi,\psi] \tag{2.9}$$

for any (t,x) *satisfying* $t \geq 0$ *and* $|x| \geq \delta t$.

Though more general result was obtained in [12], we have restricted our statement here to the estimate which will be used directly to prove Theorem 1.2.

Let $u = U_1[G]$, and suppose $\sup_{t\in[0,\infty)} N_{\rho,\kappa,2}[G](t) < \infty$ for some $\rho \in (1,2)$ and $\kappa > 1$. Then Lemmas 3.2 and 3.3 below show

$$|u(t,x)| \leq C \langle t+|x|\rangle^{-1} \langle t-|x|\rangle^{-\rho+1}, \tag{2.10}$$

$$|\partial u(t,x)| \leq C \langle t+|x|\rangle^{-1} \langle t-|x|\rangle^{-\rho} \tag{2.11}$$

near the light cone $t = r$, while Theorem 2.1 implies

$$|D_+ u(t,x)| \leq C \langle t+r\rangle^{-2} \langle t-|x|\rangle^{-\rho+1} \tag{2.12}$$

near the light cone $t = r$. Thus we find that D_+u gains

$$\langle t+|x|\rangle^{-1} \langle |t-|x|\rangle \ (\text{resp. } \langle t+r\rangle^{-1})$$

in its decay rate near the light cone, compared to ∂u (resp. u). This is exactly what we need to treat null forms, as we have stated in the previous section.

After we prove Theorem 2.1 in Section 4, we will give a proof of Theorem 1.2, where only ∂ and Ω are used, in Section 5.

3. Preliminary results

After the pioneering work of John [7], various weighted L^∞–L^∞ estimates were studied by many authors. Here we state some known results. For the proof of each lemma, consult to the works indicated therein.

Lemma 3.1 (Asakura [2]). *For $\rho > 1$ and $\mu \geq 0$, we have*

$$\langle t+|x|\rangle^{1-\mu}\langle ct-|x|\rangle^{\rho-1}|U_c^*[\phi,\psi](t,x)| \leq CA_{\rho-\mu+1,0}[\phi,\psi] \tag{3.1}$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^3$.

Lemma 3.2 (Kubota–Yokoyama [20]). *Assume $\rho > 1$, $\kappa > 1$ and $\mu \geq 0$. Then we have*

$$\langle t+|x|\rangle^{1-\mu}\langle ct-|x|\rangle^{\rho-1}|U_c[G](t,x)| \leq CN_{\rho-\mu,\kappa,0}[G](t)$$

for any $(t,x)\in[0,\infty)\times\mathbb{R}^3$.

This lemma plays an important role in the proof of Theorem 2.1. Hence we give a proof here.

Proof. Suppose that $\rho > 1$, $\kappa > 1$ and $\mu \geq 0$. Without loss of generality, we may assume $c = 1$. We write $U[G]$ for $U_1[G]$ in the following.

For $(t,r)\in[0,\infty)\times[0,\infty)$, we define

$$H(t,r) = \sup_{\omega\in S^2}|G(t,r\omega)|, \tag{3.2}$$

and we set $\widetilde{G}(t,x) = H(t,|x|)$. Then we have $|G(t,x)| \leq \widetilde{G}(t,x)$ for any $(t,x)\in(0,\infty)\times\mathbb{R}^3$. Therefore, in view of the positivity of the fundamental solution to the wave equation in $(0,\infty)\times\mathbb{R}^3$, we obtain

$$|U[G](t,x)| \leq U\big[\widetilde{G}\big](t,x). \tag{3.3}$$

Since $U\left[\widetilde{G}\right]$ is spherically symmetric, we get

$$U\left[\widetilde{G}\right](t,x) = \frac{1}{r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \lambda H(\tau,\lambda) d\lambda d\tau \tag{3.4}$$

for $r = |x| > 0$. Observing that

$$N_{\rho-\mu,\kappa,0}[G](t) = \sup_{(\tau,\lambda)\in[0,\infty)\times[0,\infty)} \langle\lambda\rangle \langle\tau+\lambda\rangle^{\rho-\mu} w(\tau,\lambda)^{\kappa} H(\tau,\lambda),$$

we obtain

$$U\left[\widetilde{G}\right](t,x) \le N_{\rho,\kappa,0}[G](t) I_{\rho-\mu,\kappa}(t,r), \tag{3.5}$$

where

$$I_{\rho-\mu,\kappa}(t,r) = \frac{1}{r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} \langle\tau+\lambda\rangle^{-\rho+\mu} w(\tau,\lambda)^{-\kappa} d\lambda d\tau. \tag{3.6}$$

If we set

$$J_{\rho-\mu,\kappa,i}(t,r) = \frac{1}{r}\int_0^t \int_{|r-t+\tau|}^{r+t-\tau} (1+\tau+\lambda)^{-\rho+\mu}(1+|c_i\tau-\lambda|)^{-\kappa} d\lambda d\tau$$

for $0 \le i \le N$, it is easy to see that $I_{\rho-\mu,\kappa}(t,r) \le C\sum_{i=0}^{N} J_{\rho-\mu,\kappa,i}(t,r)$. For each $i \in \{0,1,\ldots,N\}$, performing the change of variables $(p,q) = (\lambda+\tau, \lambda - c_i\tau)$, we obtain

$$\begin{aligned} J_{\rho-\mu,\kappa,i}(t,r) =& \frac{1}{(c_i+1)r}\int_{|t-r|}^{t+r} (1+p)^{-\rho+\mu} \left(\int_{p_i}^{p} (1+|q|)^{-\kappa} dq\right) dp \\ \le& \frac{C}{r}\int_{|t-r|}^{t+r} (1+p)^{-\rho+\mu} dp \\ \le& \frac{C\langle t+r\rangle^{\mu}}{r}\int_{|t-r|}^{t+r} (1+p)^{-\rho} dp, \end{aligned} \tag{3.7}$$

where $2p_i = (1-c_i)p + (1+c_i)(r-t)$. Here we have used the assumption $\kappa > 1$. Now we have shown

$$\begin{aligned} \langle t+r\rangle^{-\mu} |U[G](t,x)| \le& \langle t+r\rangle^{-\mu} U[\widetilde{G}](t,x) \\ \le& C N_{\rho,\kappa,0}[G](t) \frac{1}{r}\int_{|t-r|}^{t+r} (1+p)^{-\rho} dp. \end{aligned} \tag{3.8}$$

It is easy to obtain

$$\frac{1}{r}\int_{|t-r|}^{t+r}(1+p)^{-\rho}dp \le \frac{t+r-|t-r|}{r}(1+|t-r|)^{-\rho} \le 2(1+|t-r|)^{-\rho}, \tag{3.9}$$

and

$$\frac{1}{r}\int_{|t-r|}^{t+r}(1+p)^{-\rho}dp \le \frac{C}{r}(1+|t-r|)^{-\rho+1}. \tag{3.10}$$

If $r \ge (1+t)/2$, then we have $\langle t+r\rangle \le Cr$, and (3.10) implies the desired result. On the other hand, if $r < (1+t)/2$, then we have $\langle t+r\rangle \le C\langle t-r\rangle$, and (3.9) implies the desired result. This completes the proof. □

Lemma 3.3 (Katayama–Yokoyama [14]). *Assume $\rho > 1$, $\kappa > 1$ and $\mu \ge 0$. Then we have*

$$\langle t+|x|\rangle^{-\mu}\langle x\rangle\langle t-|x|\rangle^{\rho}|\partial U_c[G](t,x)| \le CN_{\rho-\mu,\kappa,1}[G](t)$$

for any $(t,x)\in[0,\infty)\times\mathbb{R}^3$, where $\partial=(\partial_t,\nabla_x)$.

This lemma will be used in the proof of Theorem 1.2, but not in that of Theorem 2.1.

To conclude this section, we introduce the following Sobolev type inequality which will be used to combine the energy inequality with weighted L^∞–L^∞ decay estimates:

Lemma 3.4 (Klainerman [16]). *We have*

$$\sup_{x\in\mathbb{R}^3}\langle x\rangle|\phi(x)| \le \sum_{|\alpha|\le 2}\|Z_*^\alpha\phi\|_{L^2(\mathbb{R}^3)} \tag{3.11}$$

for any $\phi\in C_0^\infty(\mathbb{R}^3)$, where $Z_*=(\nabla_x,\Omega)$.

4. Proof of Theorem 2.1

Without loss of generality, we may assume $c=1$. So we drop the suffix c from the notations (for example, we write U^* for U_c^*). We only give a proof for (2.8) here, because (2.9) can be proved similarly.

We have $\partial_a U[G](t,x) = U[\partial_a G](t,x) + \delta_{a0}U^*[0,G(0)](t,x)$ for $0 \le a \le 3$ with the Kronecker delta δ_{ab}. Lemma 3.2 implies

$$\langle t+|x|\rangle^{1-\mu}\langle t-|x|\rangle^{\rho-1}(t,x)|U[\partial_a G]|(t,x)| \le CN_{\rho-\mu,\kappa,1}[G](t), \tag{4.1}$$

while Lemma 3.1 leads to

$$\begin{aligned}\langle t+|x|\rangle^{1-\mu}\langle t-|x|\rangle^{\rho-1}U^*[0,G(0)] \le& CA_{\rho-\mu+1,0}[0,G(0)]\\ \le& N_{\rho-\mu,\kappa,0}[G](t)\end{aligned} \tag{4.2}$$

because we have $\langle r\rangle^{\rho-\mu+1} \le \langle r\rangle\langle 0+r\rangle^{\rho-\mu}w(0,r)^{\kappa}$. Therefore we obtain

$$\langle t+|x|\rangle^{1-\mu}\langle t-|x|\rangle^{\rho-1}|\partial_a U[G](t,x)| \le CN_{\rho-\mu,\kappa,1}[G](t). \tag{4.3}$$

If $\delta t \le r \le 1$, then we have $\langle t+r\rangle \le \sqrt{1+(1+\delta^{-1})^2}$, and (4.3) leads to the desired result.

We assume $|x| = r \ge \max\{1,\delta t\}$ in the following. Note that we have $\langle t+r\rangle \le Cr$ for such r.

For $(t,r,\omega) \in [0,\infty)\times(0,\infty)\times S^2$, we define

$$V(t,r,\omega) = rU[G](t,r\omega). \tag{4.4}$$

Then we have

$$\begin{aligned}|D_+U[G](t,r\omega)| =& \frac{1}{r}|D_+V(t,r,\omega) - U[G](t,r\omega)|\\ \le& C\langle t+r\rangle^{-1}\big(|D_+V(t,r,\omega)| + |U[G](t,r\omega)|\big).\end{aligned} \tag{4.5}$$

Since Lemma 3.2 implies

$$|U[G](t,r\omega)| \le C\langle t+r\rangle^{-1+\mu}\langle t-r\rangle^{-\rho+1}N_{\rho-\mu,\kappa,0}[G](t),$$

our task is to show

$$|D_+V(t,r,\omega)| \le C\langle t+r\rangle^{-1+\mu}\langle t-r\rangle^{-\rho+1}N_{\rho-\mu,\kappa,2}[G](t). \tag{4.6}$$

For that purpose, first we observe

$$\begin{aligned}D_-D_+V(t,r,\omega) =& r\Box U[G](t,r\omega) + \frac{1}{r}\sum_{1\le j<k\le 3}\Omega_{jk}^2 U[G](t,r\omega)\\ =& rG(t,r\omega) + \frac{1}{r}\sum_{1\le j<k\le 3}\Omega_{jk}^2 U[G](t,r\omega).\end{aligned} \tag{4.7}$$

We have

$$(4.8)\qquad |rG(t,r\omega)| \le \langle t+r\rangle^{-\rho+\mu} \langle t-r\rangle^{-\kappa} N_{\rho-\mu,\kappa,0}[G](t).$$

On the other hand, Lemma 3.2 again implies

$$(4.9)\quad \sum_{j,k} |\Omega_{jk}^2 U[G](t,r\omega)| \le C\langle t+r\rangle^{-1+\mu} \langle t-r\rangle^{-\rho+1} N_{\rho-\mu,\kappa,2}[G](t).$$

Observing that $t+r-\tau \ge \max\{1,\delta\tau\}$ for any $\tau \in [0,t]$, provided that $r \ge \max\{1,\delta t\}$, we find from (4.7)–(4.9) that

$$\begin{aligned}
|D_+V(t,r,\omega)| =& \left| \int_0^t (D_-D_+V)(\tau, t+r-\tau,\omega)d\tau \right| \\
&\le \langle t+r\rangle^{-\rho+\mu} N_{\rho-\mu,\kappa,0}[G](t) \int_0^t \langle t+r-2\tau\rangle^{-\kappa} d\tau \\
&\quad + C\langle t+r\rangle^{-2+\mu} N_{\rho-\mu,\kappa,2}[G](t) \int_0^t \langle t+r-2\tau\rangle^{-\rho+1} d\tau \\
&\le C\langle t+r\rangle^{-\rho+\mu} N_{\rho-\mu,\kappa,2}[G](t).
\end{aligned}$$

This completes the proof, because we have

$$\langle t+r\rangle^{-\rho+\mu} \le \langle t+r\rangle^{-1+\mu} \langle t-r\rangle^{-\rho+1}$$

for $\rho > 1$. □

5. Proof of Theorem 1.2

First we describe the estimates for the null forms precisely:

Lemma 5.1. *Let $u=(u_1,\dots,u_N)$, and $Q=Q_0$ or $Q=Q_{ab}$ $(0\le a< b\le 3)$. Suppose $\delta>0$. Then, for a nonnegative integer s, there exists a positive constant depending on s and δ such that*

$$\begin{aligned}
(5.1)\quad |Q(u_j,u_k)|_s \le& C\big(|u|_{[s/2],+}|\partial u|_s + |\partial u|_{[s/2]}|u|_{s,+}\big) \\
&+ C\langle t+|x|\rangle^{-1} \big(|u|_{[s/2]+1}|\partial u|_s + |\partial u|_{[s/2]}|u|_{s+1}\big)
\end{aligned}$$

at the point (t,x) satisfying $|x|\ge \delta t$, where

$$|v(t,x)|_{s,+} = \sum_{|\alpha|\le s} |D_+Z^\alpha v(t,x)|$$

for a smooth function v and a nonnegative integer s.

Proof. Since $Z^\alpha Q(u_j, u_k)$ can be written in terms of $Q_0(\Gamma^\beta u_j, \Gamma^\gamma u_k)$ and $Q_{ab}(\Gamma^\beta u_j, \Gamma^\gamma u_k)$ with $|\beta| + |\gamma| \le |\alpha|$, it suffices to show the result for $s = 0$.

Since we have $\langle t + r\rangle \le \langle r\rangle$ for $r \ge \delta t$, we obtain (5.1) for $Q = Q_0$ and $s = 0$ from (1.8).

For Q_{ab}, we have

$$
\begin{aligned}
Q_{ij}(v, w) =& \sum_{k\ne i} \frac{\omega_j\omega_k}{r}\{(\Omega_{ik}w)(\partial_r v) - (\Omega_{ik}v)(\partial_r w)\} \\
&+ \sum_{k\ne j} \frac{\omega_i\omega_k}{r}\{(\Omega_{jk}v)(\partial_r w) - (\Omega_{jk}w)(\partial_r v)\} \\
&+ \sum_{k\ne i}\sum_{l\ne j} \frac{\omega_k\omega_l}{r^2}\{(\Omega_{ik}v)(\Omega_{jl}w) - (\Omega_{jl}v)(\Omega_{ik}w)\},
\end{aligned}
\tag{5.2}
$$

$$
Q_{0k}(v, w) =(D_+v)(\partial_k w) - (\partial_k v)(D_+w) + \sum_{j=1}^{3} \omega_j Q_{kj}(v, w)
\tag{5.3}
$$

for $1 \le i < j \le 3$ and $1 \le k \le 3$, which lead to the desired result for $s = 0$. □

Suppose that all the assumptions in Theorem 1.2 be fulfilled. Since the local existence of classical solutions is well-known, what we need is some *a priori* estimate. Let $u = (u_1, \dots, u_N)$ be the solution to (1.1)–(1.2) for $0 \le t < T$ with some $T > 0$. Fix some $\rho \in (1, 2)$ and we define

$$
\begin{aligned}
e_{\rho,s,i}(t, x) =& \langle t + |x|\rangle \langle t - |x|\rangle^{\rho-1} |u_i(t, x)|_{s+1} \\
&+ \langle x\rangle \langle t - |x|\rangle^{\rho} |\partial u_i(t, x)|_s \\
&+ \chi(t, x) \langle t + |x|\rangle^2 \langle t - |x|\rangle^{\rho-1} |u_i(t, x)|_{s-1,+}
\end{aligned}
\tag{5.4}
$$

for a nonnegative integer s, where

$$
\chi(t, x) = \begin{cases} 1, & |x| \ge \delta t, \\ 0, & \text{otherwise} \end{cases}
$$

with some small $\delta > 0$.

We set

$$
E_{\rho,s}(T) = \sum_{i=1}^{N} \sup_{0\le t<T} \|e_{\rho,s,i}(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}.
\tag{5.5}
$$

Let $s \geq 4$. We are going to prove that if we assume

$$E_{\rho,s}(T) \leq M\varepsilon \tag{5.6}$$

for some large M and small $\varepsilon > 0$, then we actually have

$$E_{\rho,s}(T) \leq \frac{M}{2}\varepsilon. \tag{5.7}$$

Once we establish such property, by the so-called continuity argument (or the bootstrap argument), we find that $E_{\rho,s}$ for small ε stays bounded as far as the solution exists. This *a priori* estimate implies global existence of the solution.

In the following, we use Lemmas 3.2, 3.3 and Theorem 2.1 repeatedly with the choice of $c = c_1 = \cdots = c_N = 1$ so that we have

$$w(t,r) = \min\{\langle r\rangle, \langle t-r\rangle\}. \tag{5.8}$$

Here we note that we have $\langle r\rangle^{-1}\langle t-r\rangle^{-1} \leq C\langle t+r\rangle^{-1} w(t,r)^{-1}$ for $(t,r) \in [0,\infty) \times [0,\infty)$.

Now we are going to prove (5.7), assuming (5.6). In the following, we always assume M is large enough, and ε is small enough.

Since we have $Z^\alpha \Box u_i = \Box(Z^\alpha u_i)$, the standard energy inequality leads to

$$\begin{aligned} \|\partial u(t)\|_{2s} \leq & C\varepsilon + C\int_0^t \|F(\tau)\|_{2s} d\tau \\ \leq & C\varepsilon + C\int_0^t \||\partial u(\tau)|_s\|_{L^\infty(\mathbb{R}^3)} \|\partial u(\tau)\|_{2s} d\tau \\ \leq & C\varepsilon + CM\varepsilon\int_0^t (1+\tau)^{-1}\|\partial u(\tau)\|_{2s} d\tau \end{aligned} \tag{5.9}$$

for $0 \leq t < T$. Thus the Gronwall lemma leads to

$$\|\partial u(t)\|_{2s} \leq C\varepsilon(1+t)^{CM\varepsilon} \tag{5.10}$$

for $0 \leq t < T$.

Applying Lemma 3.4, from (5.10) we get

$$\langle x\rangle\, |\partial u(t,x)|_{2s-2} \leq C\varepsilon(1+t)^{CM\varepsilon}, \tag{5.11}$$

and we find

$$\langle x\rangle \langle t+|x|\rangle^{(1+\mu)-2\mu} w(t,x)^{\rho}|F(\partial u)(t,x)|_{2s-2} \leq CM\varepsilon^2 \langle t+|x|\rangle^{CM\varepsilon-\mu} \tag{5.12}$$

for $\mu \geq 0$. From Theorem 2.1, Lemmas 3.2 and 3.3, we obtain

$$\langle t+|x|\rangle^{-2\mu} e_{1+\mu,2s-3,i}(t,x) \leq C\varepsilon + CM\varepsilon^2 \leq M\varepsilon, \tag{5.13}$$

where $\mu = CM\varepsilon$.

Let $F_i^{(2)}$ be the quadratic part of F_i. We use Lemma 5.1 to obtain

$$|F_i^{(2)}(t,x)|_{2s-3} \leq CM^2\varepsilon^2 \langle x\rangle^{-1} \langle t+|x|\rangle^{-2+2\mu} \langle t-|x|\rangle^{-\rho-\mu} \tag{5.14}$$

for (t,x) satisfying $|x| \geq \delta t$. On the other hand, if $|x| \leq \delta t$, we have $\langle t-|x|\rangle^{-1} \leq C\langle t+|x|\rangle^{-1}$. Since $F_i^{(2)}$ are quadratic, we obtain

$$\begin{aligned} |F_i^{(2)}(t,x)|_{2s-3} &\leq CM^2\varepsilon^2 \langle x\rangle^{-2} \langle t+|x|\rangle^{2\mu} \langle t-|x|\rangle^{-1-\rho-\mu} \\ &\leq CM^2\varepsilon^2 \langle x\rangle^{-2} \langle t+|x|\rangle^{-1-\rho+\mu} \\ &\leq CM^2\varepsilon^2 \langle x\rangle^{-1} \langle t+|x|\rangle^{-1-\rho+2\mu} w(t,|x|)^{-1-\mu} \end{aligned} \tag{5.15}$$

for $|x| \leq \delta t$. We also have

$$\begin{aligned} |H_i(t,x)|_{2s-3} &\leq CM^3\varepsilon^3 \langle x\rangle^{-3} \langle t+|x|\rangle^{2\mu} \langle t-|x|\rangle^{-1-\mu-2\rho} \\ &\leq CM^3\varepsilon^3 \langle x\rangle^{-1} \langle t+|x|\rangle^{-2+2\mu} w(t,|x|)^{-1-\mu-2\rho} \end{aligned} \tag{5.16}$$

for any $(t,x) \in [0,T)\times\mathbb{R}^3$, where $H_i = F_i - F_i^{(2)}$. Gathering the above estimates, we obtain

$$|F_i(t,x)|_{2s-3} \leq CM^2\varepsilon^2 \langle x\rangle^{-1} \langle t+|x|\rangle^{-2+2\mu} w(t,|x|)^{-1-\mu} \tag{5.17}$$

Since we may assume $\rho - 2 + 2\mu \leq 0$, we obtain

$$N_{\rho,1+\mu,2s-3}[F_i](t) \leq CM^2\varepsilon^2.$$

Finally, remembering $s \geq 4$, from Theorem 2.1 and Lemma 3.3 we obtain

$$E_{\rho,s}(T) \leq E_{\rho,2s-4}(T) \leq C_0(\varepsilon + M^2\varepsilon^2) \tag{5.18}$$

with some positive constant C_0 which is independent of T, M and ε.

Finally, if we choose large M to satisfy $M \geq 4C_0$, and choose small ε_0 to satisfy $C_0M\varepsilon_0 < 1/4$, (5.18) implies (5.7) for any $\varepsilon \in (0,\varepsilon_0]$. This completes the proof. □

6. Related Results

In this article, we have observed the enhanced decay of D_+u through the weighted L^∞–L^∞ decay estimate. Here we introduce some related results.

Alinhac has obtained

$$\text{(6.1)} \qquad \|\partial u(t)\|_{L^2(\mathbb{R}^3)} + \sum_{j=1}^{3} \int_0^t \int_{\mathbb{R}^3} \frac{|T_j u(\tau, y)|^2}{\langle \tau - |y| \rangle^{\rho}} dy d\tau \\ \leq C\Big(\|\partial u(0)\|_{L^2(\mathbb{R}^3)} + \int_0^t \|(\Box u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau\Big)$$

for $\rho > 1$, where $T_j = \omega_j \partial_t + \partial_j$ for $j = 1, 2, 3$ with $\omega_j = x_j/|x|$ (see [1] for example). Without the second term on its left-hand side, (6.1) is the well–known energy inequality. The proof is also similar to that of the energy inequality: We define $\eta(a) = \int_{-\infty}^{a} \langle \tau \rangle^{-\rho} d\tau$. Then we integrate $e^{\eta(r-t)}(\partial_t u)(\Box u)$ over $\mathbb{R}^3$, perform the integration by parts, and obtain (6.1). Note that without the so-called *ghost weight* $e^{\eta(r-t)}$, which is bounded from above and below, the above argument gives the standard energy inequality.

T_j is closely connected to D_+; we have $D_+ = \sum_{j=1}^{3} \omega_j T_j$. Thus (6.1) shows the enhanced decay of D_+u implicitly. However it seems difficult to recover the pointwise decay estimate for D_+u from (6.1).

If we have the estimate for $\| \langle t + r \rangle D_+ u(t, \cdot)\|_{L^2(\mathbb{R}^3)}$, then we can easily recover the enhanced pointwise estimate through Lemma 3.4. Sideris–Thomases [23] obtained such estimate, but unfortunately S is used there.

To conclude this section, we give a few words on the mixed problem. The Dirichlet problem for nonlinear wave equations with single or multiple speed(s) in exterior domains (domains outside some obstacles) are also widely studied, and global existence results corresponding to Theorem 1.2 and its counterpart for the multiple speeds case were obtained (see Metcalfe–Nakamura–Sogge [21] and references cited therein).

Because L_j's do not preserve the boundary condition, and have unbounded coefficients near the boundary, they are unfavorable even if we consider the single speed case. This is the another reason why the vector fields method without L_j's have been widely studied. Though the same is true for S concerning the boundary condition and the unbounded coefficient near the boundary, one can manage to use S in the argument, but careful treatment of S was needed.

Our approach here for the Cauchy problem is also useful for the mixed problem, and we can simplify the argument (see Katayama–Kubo [13]). We also remark that, in [22], Metcalfe and Sogge gave a simplified approach where S can be used without special care, but their approach needs the obstacle to be star-shaped.

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