数理解析研究所講究録
第 1592 巻 2008 年 41-52

On a Stokes approximation of two dimensional exterior Oseen flow near the boundary

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1. Introduction

1.1. Background of problem. In this paper we are concerned with the motion of viscous incompressible fluid past a rigid obstacle in two space dimension.

Let $\mathscr{B} \subset \mathbb{R}^2$ be a bounded and open set whose boundary is of class C^2 and let Ω be the exterior domain to $\mathscr{B}$, i.e., $\Omega \equiv \mathbb{R}^2 \setminus \overline{\mathscr{B}}$. Here $\mathscr{B}$ and Ω denote a rigid obstacle in the plane and region which is filled with viscous incompressible fluid, respectively. We choose an $R_0 > 0$ in such a way that $B_{R_0}(0) \supset \overline{\mathscr{B}}$ and fix it throughout this paper.

The stationary motion of the fluid past obstacle $\mathscr{B}$ is governed by the following boundary value problem of the Navier-Stokes equations.

$$\text{(N-S)} \qquad \begin{cases} (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nu\Delta\boldsymbol{v} - \nabla p, \quad \operatorname{div}\boldsymbol{v} = 0, \qquad x \in \Omega, \\ \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{v}_*, \quad \displaystyle\lim_{|x|\to\infty} \boldsymbol{v} = \boldsymbol{U}_\infty. \end{cases}$$

Here $\boldsymbol{v} = {}^t(v_1, v_2)$ and p are the velocity and pressure, respectively; $\boldsymbol{v}_*$ is prescribed velocity on the boundary; $\nu > 0$ denotes the viscosity constant of fluid and $\boldsymbol{U}_\infty$ is constant vector which stands for the uniform flow; $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \sum_{j=1}^2 v_j\partial_j\boldsymbol{v}$, where $\partial_j = \partial/\partial x_j$ $(j = 1, 2)$; $\Delta = \partial_1^2 + \partial_2^2$ is the Laplace operator on $\mathbb{R}^2$, $\nabla p = {}^t(\partial_1 p, \partial_2 p)$ is the gradient of p, and $\operatorname{div}\boldsymbol{v} = \sum_{j=1}^2 \partial_j v_j$ is the divergence of $\boldsymbol{v}$. A main feature of (N-S) is that the boundary condition at space infinity is imposed on the velocity.

2000 *Mathematics Subject Classification.* 76D07, 76D05, 76M15.

Key words and phrases. Navier-Stokes equations, Oseen equations, Stokes equations, hydrodynamic potential theory, two dimensional exterior flow.

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One of the standard ways to investigate (N-S) is linear approximation. First we shall introduce the idea due to G.G. Stokes. Since the nonlinear term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ is quadratic with respect to $\boldsymbol{v}$, it may be vanishingly small in comparison to $\nu\Delta\boldsymbol{u}$ when $\nu \gg 1$. Therefore (N-S) may be reduced to the Stokes equations:

$$(1.1)\qquad \begin{cases} 0=\nu\Delta\boldsymbol{v}-\nabla p, \quad \operatorname{div}\boldsymbol{v}=0, \qquad x\in\Omega, \\ \boldsymbol{v}|_{\partial\Omega}=\boldsymbol{v}_*, \qquad \lim\limits_{|x|\to\infty}\boldsymbol{v}=\boldsymbol{U}_\infty. \end{cases}$$

To understand the slow-motion of fluid mathematically, it seems to be enough to investigate linear problem (1.1). However, does (1.1) give any approximate solution of (N-S)? It is well known that the answer is no (*Stokes paradox*). Unfortunately, Stokes's linearization does not make contribution to two dimensional flow past an obstacle. We shall explain why the Stokes paradox occurs very briefly. Let $\boldsymbol{v}_S$ denote the dominant of solutions to the Stokes equations when $|x|$ is sufficiently large. It is well known that $\boldsymbol{v}_S \sim \log|x|$. Therefore from simple calculation, we see that

$$(1.2)\qquad \frac{|(\boldsymbol{v}_S\cdot\nabla)\boldsymbol{v}_S|}{|\nu\Delta\boldsymbol{v}_S|}\sim\frac{|x|\log|x|}{\nu}\to\infty \quad (|x|\to\infty).$$

This shows that the ratio of viscous force $\nu\Delta\boldsymbol{v}_S$ to convection $(\boldsymbol{v}_S\cdot\nabla)\boldsymbol{v}_S$ diverges when $|x|$ goes to infinity even if ν is very large. This simple observation yields that the convection term cannot be negligible when we concentrate on the slow motion.

To get rid of the Stokes paradox, C.W. Oseen [3] has introduced another linearization of (N-S). Set $\boldsymbol{v}=\boldsymbol{U}_\infty+\boldsymbol{u}$. Here $\boldsymbol{u}$ stands for the flow caused by the presence of rigid obstacle $\mathscr{B}$ and it should converge to 0 when $|x|\to\infty$. Suppose that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is very small in comparison to linear terms, we obtain the Oseen equations.

$$(1.3)\qquad \begin{cases} (\boldsymbol{U}_\infty\cdot\nabla)\boldsymbol{u}=\nu\Delta\boldsymbol{u}-\nabla p, \quad \operatorname{div}\boldsymbol{u}=0, \qquad x\in\Omega, \\ \boldsymbol{u}|_{\partial\Omega}=\boldsymbol{v}_*-\boldsymbol{U}_\infty, \qquad \lim\limits_{|x|\to\infty}\boldsymbol{u}=0. \end{cases}$$

Blessedly, we do not encounter a paradox like Stokes in Oseen's approximation. Therefore to understand the motion of fluid mathematically, a study on (1.3) is important.

1.2. Mathematical problem and main result. A good understanding of (1.2) is that the ratio of viscosity to convection is quite small when $|x|$ is not large and $\nu \gg 1$. Therefore we can expect that Stokes's linearization make some contribution to mathematical analysis of the fluid even in two dimensional exterior domains, if the neighbourhood of the boundary is considered.

Our aim of the present paper is to investigate the relationship between the Oseen equations and Stokes equations near the obstacle. In particular, we would like to get an error estimate between solutions of the Oseen and Stokes equations near the obstacle. For such purpose, here we shall introduce the dimensionless form of our problem. Without loss of generality, we may assume that the uniform flow is along with x_1-axis, that is, we may take $U_\infty = |U_\infty| \cdot {}^t(1,0)$. Set $k = |U_\infty|/2\nu$. The positive real number $2k$ denotes the dimensionless *Reynolds number*. Our problems of this paper are the following boundary value problems of the Oseen equations and the Stokes equations.

$$\tag{1.4} \begin{cases} -\Delta u + 2k\dfrac{\partial}{\partial x_1}u + \nabla p = 0, \quad \operatorname{div} u = 0, \qquad x \in \Omega, \\ \qquad\qquad u|_{\partial\Omega} = \Phi. \end{cases}$$

We have the Stokes equations when we put $k \equiv 0$ in (1.4) formally.

$$\tag{1.5} \begin{cases} -\Delta u + \nabla p = 0, \quad \operatorname{div} u = 0, \qquad x \in \Omega, \\ \qquad u|_{\partial\Omega} = \Phi. \end{cases}$$

The main purpose of this paper is to get an error estimate between some solutions of the Oseen equations (1.4) and Stokes equations (1.5) for small Reynolds number. Our main result of the present paper is the following.

Theorem 1.1. *Let u_k be a solution to* (1.4) *with parameter k and u_0 be a solution to* (1.5). *For any $R > R_0$, there exists an $\varepsilon \in (0,1)$ such that if $0 < |k| < \varepsilon$, the following estimate holds.*

$$\tag{1.6} \|u_k - u_0\|_{C(\Omega\cap B_R)} \leq \frac{C_R}{|\log k|}\|\Phi\|_{C(\partial\Omega)}.$$

Here $\|\cdot\|_{C(D)} \equiv \sup_{x\in D}|\cdot|$ and C_R is a constant which depends on R and diverges when $R \to \infty$.

Remark 1.2. (i) A similar estimates as in Theorem 1.1 are well known when $\mathscr{B}$ is a disk or inside of ellipse. Indeed we can show such results by using the method of stream functions in polar coordinate and conformal mapping.

(ii) Our main theorem tells us that Stokes's linearization still works well even in the case of two dimensional exterior domain if the neighbourhoods of the obstacle is considered. The authors believe that this fact has big significance in terms of numerical study of hydrodynamics. In particular, our result and its proof are closely linked to numerical scheme so called *boundary element method*.

This paper is organized as follows. In Section 2, we will prepare some notations and preliminary results. In order to solve our problems: (1.4) and (1.5) by hydrodynamic potential theory, we need singular fundamental tensor to the formal Oseen and Stokes derivative operators which will be introduced in Section 2. In particular, asymptotic behavior of such fundamental tensor are important. In Section 3, we will introduce the layer potentials and investigate basic properties of them. After investigation of the layer potentials, we will reduce (1.4) and (1.5) to the boundary integral equations on $\partial\Omega$. We will solve associated boundary integral equations by Fredholm alternative theory. In final section, we will show our main theorem. The proof of our main theorem is based on the precise analysis for fundamental tensors which will be discussed in Section 2.

2. Preliminaries

In this section we shall introduce some notations and preliminary results.

2.1. Green's identities. We shall define the *modified* Stress tensor associated with Oseen flow and the formal Oseen operator.

Definition 2.1 (Modified Stress tensor). For a smooth vector field $\boldsymbol{u}$ and scalar function p, define the modified stress tensor by

$$\mathbf{T}_k(\boldsymbol{u}, p) = -2\mathbf{D}(\boldsymbol{u}) + pI_2 + k(\boldsymbol{u}\ 0)$$

and its formal adjoint by

$$\mathbf{T}_k^*(\boldsymbol{u}, p) = -2\mathbf{D}(\boldsymbol{u}) - pI_2 - k(\boldsymbol{u}\ 0),$$

where

$$(\boldsymbol{u}\ 0) = \begin{pmatrix} u_1 & 0 \\ u_2 & 0 \end{pmatrix}.$$

When $k = 0$, $\mathbf{T}_k(u, p)$ becomes usual stress tensor.

Definition 2.2 (Formal Oseen operator). For $k > 0$, we define the formal Oseen operator by

$$\mathcal{O}_k : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \longrightarrow \mathcal{O}_k(\boldsymbol{u}, p) = \begin{pmatrix} -\Delta \boldsymbol{u} + 2k\partial_1 \boldsymbol{u} + \nabla p \\ \operatorname{div} \boldsymbol{u} \end{pmatrix},$$

and its adjoint operator by

$$\mathcal{O}_k^* : \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} -\Delta \boldsymbol{u} - 2k\partial_1 \boldsymbol{u} - \nabla p \\ -\operatorname{div} \boldsymbol{u} \end{pmatrix}.$$

When $k = 0$, $\mathcal{O}_k$ becomes usual formal Stokes operator.

From the Gauss divergence theorem, we have the following Green's identities for O_k and O_k^*.

Lemma 2.3. *Let $G \subset \mathbb{R}^2$ be a bounded open set whose boundary is of class C^1. For smooth and solenoidal vector fields $\boldsymbol{u}, \boldsymbol{v}$ and scalar functions p, q, the* *following formulae hold.*

$$\int_G \mathcal{O}_k(\boldsymbol{u},p)\cdot\begin{pmatrix}\boldsymbol{v}\\ q\end{pmatrix}dx = \int_{\partial G}\mathbf{T}_k(\boldsymbol{u},p)\cdot\boldsymbol{v}\,d\sigma$$
$$+2\int_G \mathbf{D}(\boldsymbol{u}):\mathbf{D}(\boldsymbol{v})\,dx + k\int_G \frac{\partial \boldsymbol{u}}{\partial x_1}\cdot\boldsymbol{v}\,dx,$$
$$\int_G\left(\mathcal{O}_k(\boldsymbol{u},p)\cdot\begin{pmatrix}\boldsymbol{v}\\ q\end{pmatrix} - \begin{pmatrix}\boldsymbol{u}\\ p\end{pmatrix}\cdot\mathcal{O}_k^*(\boldsymbol{v},q)\right)dx$$
$$= \int_{\partial G}\Big(\mathbf{T}_k(\boldsymbol{u},p)\boldsymbol{n}\cdot\boldsymbol{v} - \boldsymbol{u}\cdot\mathbf{T}_k^*(\boldsymbol{v},q)\boldsymbol{n}\Big)\,d\sigma.$$

2.2. Fundamental solutions. In this subsection we shall introduce the fundamental solution to the Oseen and Stokes equations which will be needed later. The fundamental solution $E_k = (E^k_{j\ell})_{j,\ell=1,2,3}$, $k \geq 0$, is a 3×3 matrix which satisfies the following partial differential equations

$$\mathcal{O}_k E_k = \delta I_3 \quad \text{in } \mathcal{S}'(\mathbb{R}^2).$$

Here $\mathcal{S}'$ denotes the class of tempered distribution, δ denotes Dirac's delta and I_3 is 3×3 unit matrix.

It is well known that the explicit representation of $E^k_{j\ell}$ (see e.g., G.P. Galdi [2]).

$$\text{(2.1)}\quad E^k_{11}(x) = \frac{1}{4k\pi}\left(-\frac{x_1}{|x|^2} + ke^{kx_1}K_0(k|x|) + ke^{kx_1}\frac{x_1}{|x|}K_1(k|x|)\right),$$

$$\text{(2.2)}\quad E^k_{12}(x) = K^k_{21}(x) = \frac{1}{4k\pi}\left(-\frac{x_2}{|x|^2} + ke^{kx_1}\frac{x_2}{|x|}K_1(k|x|)\right),$$

$$\text{(2.3)}\quad E^k_{22}(x) = \frac{1}{4k\pi}\left(\frac{x_1}{|x|^2} + ke^{kx_1}K_0(k|x|) - ke^{kx_1}\frac{x_1}{|x|}K_1(k|x|)\right),$$

$$E^k_{3\ell}(x) = E^k_{\ell 3}(x) = \frac{1}{2\pi}\frac{x_\ell}{|x|^2},\quad \ell\neq 3,\qquad E^k_{33}(x) = \delta(x) - \frac{k}{\pi}\frac{x_1}{|x|^2}.$$

Here and hereafter $K_n(x)$, $n\in\mathbb{N}\cup\{0\}$, denote the modified Bessel function of order n.

For the Stokes equations ($k\equiv 0$ case), the following explicit representation formulae are well known.

$$\text{(2.4)}\quad \begin{aligned} E^0_{j\ell}(x) &= \frac{1}{4\pi}\left(-\delta_{j\ell}\log|x| + \frac{x_j x_\ell}{|x|^2}\right),\\ E^0_{3\ell}(x) &= E^0_{\ell 3}(x) = \frac{1}{2\pi}\frac{x_\ell}{|x|^2},\quad \ell\neq 3,\\ E^0_{33}(x) &= \delta(x).\end{aligned}$$

In order to show our main theorem, sharp analysis for the fundamental solution of the Oseen equation plays crucial role. At this point we shall investigate asymptotic behavior of $E^k_{j\ell}(x)$ $(k > 0)$ when $k|x|$ goes to 0. For such purpose, the following lemma concerning the asymptotics of the modified Bessel function is essential.

Lemma 2.4. *The modified Bessel functions have the following asymptotic behavior when $z \to 0$.*

$$K_0(z) = -\log z + \log 2 - \gamma + O(z^2)\log z,$$
$$K_1(z) = \frac{1}{z} + \frac{z}{2}\left(\log z - \log 2 + \gamma - \frac{1}{2}\right) + O(z^3)\log z,$$

where $\gamma = 0.57721\ldots$ is Euler's constant.

For (2.1)–(2.3), by Lemma 2.4 and Taylor series expansion, we have

$$E^k_{11}(x) = \frac{1}{4\pi}\left(-\log|x| + \frac{x_1^2}{|x|^2}\right) + \frac{1}{4\pi}(-\log k + \log 2 - \gamma) + k\log k C_{11}(k,x), \tag{2.5}$$

$$E^k_{12}(x) = E^k_{21}(x) = \frac{1}{4\pi}\frac{x_1x_2}{|x|^2} + k\log k C_{12}(k,x), \tag{2.6}$$

$$E^k_{22}(x) = \frac{1}{4\pi}\left(-\log|x| + \frac{x_2^2}{|x|^2}\right) + \frac{1}{4\pi}(-\log k + \log 2 - \gamma - 1) + k\log k C_{22}(k,x), \tag{2.7}$$

where $C_{j\ell}(k,x)$ $(j,\ell = 1,2)$ stand for continuous kernel with respect to k and x.

Remark 2.5. From (2.5)–(2.7) and (2.4) E_k can be decomposed into E_0 and $C(k,x)$. This fact means that the fundamental tensor of the Oseen equations has the same singularity as that of the Stokes equations when $k|x| \ll 1$. Such decomposition will play crucial role to investigate some properties of layer potentials which will be introduced in next section.

3. Layer potentials and boundary integral equations

With help of the fundamental tensor E_k $(k \geq 0)$, we shall define the layer potentials. For $k \geq 0$, let us define the single layer potential by

$$(S_k\Psi)(x) = \int_{\partial\Omega} E^{(c)}_k(x-y)\Psi(y)\,d\sigma(y),$$

and the double layer potential by

$$(D_k\Psi)(x) = \int_{\partial\Omega} D_k(x,y)\Psi(y)\,d\sigma(y).$$

Here 3×2 matrix $E_k^{(c)}$ is determined from the fundamental tensor E_k by eliminating the last column and the double layer kernel matrix $D_k(x, y)$ is given by the following formula:

$$D_k(x, y) = {}^t(-\mathbf{T}_{k,x} E_k(x - y)\boldsymbol{n}(y)) = \Big((-\mathbf{T}_{k,x} E_\ell^k(x - y))_{ij} n_j(y)\Big)_{\ell i}.$$

Here $\boldsymbol{n}(y)$ is the unit outer normal on $\partial\Omega$.

From strait-forward calculation, we see that $D_k(x, y)$ can be decomposed into $D_0(x, y)$ and continuous part when $\nu|x - y| \to 0$. This implies that the double layer kernel matrix $D_k(x, y)$ has the same singularity as $D_0(x, y)$.

Here and in what follows, let

$$(S_k^\bullet \boldsymbol{\Psi})(x) = \int_{\partial\Omega} E_k^{(r,c)}(x - y)\boldsymbol{\Psi}(y)\, d\sigma(y), \tag{3.1}$$

$$(D_k^\bullet \boldsymbol{\Psi})(x) = \int_{\partial\Omega} D_k^{(r)}(x, y)\boldsymbol{\Psi}(y)\, d\sigma(y). \tag{3.2}$$

$S_k^\bullet \boldsymbol{\Psi}$ and $D_k^\bullet \boldsymbol{\Psi}$ denote the single and double layer potentials associated with velocity, respectively. Here the 2×2 matrix $E_k^{(r,c)}$ is obtained from the fundamental tensor E_k by eliminating last row and last column, and $D_k^{(r)}$ is also obtained from D_k by eliminating the last row.

From (2.5)–(2.7) and the fact that $D_k(x, y)$ has the same singularity as $D_0(x, y)$, we have the following jump and continuity relations for the layer potentials corresponding to the velocity.

Proposition 3.1 (Jump and Continuity formulae). *Let $\boldsymbol{\Psi} \in C(\partial\Omega)^2$ and let $S_k^\bullet \boldsymbol{\Psi}$ and $D_k^\bullet \boldsymbol{\Psi}$ be the layer potentials defined by* (3.1) *and* (3.2), *respectively. Then the following formulae hold.*

$$(S_k^\bullet \boldsymbol{\Psi})^i = S_k^\bullet \boldsymbol{\Psi} = (S_k^\bullet \boldsymbol{\Psi})^e, \tag{3.3}$$

$$((S_k^\bullet)^* \boldsymbol{\Psi})^i = (S_k^\bullet)^* \boldsymbol{\Psi} = ((S_k^\bullet)^* \boldsymbol{\Psi})^e, \tag{3.4}$$

$$(D_k^\bullet \boldsymbol{\Psi})^i - D_k^\bullet \boldsymbol{\Psi} = +\frac{1}{2}\boldsymbol{\Psi} = D_k^\bullet \boldsymbol{\Psi} - (D_k^\bullet \boldsymbol{\Psi})^e, \tag{3.5}$$

$$((D_k^\bullet)^* \boldsymbol{\Psi})^i - (D_k^\bullet)^* \boldsymbol{\Psi} = -\frac{1}{2}\boldsymbol{\Psi} = (D_k^\bullet)^* \boldsymbol{\Psi} - ((D_k^\bullet)^* \boldsymbol{\Psi})^e. \tag{3.6}$$

Here $(S_k^\bullet)^*$ and $(D_k^\bullet)^*$ denote the dual operator of $S_k^\bullet$ and $D_k^\bullet$, respectively. $\boldsymbol{w}^i$ and $\boldsymbol{w}^e$ denote the limit from interior point and exterior point, respectively. Namely,

$$\boldsymbol{w}^i(z) = \lim_{\Omega^c \ni x \to z} \boldsymbol{w}(x), \qquad \boldsymbol{w}^e(z) = \lim_{\Omega \ni x \to z} \boldsymbol{w}(x).$$

Next we shall reduce our problems (1.4) and (1.5) to boundary integral equations. According to Borchers & Varnhorn [1], we choose the following ansatz.

$$(\mathrm{A}_k)\qquad \begin{pmatrix} u_k \\ p_k \end{pmatrix} = D_k \Psi - \eta S_k M \Psi + \frac{4\pi\alpha}{\log k} S_k \Psi,$$

$$(\mathrm{A}_0)\qquad \begin{pmatrix} u_0 \\ p_0 \end{pmatrix} = D_0 \Psi - \eta S_0 M \Psi - \alpha \int_{\partial\Omega} \begin{pmatrix} \Psi \\ 0 \end{pmatrix} d\sigma.$$

Here $M : \Psi \to M\Psi = \Psi - \Psi_M$, where

$$\Psi_M = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \Psi \, d\sigma, \quad |\partial\Omega|\text{: Lebesgue measure of } \partial\Omega.$$

From the boundary conditions of (1.4) and (1.5), (A_k), (A_0) and Proposition 3.1, we have the following systems of boundary equations:

$$(\mathrm{B}_k)\qquad \Phi = \left(-\frac{1}{2} I_2 + D_k^\bullet - \eta S_k^\bullet M + \frac{4\pi\alpha}{\log k} S_k^\bullet\right) \Psi \equiv K_k \Psi,$$

$$(\mathrm{B}_0)\qquad \Phi = \left(-\frac{1}{2} I_2 + D_0^\bullet - \eta S_0^\bullet M - \alpha|\partial\Omega|(I_2 - M)\right) \Psi \equiv K_0 \Psi.$$

For boundary integral equations (B_k) and (B_0), we have the following proposition.

Proposition 3.2. *Let* $\Phi \in C(\partial\Omega)^2$. *Then*

(i) *For any* $\eta, \alpha > 0$ *there exists an* $\varepsilon \in (0,1)$ *such that if* $0 < k < \varepsilon$, *then there exists exactly one solution* $\Psi \in C(\partial\Omega)^2$ *of the system of boundary integral equations* (B_k).

(ii) *For any* $\eta > 0$ *and* $\alpha \neq 0$ *there exists exactly one solution* $\Psi \in C(\partial\Omega)^2$ *of the system of boundary integral equations* (B_0).

Proof. The second assertion (ii) was already shown by [1] (see also [4]), we only show (i).

Since the operator K_k $(k > 0)$ is a compact on $C(\partial\Omega)^2$, by virtue of the Fredholm alternative theorem, solvability of (B_k) follows from the uniqueness for the adjoint problem with respect to usual inner product in $\mathbb{R}^2$. Therefore we shall investigate the following homogeneous problem:

$$(3.7)\qquad 0 = \left(-\frac{1}{2} I_2 + (D_k^\bullet)^* + M(S_k^\bullet)^* + \frac{4\pi\alpha}{\log k}(S_k^\bullet)^*\right) \Psi.$$

Let Ψ be a non-trivial solution to (3.7). Our task here is to show that $\Psi \equiv 0$. From (3.4) and (3.6), we have

$$0 = K_k^* \Psi = \left(-\frac{1}{2} I_2 + (D_k^\bullet)^* - \eta M (S_k^\bullet)^* + \frac{4\pi\alpha}{\log k} (S_k^\bullet)^* \right) \Psi$$
$$= \left(((D_k^\bullet)^*)^i - \eta M (S_k^\bullet)^* + \frac{4\pi\alpha}{\log k} (S_k^\bullet)^* \right) \Psi.$$

Therefore we obtain the equality:

$$((D_k^\bullet)^* \Psi)^i = \eta M (S_k^\bullet)^* \Psi - \frac{4\pi\alpha}{\log k} (S_k^\bullet)^* \Psi. \tag{3.8}$$

Set ${}^t(\boldsymbol{v}, q) = S_k^* \Psi(x)$. One can easily check that the pair of functions $(\boldsymbol{v}, q)$ solves the following dual Oseen problem in a bounded domain $\Omega_i \equiv \mathcal{B}$:

$$-\Delta \boldsymbol{v} - 2k \frac{\partial \boldsymbol{v}}{\partial x_1} - \nabla q = 0, \quad -\operatorname{div} \boldsymbol{v} = 0, \qquad x \in \Omega_i.$$

Hence by virtue of Lemma 2.3 in Ω_i and (3.8), we have

$$0 = \int_{\Omega_i} \left(-\Delta \boldsymbol{v} - \nabla q - 2k \frac{\partial \boldsymbol{v}}{\partial x_1} \right) \cdot \boldsymbol{v} \, dx$$
$$= \int_{\partial\Omega} \mathbf{T}_{k,x}^*(\boldsymbol{v}, q) \boldsymbol{n} \cdot \boldsymbol{v} \, d\sigma + 2 \int_{\Omega} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{v}) \, dx - k \int_{\Omega_i} \frac{\partial \boldsymbol{v}}{\partial x_1} \cdot \boldsymbol{v} \, dx$$
$$= \int_{\partial\Omega} ((D_k^\bullet)^* \Psi)^i \cdot \boldsymbol{v} \, d\sigma + 2 \int_{\Omega_i} \mathbf{D}(\boldsymbol{v}) : \mathbf{D}(\boldsymbol{v}) \, dx - \frac{k}{2} \int_{\partial\Omega} n_1 |\boldsymbol{v}|^2 \, d\sigma$$
$$= \eta \|M \boldsymbol{v}\|_{L^2(\partial\Omega)}^2 - \frac{4\pi\alpha}{\log k} \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + 2 \|\mathbf{D}(\boldsymbol{v})\|_{L^2(\Omega_i)}^2 - \frac{k}{2} \int_{\partial\Omega} n_1 |\boldsymbol{v}|^2 \, d\sigma$$
$$\geq \eta \|M \boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + \frac{1}{2} \left(\frac{8\pi\alpha}{\log(1/k)} - k \right) \|\boldsymbol{v}\|_{L^2(\partial\Omega)}^2 + 2 \|\mathbf{D}(\boldsymbol{v})\|_{L^2(\Omega_i)}^2.$$

Choose $k \in (0,1)$ in such a way that

$$\frac{8\pi\alpha}{\log(1/k)} - k > 0,$$

we can conclude that $\boldsymbol{v} = 0$ in $\overline{\Omega_i}$. Therefore we have

$$((D_k^\bullet)^* \Psi)^i = (q\boldsymbol{n})^i = \eta M \boldsymbol{v} - \frac{4\pi\alpha}{\log k} \boldsymbol{v} = 0.$$

On the other hand $S_k^* \Psi$ solves the following boundary value problem in the exterior domain $\Omega_e \equiv \Omega$:

$$\begin{cases} -\Delta \boldsymbol{v} - 2k \dfrac{\partial \boldsymbol{v}}{\partial x_1} - \nabla q = 0, \quad -\operatorname{div} \boldsymbol{v} = 0, & x \in \Omega_e, \\ \boldsymbol{v}|_{\partial\Omega_e} = 0. \end{cases}$$

Therefore from the uniqueness result due to Galdi [2], we have $(v, q) \equiv (0,0)$ in Ω_e. This yields that $((D_k^\bullet)^*\Psi)^e = 0$.

Since $\Psi = ((D_k^\bullet)^*\Psi)^e - ((D_k^\bullet)^*\Psi)^i$ (see (3.6)), we conclude that $\Psi = 0$. This completes the proof. □

4. Proof of Theorem 1.1

This section is devoted to the proof of our main result. First of all we shall show two key lemmas.

Lemma 4.1. *For K_k and K_0, there exists a $k_0 \in (0,1)$ such that if $|k| < k_0$ the following estimate holds.*

$$\|K_k - K_0\|_{\mathcal{L}} \leq \frac{C}{|\log k|}. \tag{4.1}$$

Here and hereafter $\|\cdot\|_{\mathcal{L}}$ stands for the operator norm of the space $\mathcal{L}(C(\partial\Omega), C(\partial\Omega))$.

Proof. Let $\Psi \in C(\partial\Omega)^2$. Then from ($B_k$) and ($B_0$),

$$\begin{aligned}\|K_k\Psi - K_0\Psi\|_{C(\partial\Omega)} &\leq \|D_k^\bullet\Psi - D_0^\bullet\Psi\|_{C(\partial\Omega)} + |\eta|\|S_k^\bullet M\Psi - S_0^\bullet M\Psi\|_{C(\partial\Omega)} \\ &\quad + |\alpha|\left\|\frac{4\pi}{\log k}S_k^\bullet\Psi + |\partial\Omega|(I_2 - M)\Psi\right\|_{C(\partial\Omega)} \\ &\equiv J_1 + |\eta|J_2 + |\alpha|J_3.\end{aligned}$$

J_1 and J_2 can be estimated by (2.5)–(2.7) and the fact that $D_k(x,y)$ can be decomposed into $D_0(x,y)$ and continuous parts. Estimate for J_3 also follows from (2.5)–(2.7) and decomposition of double layer kernel. In fact, from direct calculation, J_3 can be estimated as follows.

$$\begin{aligned}J_3 &= \sup_{x\in\partial\Omega}\left|\frac{4\pi}{\log k}S_k^\bullet\Psi + |\partial\Omega|(I_2 - M)\Psi\right| \\ &\leq \|\Psi\|_{C(\partial\Omega)}\sup_{x\in\partial\Omega}\left|\int_{\partial\Omega}\left(\frac{4\pi}{\log k}E_k^{(r,c)}(x-y) + I_2\right)d\sigma\right|.\end{aligned}$$

Set

$$A(x,y) = (a_{j\ell}(x,y))_{1\leq i,j\leq 2} = \frac{4\pi}{\log k}E_{j\ell}^k(x-y) + \delta_{j\ell}, \quad j,\ell = 1,2.$$

We shall estimate $\sup_{x\in\partial\Omega}\int_{\partial\Omega}a_{j\ell}(x,y)\,d\sigma$ for each j,ℓ. From (2.5), we obtain

$$\begin{aligned}a_{11}(x) &= \frac{4\pi}{\log k}E_{11}^k(x) + 1 \\ &= \frac{1}{\log k}\left(-\log|x| + \frac{x_1^2}{|x|^2} + \log 2 - \gamma\right) + kC_{11}(k,x).\end{aligned}$$

Therefore we have

$$\text{(4.2)} \qquad \sup_{x\in\partial\Omega}\left|\int_{\partial\Omega} a_{11}(x,y)\,d\sigma\right| \le \frac{C_1}{|\log k|} + C_2 k.$$

From a similar manner, we can conclude that

$$\text{(4.3)} \qquad \sup_{x\in\partial\Omega}\left|\int_{\partial\Omega} a_{22}(x,y)\,d\sigma\right| \le \frac{C_1}{|\log k|} + C_2 k.$$

Next we shall consider estimate of $a_{12}(x,y)$. From (2.2), we see that

$$a_{12}(x,y) = \frac{4\pi}{\log k} E_{12}^k(x-y) = \frac{1}{\log k}\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2} + 4\pi k C_{12}(k, x-y).$$

This implies that

$$\text{(4.4)} \qquad \sup_{x\in\partial\Omega}\left|\int_{\partial\Omega} a_{12}(x,y)\,d\sigma\right| \le \frac{C_1}{|\log k|} + C_2 k.$$

Combining (4.2)–(4.4), we have desired estimate for some $k \in (0,1)$. $\square$

Lemma 4.2. *For sufficiently small $k \in (0,1)$, we have*

$$\|K_k^{-1}\|_{\mathfrak{L}} \le 2\|K_0^{-1}\|_{\mathfrak{L}}.$$

Proof. From Theorem 3.2 (ii), the operator K_0 has bounded inverse. In view of (4.1), we shall choose k is sufficiently small in such a way that

$$\|K_k - K_0\|_{\mathfrak{L}} \le \frac{1}{2\|K_0^{-1}\|_{\mathfrak{L}}}.$$

Then the Neumann series

$$(I_2 - A_k)^{-1} = \sum_{\ell=0}^{\infty} A_k^{\ell} \quad \text{with } A_k = K_0^{-1}(K_0 - K_k) = I_2 - K_0^{-1}K_k,$$

converges absolutely in $\mathfrak{L}(C(\partial\Omega), C(\partial\Omega))$ with

$$\|(I_2 - A_K)^{-1}\|_{\mathfrak{L}} \le (1 - \|A_k\|_{\mathfrak{L}})^{-1} \le (1 - \|K_0^{-1}\|_{\mathfrak{L}}\|K_0 - K_k\|_{\mathfrak{L}})^{-1} \le 2.$$

Since $A_k = I_2 - K_0^{-1}K_k$ is equivalent to $K_k^{-1} = (I_2 - A_k)^{-1}K_0^{-1}$, we have

$$\|K_k^{-1}\|_{\mathfrak{L}} \le \|(I_2 - A_k)^{-1}\|_{\mathfrak{L}}\|K_0^{-1}\|_{\mathfrak{L}} \le 2\|K_0^{-1}\|_{\mathfrak{L}}.$$

This is desired estimate. $\square$

We are now in a position to show our main theorem.

Proof of Theorem 1.1. Let $\boldsymbol{u}_k$ and $\boldsymbol{u}_0$ be solutions of (1.4) and (1.5), respectively,

$$\boldsymbol{u}_k(x) = L_k\boldsymbol{\Psi} = \left(D_k^{\bullet} - \eta S_k^{\bullet} M + \frac{4\pi\alpha}{\log k} S_k^{\bullet}\right)\boldsymbol{\Psi}, \quad \text{with } \boldsymbol{\Psi} = K_k^{-1}\boldsymbol{\Phi}$$

$$\boldsymbol{u}_0(x) = L_0\boldsymbol{\Psi} = (D_0^{\bullet} - \eta S_0^{\bullet} M - \alpha|\partial\Omega|(I - M))\boldsymbol{\Psi}, \quad \text{with } \boldsymbol{\Psi} = K_0^{-1}\boldsymbol{\Phi}.$$

Then we have

$$\begin{aligned}|u_k(x) - u_0(x)| &= |L_k K_k^{-1}\Phi - L_0 K_0^{-1}\Phi| \\ &= |(L_k - L_0)K_k^{-1}\Phi| + |L_0(K_k^{-1} - K_0^{-1})\Phi| \equiv G_1 + G_2.\end{aligned}$$

We shall estimate G_1 and G_2 over $\Omega \cap B_R$ with $R > R_0$. First we consider G_1. From Lemmas 4.1 and 4.2

$$\sup_{x\in\Omega\cap B_R} |G_1| \le \frac{C_R}{|\log k|}\|K_k^{-1}\Phi\|_{C(\partial\Omega)} \le \frac{C_R}{|\log k|}\|\Phi\|_{C(\partial\Omega)}. \tag{4.5}$$

Here C_R is a constant depends on R which diverges when $R \to \infty$.

Next we shall estimate G_2. Since $K_k^{-1} = (I_2 - A_k)^{-1}K_0^{-1}$ with $A_k = I_2 - K_0^{-1}K_k$,

$$K_k^{-1} = \sum_{\ell=0}^{\infty} A_k^{\ell} \cdot K_0^{-1} = K_0^{-1} + \sum_{\ell=1}^{\infty} A_k^{\ell} \cdot K_0^{-1}.$$

Hence, from Lemmas 4.1 and 4.2, we see that

$$\begin{aligned}\|K_k^{-1} - K_0^{-1}\|_{\mathfrak{L}} &\le \left\|\sum_{\ell=1}^{\infty} A_k^{\ell}\right\|_{\mathfrak{L}} \cdot \|K_0^{-1}\|_{\mathfrak{L}} \le \|A_k\|_{\mathfrak{L}} \left\|\sum_{\ell=0}^{\infty} A_k^{\ell}\right\|_{\mathfrak{L}} \cdot \|K_0^{-1}\|_{\mathfrak{L}} \\ &\le C\|K_k - K_0\|_{\mathfrak{L}} \le \frac{C}{|\log k|}.\end{aligned}$$

Therefore, we have

$$\sup_{x\in\Omega\cap B_R} |G_2| \le C_R\|(K_k^{-1} - K_0^{-1})\Phi\|_{C(\partial\Omega)} \le \frac{C_R}{|\log k|}\|\Phi\|_{C(\partial\Omega)}. \tag{4.6}$$

Combining (4.5) and (4.6), we have our main theorem. $\square$

Acknowledgement. Research of the third author was supported by Waseda University Grant for Special Research Project (2007-B-177).

References

[1] W. Borchers and W. Varnhorn. On the boundedness of the Stokes semigroup in two-dimensional exterior domains. *Math. Z.*, 213(2):275–299, 1993.

[2] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.

[3] C. W. Oseen. *Neuere Methoden und Ergebnisse in der Hydrodynamik*. Akademische Verlagsgesellshaft, Leipzig, 1927.

[4] W. Varnhorn. *The Stokes equations*, volume 76 of *Mathematical Research*. Akademie-Verlag, Berlin, 1994.