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The Stokes and Navier-Stokes equations in an aperture domain

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Abstract

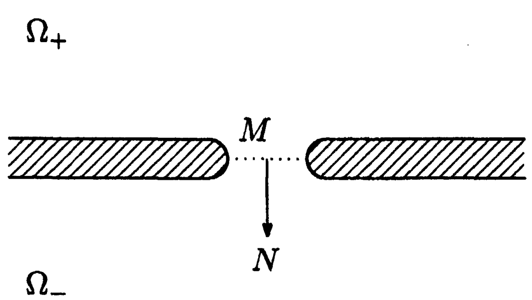
We consider the nonstationary Navier-Stokes equations in an aperture domain $\Omega\subset\mathbb{R}^{n}$, $n\geq 2$. Main purpose of this paper is to discuss the existence of a unique solution to the Navier-Stokes problem with a zero and a non-zero flux condition through the aperture.

To this end, we prove $L^{p}-L^{q}$ type estimate of the Stokes semigroup in the aperture domain. Applying them to the Navier-Stokes initial value problem in the aperture domain, we can prove the global existence of a unique solution to the Navier-Stokes problem with the zero-flux condition and some decay properties as $t\rightarrow\infty$, when the initial velocity is sufficiently small in the L^{n} space. Moreover we can prove the time-local existence of a unique solution to the Navier-Stokes problem with the non-trivial flux condition.

1 Introduction

An aperture domain $\Omega\subset\mathbb{R}^{n}$ $(n\geq 2)$ is an unbounded domain with noncompact boundary $\partial\Omega$. Roughly speaking, Ω consists of two disjoint half-spaces separated by a wall and connected by a hole (aperture) through this wall (see section 2 for detail).

We assume that $\partial\Omega$ is smooth enough, $\partial\Omega\in C^{1}$ for the Helmholtz decomposition, $\partial\Omega\in C^{2,\mu}(0<\mu<1)$ for the Stokes resolvent system and that Ω is divided into some upper domain Ω_{+}, some lower domain Ω_{-} and some smooth $(n-1)$-dimensional manifold M in the hole such that $\Omega=\Omega_{+}\cup M\cup\Omega_{-}$.



In $\Omega \times (0,\infty)$, we consider the nonstationary Navier-Stokes initial boundary value problem:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla\pi = 0 & \text{in} \quad \Omega\times(0,\infty), \\ \nabla\cdot u = 0 & \text{in} \quad \Omega\times(0,\infty), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega\times(0,\infty), \\ u(x,0) = a(x) & \text{in} \quad \Omega \end{cases} \tag{NS}$$

for the unknown velocity field $u = (u_1,\dots,u_n) \in W^{2,p}(\Omega)^n$ and the unknown scalar pressure term $\nabla\pi \in L^p(\Omega)^n$, where $1 < p < \infty$.

The aperture domain is a particularly interesting class of domains with noncompact boundaries. In 1976, Heywood [23] pointed out that the solution may not be uniquely determined by usual boundary conditions in this domain and therefore in order to get a unique solution u we may have to prescribe either the pressure drop $[\pi]$ at infinity between the upper and lower subdomains $\Omega_\pm$:

$$[\pi] = \lim_{|x|\to\infty,\ x\in\Omega_+} \pi(x) - \lim_{|x|\to\infty,\ x\in\Omega_-} \pi(x)$$

or the flux $\phi(u)$ through the aperture M :

$$\phi(u) = \int_M N\cdot u\, d\sigma,$$

where N denotes the normal vector on M directed to Ω_-, as an additional boundary condition. When $n = 2$, for $1 < p \le 2$ the solution is unique and the flux vanishes, whereas for $p > 2$ the flux has to be given. When $n \ge 3$, for $1 < p \le \frac{n}{n-1} (=: n')$ the solution is unique, without claiming any additional boundary condition. If $n' < p < n$, either the flux or the pressure drop can be prescribed, whereas for $p \ge n$ only the flux can be given (see Farwig [15]).

We shall introduce the known results concerning the aperture domain Ω. The results of Farwig and Sohr [17] and Miyakawa [34] are the first step to discuss the nonstationary problem (NS) in the L^p-space. They showed the Helmholtz decomposition of the L^p-space of vector fields $L^p(\Omega)^n = J^p(\Omega) \oplus G^p(\Omega)$ for $n \ge 2$ and $1 < p < \infty$, where $J^p(\Omega)$ and $G^p(\Omega)$ denote as follows:

$$J^p(\Omega) = \overline{\{u \in C_0^\infty(\Omega)^n \mid \nabla\cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^p(\Omega)^n}},$$
$$G^p(\Omega) = \{\nabla\pi \in L^p(\Omega)^n \mid \pi \in L^p_{loc}(\overline{\Omega})\}.$$

The space $J^p(\Omega)$ is characterized as

$$J^p(\Omega) = \{u \in L^p(\Omega) \mid \nabla\cdot u = 0,\ \ \nu\cdot u|_{\partial\Omega} = 0,\ \ \phi(u) = 0\},$$

where ν is the unit outer normal vector on $\partial\Omega$ (see [17, Lemma 3.1]) . Here the condition $\phi(u) = 0$ is automatically satisfied and may be omitted if $1 < q \le n'$ but otherwise the element of $J^p(\Omega)$ have to possess this condition $\phi(u) = 0$.

Let P be a continuous projection from $L^p(\Omega)^n$ to $J^p(\Omega)$ associated with the Helmholtz decomposition. The Stokes operator A is defined by $A = -P\Delta$ with a domain which is introduced in section 2. It is proved by Farwig and Sohr [17] that $-A$ generates a bounded analytic semigroup $T(t)$ on $J^p(\Omega)$.

The main purpose of this paper is to prove the global existence of a unique solution to the Navier-Stokes problem with the zero-flux condition through the aperture when the initial velocity is sufficiently small in $L^n(\Omega)$ and the local-existence of a unique solution to the Navier-Stokes problem with the non-trivial flux condition. The main step of the proof is to show the following $L^p - L^q$ estimates of the Stokes semigroup:

$$\|T(t)a\|_{L^q(\Omega)^n} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|a\|_{L^p(\Omega)^n}, \tag{1.1}$$

$$\|\nabla T(t)a\|_{L^q(\Omega)^{n^2}} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|a\|_{L^p(\Omega)^n} \tag{1.2}$$

for $a \in J^p(\Omega)$ and $t > 0$, where $1 \leq p \leq q \leq \infty$ ($p \neq \infty, q \neq 1$) for (1.1) and $1 \leq p \leq q < \infty$ ($q \neq 1$) for (1.2).

The $L^p - L^q$ estimates of the Stokes semigroup have been already studied by many authors in some cases of other domains. In fact, when Ω is the whole space, applying the Young inequality to the concrete solution formula, we have (1.1) and (1.2) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$). When Ω is the half-space, it is proved by Ukai [40] and Borchers and Miyakawa [5] that (1.1) and (1.2) hold for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) (cf. Desch, Hieber and Prüss [12]). When Ω is an infinite layer case, Abe and Shibata [1] proved that (1.1) and (1.2) hold for $1 < p \leq q < \infty$. When Ω is a bounded domain, (1.1) and (1.2) for $1 < p \leq q < \infty$ follow from the result of Giga [20] on a characterization of the domains of fractional powers of the Stokes operator. In an infinite layer case and a bounded domain case, an exponential decay property of the semigroup is available.

When Ω is an exterior domain, (1.1) holds for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) but (1.2) holds only for $1 \leq p \leq q \leq n$ ($q \neq 1$). At first Iwashita [24] proved that (1.1) holds for $1 < p \leq q < \infty$ and (1.2) for $1 < p \leq q \leq n$ when $n \geq 3$. The refinement of his result was done by the following authors: Chen [8] ($n = 3$, $q = \infty$), Shibata [37] ($n = 3$, $q = \infty$), Borchers and Varnhorn [7] ($n = 2$, (1.1) for $p = q$), Dan and Shibata [9], [10] ($n = 2$), Dan, Kobayashi and Shibata [11] ($n = 2, 3$), and Maremonti and Solonnikov [32] ($n \geq 2$). Especially, it was shown by Maremonti and Solonnikov [32] that Iwashita's restriction $q \leq n$ in (1.2) is unavoidable.

When Ω is a perturbed half-space, Kubo and Shibata [31] proved (1.1) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$) and (1.2) for $1 \leq p \leq q < \infty (q \neq 1)$ when $n \geq 2$.

When Ω is an aperture domain, Abels [2] proved (1.1) for $1 < p \leq q < \infty$ and (1.2) for $1 < p \leq q < n$ when $n \geq 3$; and Hishida [22] proved (1.1) for $1 \leq p \leq q \leq \infty$ ($p \neq \infty, q \neq 1$) and (1.2) for $1 \leq p \leq q \leq n$ ($q \neq 1$) and $1 \leq p < n < q < \infty$ when $n \geq 3$.

This paper reports that (1.1) holds for $1 \leq p \leq q \leq \infty$ ($p \neq \infty, q \neq 1$) and (1.2) holds for $1 \leq p \leq q < \infty$ ($q \neq 1$) when $n \geq 2$. In particular, the gradient estimate (1.2) without any restriction on (p, q) is our important contribution and also our result covers the case $n = 2$. Although the result of [22] is sufficient for the proof of the global existence of the Navier-Stokes flow with small L^n data ($n \geq 3$), the improvement above of the gradient estimate is of own interest and also implies optimal decay rates of the gradient of the global solution of [22] in L^r with $r > n$; see Theorem 2.3. Recently in [31] the author and Shibata proved the $L^p - L^q$ estimates of the Stokes semigroup for the same (p, q) as above and $n \geq 2$ in the case of a perturbed half-space by using a precise analysis of the resolvent for the half-space problem due to ourselves [30]. Since the aperture domain is obtained from upper and lower half-spaces by a perturbation within a bounded region, one can exactly follow the argument of [31] in the proof of (1.1) and (1.2). In this paper, we give the outline of the proof in our context of the aperture domain. As explained above, the aperture domain is physically more interesting than the perturbed half-space; for instance, one can discuss the fluid motion when a non-trivial flux $\phi(u)$ through the aperture is prescribed.

Lastly, we introduce the known result concerning the global existence of the solution to the Navier-Stokes problem with small L^n data. It is well-known that we can prove the global existence as an application of the $L^p - L^q$ estimate of the Stokes semigroup. In fact, the time-global existence was proved by many authors in the following domain cases: Giga and Miyakawa [21] for bounded domains, Kato [25] for the whole space, Ukai [40] and Kozono [26] for the half-space, Iwashita [24] and Wiegner [41] for the exterior domain, Abe and Shibata [1] for the infinite layer, Kubo and Shibata [31] for the perturbed half-space and Hishida [22] for the aperture domain. On the other hand, concerning the local existence of strong solutions with a non-trivial flux through the aperture, we refer to Heywood [23] and Franzke [18], both of which are L^2 theory.

This paper reports that we can prove the global existence of a unique solution to (NS) with $\phi(u) = 0$ when the initial velocity is sufficiently small in $L^n(\Omega)$ and the local-existence of a unique solution to (NS) with $\phi(u) \neq 0$ in $L^p(p > 2)$ framework.

2 Main theorems and notations

First of all, in order to discuss our results more precisely we outline our notation used throughout this paper. We define upper and lower half-spaces by $H_\pm = \{x \in \mathbb{R}^n \mid \pm x_n > 1\}$, and sometimes write $H = H_+$ or H_- to describe some assertions for the half-space. To denote the special sets we use the following symbols:

$$B_R = \{x \in \mathbb{R}^n \mid \ |x| < R\}, \ \ \Omega_R = \Omega \cap B_R, \ \ H_R = H \cap B_R, \tag{2.1}$$

where $x' = (x_1, \dots, x_{n-1})$. Let $\Omega \subset \mathbb{R}^n$ be an aperture domain with smooth enough boundary $\partial\Omega$, namely, there is a positive number R_0 such that

$$\Omega \backslash B_{R_0} = (H_+ \cup H_-) \backslash B_{R_0} \tag{2.2}$$

In what follows we fix such R_0. Ω is divided into some upper domain Ω_+, some lower domain Ω_- and some smooth $(n-1)$-dimensional manifold M in the hole such that $\Omega = \Omega_+ \cup M \cup \Omega_-$, $\Omega_\pm \backslash B_{R_0} = H_\pm \backslash B_{R_0}$ and $M \cup \partial M = \partial\Omega_+ \cap \partial\Omega_- \subset \overline{B_{R_0}}$.

For a domain $G \subset \mathbb{R}^n$ we will use the standard symbols: for example, $L^p(G)$ denotes the Lebesgue space with norm $\|\cdot\|_{L^p(G)}$ and $W^{m,p}(G)$ denotes the Sobolev space with norm $\|\cdot\|_{W^{m,p}(G)}$. We set

$$L^p_R(G) = \{f \in L^p(G) \mid \ f(x) = 0 \text{ for } |x| > R\}.$$

We often use the same symbols for denoting the vector and scalar function spaces if there is no confusion.

For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear operators from X to Y. We write $\mathcal{L}(X) = \mathcal{L}(X,X)$. $\mathcal{B}(U;X)$ denotes the set of all X-valued bounded holomorphic functions on U. And $BC([0,T);X)$ denotes the class of X-valued bounded continuous function on $[0,T)$.

When Ω is the half-space or the aperture domain, the space $L^p(\Omega)$ admits the Helmholtz decomposition

$$L^p(\Omega) = J^p(\Omega) \oplus G^p(\Omega)$$

for $1 < p < \infty$ and $n \geq 2$, where $J^p(\Omega)$ and $G^p(\Omega)$ are defined by the following relation respectively:

$$J^p(\Omega) = \overline{\{u \in C_0^\infty(\Omega) \mid \nabla \cdot u = 0 \text{ in } \Omega\}}^{\|\cdot\|_{L^p(\Omega)}},$$
$$G^p(\Omega) = \{\nabla\pi \in L^p(\Omega) \mid \pi \in L^p_{loc}(\overline{\Omega})\}.$$

Let $P_{p,\Omega}$ be a continuous projection from $L^p(\Omega)$ to $J^p(\Omega)$ associated with the Helmholtz decomposition. The Stokes operator $A_{p,\Omega}$ is defined by $A_{p,\Omega} = -P_{p,\Omega}\Delta$ with a domain

$$D(A_{p,\Omega}) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \cap J^p(\Omega),$$

where $1 < p < \infty$. For simplicity we use the abbreviations P_p for $P_{p,\Omega}$ and A_p for $A_{p,\Omega}$ when Ω is an aperture domain and the subscript p is also often omitted if there is no confusion. The Stokes operator satisfies the parabolic resolvent estimate

$$\|(\lambda + A_\Omega)^{-1}\|_{\mathcal{L}(J^p(\Omega))} \leq \frac{C_\varepsilon}{|\lambda|} \tag{2.3}$$

for $|\arg\lambda| \leq \pi - \varepsilon$ ($\lambda \neq 0$), where $\varepsilon > 0$ is arbitrary (see Farwig [15] and Farwig and Sohr [17] for the aperture domain, McCracken [33] and Farwig and Sohr [16] for the half-space). Estimate (2.3) implies that $-A_\Omega$ generates a bounded analytic semigroup $T_{A_\Omega}(t)$ of class C_0 in each $J^p(\Omega)$. We write $E_\pm(t) = T_{A_{H_\pm}}(t)$ and $T(t) = T_A(t)$ as the Stokes semigroup for the half-space and the one for the aperture domain respectively.

The following theorem provides the $L^p - L^q$ estimates of Stokes semigroup $T(t)$ for the aperture domain.

Theorem 2.1 (**$L^p - L^q$ estimates**). *Let $n \geq 2$.*
(i) *Let $1 \leq p \leq q \leq \infty$ ($p \neq \infty$, $q \neq 1$). There exists a positive constant $C_{p,q}$ such that*

$$\|T(t)f\|_{L^q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\Omega)} \tag{2.4}$$

for all $t > 0$ and $f \in J^p(\Omega)$. When $p = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap J^s(\Omega)$ for some $s \in (1,\infty)$.
(ii) *Let $1 \leq p \leq q < \infty$ ($q \neq 1$), there holds the estimate:*

$$\|\nabla T(t)f\|_{L^q(\Omega)} \leq C_{p,q} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|f\|_{L^p(\Omega)} \tag{2.5}$$

for all $t > 0$ and $f \in J^p(\Omega)$. When $p = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap J^s(\Omega)$ for some $s \in (1,\infty)$.

Next we consider an application of the $L^p - L^q$ estimates to the Navier-Stokes initial value problem (NS). Applying the solenoidal projection P to (NS), we can rewrite (NS) with $\phi(u) = 0$ as follows:

$$\partial_t u + Au + P((u \cdot \nabla)u) = 0, \quad u(0) = a, \tag{PNS}$$

where $A = -P\Delta$ is the Stokes operator.

For given $a \in J^n(\Omega)$ and $0 < T \leq \infty$, a measurable function u defined on $\Omega \times (0,T)$ is called a strong solution of (NS) on $(0,T)$ satisfying $\phi(u) = 0$ if u belongs to

$$u \in C([0,T); J^n(\Omega)) \cap C((0,T); D(A)) \cap C^1((0,T); J^n(\Omega))$$

together with $\lim_{t\to 0} \|u(t) - a\|_{L^n} = 0$ and satisfies (PNS) for $0 < t < T$ in $J^n(\Omega)$.

In the same way as Hishida's argument [22], we can show the following theorem which tells us the global existence of a strong solution to (NS) with $\phi(u) = 0$ and several decay properties when the initial data a are small:

Theorem 2.2. *Let $n \geq 2$. There exists a constant $\delta = \delta(\Omega, n) > 0$ with the following property: if $a \in J^n(\Omega)$ satisfies $\|a\|_{L^n} \leq \delta$, the problem* (NS) *with $\phi(u) = 0$ admits a unique strong solution $u(t)$ on $(0,\infty)$. Moreover as $t \to \infty$,*

$$\|u(t)\|_{L^r} = o(t^{-\frac{1}{2}+\frac{n}{2r}}) \qquad \text{for} \quad n \leq r \leq \infty, \tag{2.6}$$

$$\|\nabla u(t)\|_{L^r} = o(t^{-1+\frac{n}{2r}}) \qquad \text{for} \quad n \leq r < \infty, \tag{2.7}$$

$$\|\partial_t u(t)\|_{L^r} + \|Au(t)\|_{L^r} = o(t^{-\frac{3}{2}+\frac{n}{2r}}) \qquad \text{for} \quad n \leq r < \infty. \tag{2.8}$$

For $n = 2$, the smallness of $\|a\|_{L^2(\Omega)}$ is redundant.

Moreover if $a \in L^1(\Omega) \cap J^n(\Omega)$ has small $\|a\|_{L^n}$, then we can show the following theorem. For the case $n \geq 3$, the results are exactly the same as those in [22].

Theorem 2.3. *Let $n \geq 2$. There exists a constant $\eta = \eta(\Omega, n) \in (0, \delta]$ with the following property: if $a \in L^1(\Omega) \cap J^n(\Omega)$ satisfies $\|a\|_{L^n} \leq \eta$, then the solution $u(t)$ obtained in Theorem 2.2 enjoys*

$$\|u(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \qquad \text{for } 1 < r \leq \infty, \tag{2.9}$$

$$\|\nabla u(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \qquad \text{for } 1 < r < \infty, \tag{2.10}$$

$$\|\partial_t u(t)\|_{L^r} + \|Au(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})-1}) \qquad \text{for } 1 < r < \infty, \tag{2.11}$$

$$\|\nabla^2 u(t)\|_{L^r} + \|\nabla \pi(t)\|_{L^r} = O(t^{-\frac{n}{2}(1-\frac{1}{r})-1}) \qquad \text{for } 1 < r < n \tag{2.12}$$

as $t \to \infty$. Moreover, for each $t > 0$ there exist two constants $\pi_\pm(t) \in \mathbb{R}$ such that $\pi(t) - \pi_\pm(t) \in L^r(\Omega_\pm)$ with

$$\|\pi(t) - \pi_\pm\|_{L^r(\Omega_\pm)} + |[\pi(t)]| = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \qquad \text{for } n' < r < \infty \tag{2.13}$$

as $t \to \infty$ where $[\pi(t)] = \pi_+(t) - \pi_-(t)$. For $n = 2$, the smallness of $\|a\|_{L^2(\Omega)}$ is redundant.

Remark 2.4. In the two dimensional case, Kozono and Ogawa [27] established the global existence result without the smallness of $\|a\|_{L^2}$ for an arbitrary unbounded domain, which covers the aperture domain with the hidden flux condition $\phi(u) = 0$. But (2.6) with $r = \infty$ was not obtained in [27]. In [28] they derived various decay properties of the global solution when $a \in L^p(\Omega) \cap J^2(\Omega)$ with $1 < p < 2$.

Next, we shall consider the case where the flux through the aperture is non-trivial. We fix an auxiliary function $\chi \in C^\infty(\Omega) \cap W^{2,p}(\Omega)(n' < p < \infty)$ satisfying

$$\chi|_{\partial\Omega} = 0, \ \nabla \cdot \chi = 0, \ \phi(\chi) = 1$$

(see Heywood [23, Lemma 11] and Galdi [19, III.4.3]). Given a flux $\phi(v(t)) = \alpha(t)$, we study the problem (NSf) (see section 4). We set $u(t,x) = v(t,x) - \alpha(t)\chi(x)$ and reduce (NSf) to (NS′) with vanishing flux condition (see section 4). For $n \geq 3$, the notion of strong solution u to (NS′) with $\phi(u) = 0$ is defined similarly to that given above for (NS) with $\phi(u) = 0$. For $n = 2$, the auxiliary function χ does not belong to $L^2(\Omega)$ and the force term includes $\alpha'\chi$ in (NS′); thus, the solution u to (NS′) can't belong to $J^2(\Omega)$. Therefore we must change the definition of the strong solution $u(t)$ to (NS′) with $\phi(u) = 0$ as follows: for given $a \in J^p(\Omega)$ $(n = 2 < p < \infty)$ and $0 < T \leq \infty$, a measurable function u defined on $\Omega \times (0,T)$ is called a strong solution of (NS′) on $(0,T)$ satisfying $\phi(u) = 0$ if u belongs to

$$u \in C([0,T); J^p(\Omega)) \cap C((0,T); D(A)) \cap C^1((0,T); J^p(\Omega))$$

together with $\lim_{t\to 0} \|u(t) - a\|_{L^p} = 0$ and satisfies (PNS′) for $0 < t < T$ in $J^p(\Omega)$.

The following theorem gives us the time-local solution to the Navier-Stokes problem with a non-flux condition:

Theorem 2.5. *Suppose that the flux $\phi(v(t)) = \alpha(t)$ belongs to $C^{1,\theta}([0,T])$ with some $T > 0$ and $0 < \theta < 1$ in the problem* (NSf).
(i) *Let $n \geq 3$. If $a - \alpha(0)\chi \in J^n(\Omega)$, then there exists $T_* \in (0,T]$ such that the reduced problem* (NS′) *admits a unique strong solution $u(t)$ on $(0,T_*)$. Moreover the solution $u(t)$ satisfies*

$$t^{\frac{1}{2}-\frac{n}{2r}} u \in BC([0,T_*); J^r(\Omega)) \quad \text{for} \quad n \leq r \leq \infty, \tag{2.14}$$
$$t^{1-\frac{n}{2r}} \nabla u \in BC([0,T_*); L^r(\Omega)) \quad \text{for} \quad n \leq r < \infty. \tag{2.15}$$

The values of $t^{\frac{1}{2}-\frac{n}{2r}}u(t)$ and $t^{1-\frac{n}{2r}}\nabla u(t)$ at $t = 0$ vanish except for $r = n$ in (2.14), *in which $u(0) = a - \alpha(0)\chi$.*
(ii) *Let $n = 2 < p < \infty$. If $a - \alpha(0)\chi \in J^p(\Omega)$, then there is $T_* \in (0,T]$ such that the reduced problem* (NS′) *admits a unique strong solution $u(t)$ on $(0,T_*)$. Moreover the solution $u(t)$ satisfies*

$$t^{\frac{1}{p}-\frac{1}{r}} u \in BC([0,T_*); J^r(\Omega)) \quad \text{for} \quad p \leq r \leq \infty, \tag{2.16}$$
$$t^{\frac{1}{p}-\frac{1}{r}+\frac{1}{2}} \nabla u \in BC([0,T_*); L^r(\Omega)) \quad \text{for} \quad p \leq r < \infty. \tag{2.17}$$

The values of $t^{\frac{1}{p}-\frac{1}{r}}u(t)$ and $t^{\frac{1}{p}-\frac{1}{r}-\frac{1}{2}}\nabla u(t)$ at $t = 0$ vanish except for $r = p$ in (2.16), *in which $u(0) = a - \alpha(0)\chi$.*

3 Outline of the proof of Theorem 2.1

In this section, we shall describe the outline of the proof of $L^p - L^q$ estimates of Stokes semigroup in the aperture domain (Theorem 2.1). Our proof is based on the following local energy decay estimate.

Lemma 3.1 (Local energy decay). *Let $n \geq 2$, $1 < p < \infty$ and $R > R_0$. Then there exists a positive constant C_p such that the inequality*

$$\|T(t)Pf\|_{L^p(\Omega_R)} \leq C_p t^{-\frac{n+1}{2}} \|f\|_{L^p(\Omega)} \tag{3.1}$$

is valid for any $f \in L^p_R(\Omega)$ and $t \geq 1$.

In order to prove the local energy decay estimate in the same way as Iwashita [24] , we need the expansion formula of the solution operator near the origin as follows:

Lemma 3.2. *Let $n \geq 2$ and $(R(\lambda), \Pi(\lambda))$ be the solution operator to resolvent Stokes problem. We set $B_H = \mathcal{L}(L^p_R(H), W^{2,p}(H_R) \times W^{1,p}(H_R))$. Then $(R(\lambda), \Pi(\lambda))$ has the following expansion formula with respect to $\lambda \in \{\lambda \in \mathbb{C} \backslash (-\infty, 0] \mid |\lambda| < 1/2\}$:*

$$(R(\lambda), \Pi(\lambda)) = \begin{cases} G_1(\lambda)\lambda^{\frac{n-1}{2}} + G_2(\lambda)\lambda^{\frac{n}{2}} \log \lambda + G_3(\lambda) & \text{where } n \text{ is even,} \\ G_1(\lambda)\lambda^{\frac{n}{2}} + G_2(\lambda)\lambda^{\frac{n-1}{2}} \log \lambda + G_3(\lambda) & \text{where } n \text{ is odd,} \end{cases} \tag{3.2}$$

where $G_1(\lambda), G_2(\lambda)$ and $G_3(\lambda)$ are B_H-valued holomorphic functions in $\{\lambda \in \mathbb{C} \mid |\lambda| < 1/2\}$.

By using the Dunford integral representation of the Stokes semigroup in terms of the resolvent together with a formula of the gamma function, we can obtain Lemma 3.1. We refer to Kubo and Shibata [31] and Kubo [29] for details

Remark 3.3. Higher order derivatives of $T(t)Pf$ in t and x are discussed similarly. For example, we can prove the estimates:

$$\|\partial_t^m T(t)Pf\|_{W^{2,p}(\Omega_R)} \leq C_p t^{-\frac{n+1}{2}-m} \|f\|_{L^p(\Omega)}$$

for nonnegative integer m.

Remark 3.4. For the exterior domain case, Iwashita [24] proved that there holds the estimate:

$$\|T(t)Pf\|_{L^p(\Omega_R)} \leq Ct^{-\frac{n}{2}} \|f\|_{L^p(\Omega)}.$$

The reason why the rate of decay for the aperture domain case is one-half better than the one for the exterior domain case is that the worst term in expansion is canceled out by the reflection at the boundary.

Next we shall go on showing the $L^p - L^q$ estimate in an aperture domain Ω by using the cut-off technique. First we show the decay estimate of the Stokes semigroup in Ω_R for general data. By using Lemma 3.1 and the $L^p - L^q$ estimate of Stokes semigroup $E_\pm(t)$

in the half-space proved by Ukai [40] and Borchers and Miyakawa [5], together with a Poincare type inequality:

$$\|E_\pm(t)f\|_{L^p(C_R^\pm)} \leq R\|\nabla E_\pm(t)\|_{L^p(C_R^\pm)} \tag{3.3}$$

for the cylinder $C_R^\pm = \{x \in H_\pm \mid |x'| \leq R,\ \pm x_n \leq R\}$, we obtain the following lemma :

Lemma 3.5. *Let $n \geq 2$, $1 < p < \infty$ and $R \geq R_0$. Then there exists a positive number $C = C(\Omega, n, p, R)$ such that*

$$\|\partial_t T(t)f\|_{W^{1,p}(\Omega_R)} + \|T(t)f\|_{W^{1,p}(\Omega_R)} \leq Ct^{-\frac{n}{2p}-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ and $t \geq 2$.

Remark 3.6. We know that in the exterior domain, there holds the following estimate:

$$\|T(t)f\|_{W^{1,p}(\Omega_R)} \leq Ct^{-\frac{n}{2p}}\|f\|_{L^p(\Omega)}.$$

The reason why the rate of decay for the aperture domain case is one half better than the one for the exterior domain case is that the better decay obtained in Theorem 3.1 and the Poincare type inequality (3.3) hold.

Secondly we show the $L^p - L^q$ estimates of Stokes semigroup in $\Omega_\pm \backslash \Omega_R$. By using the cut-off technique and the $L^p - L^q$ estimates of Stokes semigroup $E(t)$ in the half-space, we obtain the following lemma:

Lemma 3.7. (i) *Let $1 < p \leq q \leq \infty$ $(p \neq \infty)$ with $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$. Then there exists a positive number $C = C(p, q, R)$ such that*

$$\|T(t)f\|_{L^q(\Omega_\pm \backslash \Omega_R)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ and $t \geq 2$.
(ii) *Let $1 < p < \infty$. Then there exists a positive number $C = C(p, R)$ such that*

$$\|\nabla T(t)f\|_{L^p(\Omega_\pm \backslash \Omega_R)} \leq Ct^{-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ and $t \geq 2$.

Thirdly we prove the $L^p - L^q$ estimates of Stokes semigroup $T(t)$ in the aperture domain near $t = 0$. By using the interpolation theory and the resolvent estimate of Stokes semigroup, we obtain the following lemma:

Lemma 3.8. *Let $1 < p \leq q \leq \infty$ $(p \neq \infty)$ with $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$. Then there exists a positive number $C = C(p, q, R)$ such that*

$$\|T(t)f\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\Omega)},$$
$$\|\nabla T(t)f\|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|f\|_{L^p(\Omega)}$$

for $f \in J^p(\Omega)$ and $0 < t < 2$.

We can immediately show Theorem 2.1 from the three lemmas above.

4 The Navier-Stokes flow in an aperture domain

In this section, we shall apply the $L^p - L^q$ estimate to the Navier-Stokes equation. We begin with the proof of Theorem 2.2.

Proof of Theorem 2.2. By means of a standard contraction mapping principle in the same way as Kato [25], we can construct a unique global solution $u(t)$ of the integral equation

$$u(t) = T(t)a - \int_0^t T(t-\tau)P((u\cdot\nabla)u)(\tau)d\tau,$$

provided that $\|a\|_{L^n} \le \delta_0$, where $\delta_0 = \delta_0(\Omega, n)$ is a positive constant. The solution $u(t)$ enjoys

$$\begin{aligned} &\|u(t)\|_{L^r} \le Ct^{-\frac{1}{2}+\frac{n}{2r}}\|a\|_{L^n} && \text{for} \quad n \le r \le \infty,\\ &\|\nabla u(t)\|_{L^r} \le Ct^{-1+\frac{n}{2r}}\|a\|_{L^n} && \text{for} \quad n \le r < \infty \end{aligned}$$

for $t > 0$, which imply the Hölder estimate:

$$\|u(t) - u(\tau)\|_{L^\infty} + \|\nabla u(t) - \nabla u(\tau)\|_{L^n} \le C(t-\tau)^\theta \tau^{-\frac{1}{2}-\theta}\|a\|_{L^n}$$

for $0 < \tau < t$ and $0 < \theta < \frac{1}{2}$. Due to the Hölder estimate, the solution $u(t)$ becomes actually a strong one of (NS) (see Tanabe [39]). Furthermore, in a similar way to Hishida [22], we can obtain the decay properties (2.6) and (2.7). □

Since Hishida [22] proved Theorem 2.3 for $n \ge 3$, we have only to give a comment on the case $n = 2$. The key of his proof is to show the following lemma.

Lemma 4.1. *Let $n \ge 2$ and $a \in L^1(\Omega) \cap J^n(\Omega)$. When $n \ge 3$, for any small $\varepsilon > 0$ there are constants $\eta_* = \eta_*(\Omega, n, \varepsilon) \in (0, \delta]$ and $C = C(\Omega, n, \|a\|_{L^1}, \|a\|_{L^n}, \varepsilon)$ such that if $\|a\|_{L^n} \le \eta_*$, then the solution $u(t)$ obtained in Theorem 2.2 satisfies*

$$\|u(t)\|_{L^{\frac{n}{n-1}}} \le C(1+t)^{-\frac{1}{2}+\varepsilon}, \tag{4.1}$$

$$\|u(t)\|_{L^{2n}} \le Ct^{-\frac{1}{4}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon}, \tag{4.2}$$

$$\|\nabla u(t)\|_{L^n} \le Ct^{-\frac{1}{2}}(1+t)^{-\frac{n}{2}+\frac{1}{2}+\varepsilon} \tag{4.3}$$

for $t > 0$. When $n = 2$, without the assumption that a is small, the solution $u(t)$ obtained in Theorem 2.2 satisfies (4.1)-(4.3).

Remark 4.2. When $n = 2$, Kozono and Ogawa [28] proved that if $a \in J^2(\Omega) \cap L^p(\Omega)$ with $p = 1/(1-\varepsilon)$, then the solution $u(t)$ obtained in Theorem 2.2 enjoys (4.1)–(4.3) for $t \ge 1$ without any smallness condition on the initial data. We thus obtain Lemma 4.1 for $n = 2$.

Next we shall show the time-local existence of the strong solution $v(t)$ to the following Navier-Stokes problem with the non-trivial flux $\alpha(t) \not\equiv 0$ in $[0, \infty)$:

$$\begin{cases} \partial_t v - \Delta v + (v\cdot\nabla)v + \nabla\pi = 0 & \text{in} \quad \Omega\times(0,\infty),\\ \nabla\cdot v = 0 & \text{in} \quad \Omega\times(0,\infty),\\ v(x,t) = 0 & \text{on} \quad \partial\Omega\times(0,\infty),\\ v(x,0) = a(x) & \text{in} \quad \Omega,\\ \phi(v) = \alpha(t). \end{cases} \tag{NSf}$$

To this end, we prepare the auxiliary function. Heywood [23] showed that there exists $\chi = \chi(x) \in C^\infty(\Omega) \cap W^{2,q}(\Omega)$ $(n' < q < \infty)$ enjoying the following equations:

$$\chi|_{\partial\Omega} = 0, \ \nabla \cdot \chi = 0, \quad \phi(\chi) = 1. \tag{4.4}$$

Now by using the auxiliary function above, we set $u(x,t) = v(x,t) - \alpha(t)\chi(x)$. We see that u enjoys the following equations:

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla\pi = -F(u) + G(\alpha,\chi), \quad \nabla \cdot u = 0 \quad \text{in } \Omega \times (0,\infty) \tag{NS'}$$

subject to $u|_{\partial\Omega} = 0$, $\phi(u) = 0$ and $u(0) = v(0) - \alpha(0)\chi$, where

$$F(u) = \alpha(\chi \cdot \nabla)u + \alpha(u \cdot \nabla)\chi, \qquad G(\alpha,\chi) = -\alpha'\chi + \alpha\Delta\chi - \alpha^2(\chi \cdot \nabla)\chi.$$

Applying the solenoidal projection P to (NS'), we can rewrite (NS') as follows:

$$\partial_t u + Au = -P((u \cdot \nabla)u) - PF(u) + PG(\alpha,\chi), \qquad u(0) = v(0) - \alpha(0)\chi, \tag{PNS'}$$

where $A = -P\Delta$ is the Stokes operator. This is further transformed into the nonlinear integral equation:

$$\begin{aligned} u(t) = T(t)u(0) - &\int_0^t T(t-s)P((u \cdot \nabla)u)(s)ds \\ &- \int_0^t T(t-s)PF(u)(s)ds + \int_0^t T(t-s)PG(\alpha,\chi)(s)ds. \end{aligned} \tag{IE}$$

We shall construct a unique time-local solution $u(t)$ of the integral solution (IE) by successive approximation, according to the following scheme:

$$\begin{aligned} u_0(t) &= T(t)u(0) + \int_0^t T(t-s)PG(\alpha,\chi)(s)ds, \\ u_{m+1}(t) &= u_0(t) - \int_0^t T(t-s)P((u_m \cdot \nabla)u_m)(s)ds - \int_0^t T(t-s)PF(u_m)(s)ds. \end{aligned} \tag{INT}$$

Before we estimate $u_0(t)$ and $u_{m+1}(t)$, we ready for the following proposition which is proved by elementary calculation.

Proposition 4.3. *Let $1 < q \le r < \infty$ such that $1/q - 1/r < 1/n$. There holds the following estimate:*

$$\int_0^t \|\nabla^j T(t-s)P(g(s)f(\cdot))\|_{L^r} ds \le C_{q,r}\mathcal{A}\|f\|_{L^q} B\left(-\frac{n}{2q} + \frac{n}{2r} + 1 - \frac{j}{2}, 1\right) t^{-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2}}$$

for $f \in L^q(\Omega)$ and g with $\sup_{0<s<t}|g(s)| \le \mathcal{A}$, where $B(\cdot,\cdot)$ denotes the beta function.

Proof of Theorem 2.5. (i) We shall solve (INT) for $n \ge 3$ by successive approximation. To this end we show by induction that the u_m exist and satisfy the following relations:

$$t^{\frac{1}{4}} u_m \in BC([0,T]; J^{2n}(\Omega)), \tag{4.5}$$

$$t^{\frac{1}{2}} \nabla u_m \in BC([0,T]; L^n(\Omega)) \tag{4.6}$$

with value zero at $t = 0$ and

$$\sup_{0<t\leq T}\left(t^{\frac{1}{4}}\|u_m(t)\|_{L^{2n}} + t^{\frac{1}{2}}\|\nabla u_m(t)\|_{L^n}\right) \leq K_m. \tag{4.7}$$

In order to estimate $u_0(t)$, we set

$$\begin{aligned} u_0(t) = T(t)u(0) &+ \int_0^t T(t-s)P(\alpha\Delta\chi)(s)ds \\ &- \int_0^t T(t-s)P(\alpha^2(\chi\cdot\nabla)\chi)ds - \int_0^t T(t-s)P(\alpha'\chi)(s)ds. \end{aligned} \tag{4.8}$$

We shall show the estimate of u_0^j $(j = 1,2,3)$. Setting

$$\mathcal{A} = \max\left(\max_{0\leq t\leq T}|\alpha(t)|,\ \max_{0\leq t\leq T}|\alpha'(t)|\right),\ \mathcal{B}^j_{q,r} = B\left(-\frac{n}{2q}+\frac{n}{2r}+1-\frac{j}{2},1\right)$$

and using Proposition 4.3, we can show that for $n' \leq \frac{n}{2} < q \leq n$, there exists the positive number K_0 enjoying the following inequality:

$$\sup_{0<t\leq T}\left(t^{\frac{1}{4}}\|u_0(t)\|_{L^{2n}} + t^{\frac{1}{2}}\|\nabla u_0(t)\|_{L^n}\right) \leq K_0 \tag{4.9}$$

with

$$\begin{aligned} K_0 = K_0(T) = &\sup_{0<t\leq T}\left(t^{\frac{1}{4}}\|T(t)u(0)\|_{L^{2n}} + t^{\frac{1}{2}}\|\nabla T(t)u(0)\|_{L^n}\right) \\ &+ C_{q,n}\mathcal{A}(\mathcal{B}^0_{q,2n}+\mathcal{B}^1_{q,n})\left(\|\Delta\chi\|_{L^q}+\|\chi\|_{L^q}+\mathcal{A}\|\chi\|_{L^{2q}}\|\nabla\chi\|_{L^{2q}}\right)T^{\frac{3}{2}-\frac{n}{2q}}. \end{aligned}$$

Note that we can take small $K_0 = K_0(T_*)$ when we restrict the time to some short interval $[0, T_*]$ since $u(0) \in J^n(\Omega)$.

The continuity at $t = 0$, with value zero, of the function (4.5) with $n = 0$ follows from the facts that the operator $t^{\frac{1}{4}}T(t)$ is uniformly bounded from J^n to J^{2n} and tends to zero strongly as $t \to 0$. The similar continuous property of (4.6) is shown similarly.

We shall proceed to the next step. Assuming now that (4.5) and (4.6) with (4.7) are true for m, we shall show those for $m+1$. For simplicity, we set

$$\begin{aligned} u_{m+1}(t) = u_0(t) &- \int_0^t T(t-s)P((u_m\cdot\nabla)u_m)(s)ds \\ &- \int_0^t T(t-s)P(\alpha(\chi\cdot\nabla)u_m)(s)ds - \int_0^t T(t-s)P(\alpha(u_m\cdot\nabla)\chi)(s)ds. \end{aligned} \tag{4.10}$$

By Theorem 2.1 and Hölder inequality, we can obtain

$$\sup_{0<t\leq T}\left(t^{\frac{1}{4}}\|u_{m+1}(t)\|_{L^{2n}} + t^{\frac{1}{2}}\|\nabla u_m(t)\|_{L^n}\right) \leq K_{m+1}$$

with

$$K_{m+1} = K_0 + LK_m + NK_m^2,$$

where

$$\begin{aligned} L =& C_{\tilde{q}} A \|\chi\|_{L^{\tilde{q}}} \left(B\left(\frac{3}{4} - \frac{n}{2\tilde{q}}, \frac{1}{2}\right) + B\left(\frac{1}{2} - \frac{n}{2\tilde{q}}, \frac{1}{2}\right)\right) T^{\frac{1}{2}-\frac{n}{2\tilde{q}}} \\ &+ C_{\tilde{q}} A \|\nabla\chi\|_{L^{\tilde{q}}} \left(B\left(1 - \frac{n}{2\tilde{q}}, \frac{3}{4}\right) + B\left(\frac{3}{4} - \frac{n}{2\tilde{q}}, \frac{3}{4}\right)\right) T^{1-\frac{n}{2\tilde{q}}}, \\ N =& C_{n,r} \left(B\left(\frac{1}{2}, \frac{1}{4}\right) + B\left(\frac{1}{4}, \frac{1}{4}\right)\right). \end{aligned}$$

One can replace T by some small $T_* \in (0, T]$ so that $L < 1$ and $K_0 < \frac{(1-L)^2}{4N}$. Set

$$K := \frac{(1-L) - \sqrt{(1-L)^2 - 4NK_0}}{2N}.$$

We easily find that $K_0 < K$ and that $K_m \leq K$ implies

$$K_{m+1} \leq K_0 + LK + NK^2 = K.$$

We thus obtain

$$\sup_{0<t\leq T_*} \left(t^{\frac{1}{4}} \|u_m(t)\|_{L^{2n}} + t^{\frac{1}{2}} \|\nabla u_m(t)\|_{L^n} \right) \leq K$$

for all m. This together with the same calculations for

$$\gamma_m(T_*) := \sup_{0<t\leq T_*} \left(t^{\frac{1}{4}} \|u_m(t) - u_{m-1}(t)\|_{L^{2n}} + t^{\frac{1}{2}} \|\nabla u_m(t) - \nabla u_{m-1}(t)\|_{L^n} \right)$$

as above yields

$$\gamma_{m+1}(T_*) \leq \left\{ CK_0(T_*) + C_{\tilde{q}} A \left(\|\chi\|_{L^{\tilde{q}}} T_*^{\frac{1}{2}-\frac{n}{2\tilde{q}}} + \|\nabla\chi\|_{L^{\tilde{q}}} T_*^{1-\frac{n}{2\tilde{q}}} \right) \right\} \gamma_m(T_*)$$

for all m. When we take still smaller T_* (if necessary), we see that the sequence $\{u_m\}$ converges uniformly in t as $m \to \infty$ to a function u, which satisfies (IE) for $0 < t \leq T_*$ and is of class

$$t^{\frac{1}{4}} u \in BC([0, T_*]; J^{2n}(\Omega)), \ t^{\frac{1}{2}} \nabla u \in BC([0, T_*]; L^n(\Omega))$$

with

$$\sup_{0<t\leq T_*} \left(t^{\frac{1}{4}} \|u(t)\|_{L^{2n}} + t^{\frac{1}{2}} \|\nabla u(t)\|_{L^n} \right) \leq K.$$

By use of this we estimate (IE) to obtain (2.14) for $n \leq r \leq \infty$ with initial condition and (2.15) for $n \leq r < 2n$; and then, a bootstrap argument yields (2.15) for any $r < \infty$. This leads to a local solution $u(t)$ to (IE) with desired estimates. Since $\alpha \in C^{1,\theta}$, the solution $u(t)$ actually becomes a strong one (see Tanabe [39]). We thus complete the proof of Theorem 2.5 for $n \geq 3$.

(ii) We shall show the outline of the proof. Let $n = 2 < p < \infty$ and $u(0) \in J^p(\Omega)$. Then, by using successive approximation scheme (INT) again, we can show the existence of a unique solution u to (IE), which satisfies

$$t^{\frac{1}{2p}}u \in BC([0,T_*]; J^{2p}(\Omega)),\ t^{\frac{1}{2}}\nabla u \in BC([0,T_*]; L^p(\Omega))$$

with

$$\sup_{0<t\leq T_*} \left(t^{\frac{1}{2p}}\|u(t)\|_{L^{2p}} + t^{\frac{1}{2}}\|\nabla u(t)\|_{L^p}\right) \leq K.$$

Theorem 2.5 (ii) is thus proved in the same way as the case where $n \geq 3$. □

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