数理解析研究所講究録
第 1592 巻 2008 年 116-129

Global Solvability and Some Uniform-in-Time Bounds of the Solution for Self-Gravitating Viscous Stellar Models

梅原 守道 (Morimichi UMEHARA), 谷 温之 (Atusi TANI)
慶應義塾大学 理工学部
Department of Mathematics, Keio University

1 Introduction

We consider the one-dimensional motion of a compressible, viscous and heat-conductive gas driven by the self-gravitation. In addition, let the two edges of the medium be fluctuating and thermally-insulated and the gas be chemically reactive. Such a gaseous motion, especially in the processes of the unimolecular reactions of kinetic order one, is described by the following four equations in the Eulerian coordinate system corresponding to the conservation laws of mass, momentum and energy, and an equation of reaction-diffusion type ([38]):

$$\begin{cases} \rho_t + (\rho v)_y = 0, \\ \rho(v_t + vv_y) = \big(-p + \mu v_y\big)_y + \rho f, \\ \rho(e_t + ve_y) = \big(-p + \mu v_y\big)v_y + \kappa\theta_{yy} + \lambda\rho\phi z, \\ \rho(z_t + vz_y) = d(\rho z_y)_y - \rho\phi z \end{cases} \tag{1.1}$$

in $\bigcup_{t>0}\big(D_t \times \{t\}\big)$ with $D_t := \{y \in \mathbb{R} \,|\, y_1(t) < y < y_2(t)\}$ for any $t \geq 0$, and the boundary conditions for $t > 0$, $i = 1,2$,

$$\begin{cases} \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = v(y_i(t), t), \\ (-p + \mu v_y,\ \theta_y,\ z_y)|_{y=y_i(t)} = (-p_e,\ 0,\ 0). \end{cases} \tag{1.2}$$

Our problem is: To find, for any time $t > 0$, the fluctuating boundary functions $y_i(t)$ $(i = 1,2)$ and the distributions of the density $\rho = \rho(y,t)$, the velocity $v = v(y,t)$, the absolute temperature $\theta = \theta(y,t)$ and the mass fraction of the reactant $z = z(y,t)$ satisfying (1.1) and (1.2) for the given initial conditions

$$(\rho, v, \theta, z)|_{t=0} = \big(\rho_0(y), v_0(y), \theta_0(y), z_0(y)\big), \quad y \in \overline{D_0}. \tag{1.3}$$

Here μ, κ, d and λ are the coefficients of the bulk viscosity, the thermal conduction, the species diffusion and the difference in heat between the reactant and the product, respectively, which are assumed as positive constants; a constant $p_e \geq 0$ is the external pressure. Moreover we assume that the gas is polytropic and ideal, namely, the pressure $p = p(\rho, \theta)$ and the internal energy per unit mass $e = e(\rho, \theta)$ are given by the equations of state

$$p(\rho, \theta) = R\rho\theta, \qquad e(\rho, \theta) = c_v\theta$$

with two positive constants R and c_v, the perfect gas constant and the specific heat capacity at constant volume, respectively. The external force per unit mass $f = f(y,t)$ is defined by $f = -U_y$, where $U = U(y,t)$ is the solution of the boundary value problem

$$\begin{cases} U_{yy} = G\rho & \text{in } D_t, \\ U|_{y=y_1(t)} = U|_{y=y_2(t)} = 0 \end{cases} \tag{1.4}$$

for each $t > 0$ with a positive constant G.

Remark. In the three-dimensional case, for space variable $y \in \mathbb{R}^3$ and time variable $t > 0$ the self-gravitation per unit mass $\boldsymbol{f} = \boldsymbol{f}(y,t) = (f_1, f_2, f_3)(y,t)$ is given by the formula

$$\boldsymbol{f} = -\nabla U_g,$$

where $U_g = U_g(y,t)$, the gravitational potential, is defined by

$$U_g(y,t) = -G_0 \int_{\Omega_t} \frac{\rho(s,t)}{|y-s|} \, ds$$

with the domain $\Omega_t \subset \mathbb{R}^3$ occupied by the gas at t, the density $\rho(y,t)$ and the Newtonian gravitational constant G_0. It is also known that U_g satisfies the Poisson equation

$$\Delta U_g = 4\pi G_0 \rho$$

in Ω_t for each $t > 0$. From these arguments one can regard that f defined above is the one-dimensional version of the "self-gravitation", U the "self-gravitational potential" and G the corresponding "Newtonian gravitational constant".

Furthermore we assume that the reaction rate function $\phi = \phi(\theta)$ is defined by the Arrhenius law (see for example, [22])

$$\phi(\theta) = K e^{-A/\theta} \tag{1.5}$$

with the coefficent of rate of the reactant K and the activation energy A, assumed as positive constants in this paper.

We transform this problem into the one in the Lagrangian coordinate. For a given smooth velocity field $v(y,t)$ and for any fixed point $(y,t) \in \bigcup_{t>0} \left(\overline{D_t} \times \{t\}\right)$, finding the solution $Y_{y,t}(\tau)$ of the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}Y_{y,t}(\tau)}{\mathrm{d}\tau} = v\big(Y_{y,t}(\tau),\tau\big) & \text{for } 0<\tau<t, \\ Y_{y,t}(t) = y \end{cases}$$

and putting $Y_{y,t}(0) = \xi$, we have

$$y = Y_{\xi,0}(t) = \xi + \int_0^t v\big(Y_{\xi,0}(\tau),\tau\big)\,\mathrm{d}\tau.$$

Then we introduce the mass transformation

$$\xi \mapsto x = \int_0^\xi \rho_0(s)\,\mathrm{d}s.$$

From these changes of variable problem (1.4) is reduced to for each $t>0$

$$\begin{cases} (\hat\rho\,\hat U_x)_x = G & \text{in } (0,M), \\ \hat U|_{x=0} = \hat U|_{x=M} = 0, \end{cases}$$

where $M = \int_{y_1(0)}^{y_2(0)} \rho_0(\xi)\,\mathrm{d}\xi$ and hat "^" represents the transformed functions. Through the relation $\hat f = -\hat\rho\,\hat U_x$ we can get the explicit formula

$$\hat f(x,t) = -G\left(x - \frac{\int_0^M \eta\hat\rho(\eta,t)^{-1}\,\mathrm{d}\eta}{\int_0^M \hat\rho(\eta,t)^{-1}\,\mathrm{d}\eta}\right). \tag{1.6}$$

Consequently, by putting the specific volume $v(x,t) := 1/\hat\rho(x,t)$ and the velocity $u(x,t) := \hat v(x,t)$, by using $\big(\theta,z,\phi(\theta)\big)(x,t)$ in place of $\big(\hat\theta,\hat z,\phi(\hat\theta)\big)(x,t)$ and by normalizing $M=1$, our problem becomes

$$\begin{cases} v_t = u_x, \\ u_t = \left(-R\dfrac{\theta}{v} + \mu\dfrac{u_x}{v}\right)_x - G\left(x - \dfrac{\int_0^1 \eta v(\eta,t)\,\mathrm{d}\eta}{\int_0^1 v(\eta,t)\,\mathrm{d}\eta}\right), \\ c_v\theta_t = \left(-R\dfrac{\theta}{v} + \mu\dfrac{u_x}{v}\right)u_x + \kappa\left(\dfrac{\theta_x}{v}\right)_x + \lambda\phi z, \\ z_t = d\left(\dfrac{z_x}{v^2}\right)_x - \phi z \end{cases} \tag{1.7}$$

in $(0,1)\times(0,\infty)$ with the boundary conditions for $t>0$

$$\begin{cases} \left(-R\dfrac{\theta}{v} + \mu\dfrac{u_x}{v},\, \theta_x\right)\Big|_{x=0,1} = (-p_e,\,0), \\ z_x|_{x=0,1} = 0 \end{cases} \tag{1.8}$$

and the initial conditions for $x \in [0,1]$

$$\begin{cases} (v,u,\theta)|_{t=0} = \big(v_0(x), u_0(x), \theta_0(x)\big), \\ z|_{t=0} = z_0(x). \end{cases} \tag{1.9}$$

Now, by integration of $(1.7)^2$ with respect to x over $[0,1]$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u\,\mathrm{d}x = -G\left(\frac{1}{2} - \frac{\int_0^1 \eta v(\eta,t)\,\mathrm{d}\eta}{\int_0^1 v(\eta,t)\,\mathrm{d}\eta}\right). \tag{1.10}$$

Denoting $u - \int_0^1 u\,\mathrm{d}x$ by u again, we obtain the final form:

$$\begin{cases} v_t = u_x, \\ u_t = \left(-R\dfrac{\theta}{v} + \mu\dfrac{u_x}{v}\right)_x - G\left(x - \dfrac{1}{2}\right), \\ c_v\theta_t = \left(-R\dfrac{\theta}{v} + \mu\dfrac{u_x}{v}\right)u_x + \kappa\left(\dfrac{\theta_x}{v}\right)_x + \lambda\phi z, \\ z_t = d\left(\dfrac{z_x}{v^2}\right)_x - \phi z \end{cases} \tag{1.11}$$

in $(0,1)\times(0,\infty)$ with the same initial-boundary conditions (1.8) and (1.9). In addition, integrating $(1.11)^2$ over $[0,1]\times[0,t]$ under $(1.8)^1$ gives $\int_0^1 u(x,t)\,\mathrm{d}x = \int_0^1 u(x,0)\,\mathrm{d}x$, whose left-hand side is identically equal to zero. Hence, it is natural for us to take the condition

$$\int_0^1 u_0(x)\,\mathrm{d}x = 0. \tag{1.12}$$

We also assume the compatibility conditions

$$\begin{cases} \left(-R\dfrac{\theta_0}{v_0} + \mu\dfrac{u_0'}{v_0},\ \theta_0'\right)\Big|_{x=0,1} = (-p_e, 0), \\ z_0'|_{x=0,1} = 0 \end{cases} \tag{1.13}$$

For the system (1.11), (1.8), (1.9) with (1.5) under the hypotheses (1.12), (1.13) we shall establish the unique existence, global in time, of a classical solution and obtain the temporally asymptotic behaviour of the solution. One can easily see that from (1.10) this solution leads to the one for the problem (1.7)-(1.9), and that when $\rho(x,t) := 1/v(x,t)$ is strictry positive for any $(x,t) \in [0,1]\times[0,T]$ with any $T \geq 0$, that solution corresponds to the one for the original problem (1.1)-(1.3).

We introduce some function spaces (see for example, [14]). Let $\Omega := (0,1)$, m be a nonnegative integer and $0 < \sigma$, $\sigma' < 1$. By $C(\Omega)$ the space of continuous functions defined in Ω is denoted. We use $H^m(\Omega)$ as the usual Sobolev spaces of order m in the L^2

sense. $C^{m+\sigma}(\Omega)$ denote the spaces of functions $u = u(x)$ which have bounded derivatives up to order m, and whose m-th order derivative is uniformly Hölder continuous with exponent σ. Its norm is defined by

$$|u|_{m+\sigma} := \sum_{k=0}^{m} \sup_{x\in\Omega} |D^k u(x)| + \sup_{\substack{x,x'\in\Omega \\ x\neq x'}} \frac{|D^m u(x) - D^m u(x')|}{|x-x'|^\sigma}$$

with $D = \mathrm{d}/\mathrm{d}x$. Let T be a positive constant and $Q_T := \Omega \times (0,T)$. For a function u defined on Q_T,

$$|u|^{(0)} := \sup_{(x,t)\in Q_T} |u(x,t)|,$$

$$|u|_x^{(\sigma)} := \sup_{\substack{(x,t),(x',t)\in Q_T \\ x\neq x'}} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\sigma}, \quad |u|_t^{(\sigma)} := \sup_{\substack{(x,t),(x,t')\in Q_T \\ t\neq t'}} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\sigma}.$$

We say that $u \in C_{x,t}^{\sigma,\sigma'}(Q_T)$ if u is bounded and uniformly Hölder continuous in x and t with exponents σ and σ', respectively, with the norm

$$|u|_{\sigma,\sigma'} := |u|^{(0)} + |u|_x^{(\sigma)} + |u|_t^{(\sigma')}.$$

We also say that $u \in C_{x,t}^{2+\sigma,1+\sigma/2}(Q_T)$ if u is bounded, has bounded derivative u_x and $(u_{xx}, u_t) \in \left(C_{x,t}^{\sigma,\sigma/2}(Q_T)\right)^2$. Its norm is defined by

$$|u|_{2+\sigma,1+\sigma/2} := |u|^{(0)} + |u_x|^{(0)} + |u_{xx}|_{\sigma,\sigma/2} + |u_t|_{\sigma,\sigma/2}.$$

Our first result is

Theorem 1 (Global Solution) *Let $\alpha \in (0,1)$. Assume that*

$$(v_0,\, u_0,\, \theta_0,\, z_0) \in C^{1+\alpha}(\Omega) \times \left(C^{2+\alpha}(\Omega)\right)^3$$

satisfies (1.13), (1.12) *and $v_0(x) > 0$, $\theta_0(x) > 0$, $0 \le z_0(x) \le 1$ for any $x \in \overline{\Omega}$, and that $p_e > 0$. Then there exists a unique solution (v,u,θ,z) of the initial-boundary value problem* (1.11), (1.8), (1.9) *with* (1.5) *such that for any positive number T*

$$(v,\, v_x,\, v_t,\, u,\, \theta, z) \in \left(C_{x,t}^{\alpha,\alpha/2}(Q_T)\right)^3 \times \left(C_{x,t}^{2+\alpha,1+\alpha/2}(Q_T)\right)^3.$$

Moreover for any $(x,t) \in \overline{Q_T}$

$$v(x,t) > 0, \quad \theta(x,t) > 0, \quad 0 \le z(x,t) \le 1.$$

Simultaneously, we obtain a temporally asymptotic profile of the specific volume.

Proposition 1 (Asymptotic Behaviour of the Specific Volume) *Assume that the initial data and p_e satisfy the hypotheses of Theorem 1. Then v has a limit as $t \to +\infty$ uniformly in $x \in \overline{\Omega}$:*

$$v(x,t) \longrightarrow \tilde{v}^*(x) := \frac{R}{f(x)} \frac{E_0 - \lambda \bar{z}}{c_v + R},$$

where $\bar{z}$ is the limit of $\int_0^1 z(x,t)\,\mathrm{d}x$ as $t \to +\infty$ satisfying $0 \le \bar{z} \le 1$,

$$E_0 := \int_0^1 \left(\frac{1}{2} {u_0}^2 + c_v \theta_0 + f(x) v_0 + \lambda z_0 \right) \mathrm{d}x,$$

$$f(x) := p_e + \frac{1}{2} G x (1-x).$$

Remark. From the physical point of view one expects that the solution $(v, u, \theta, z)(x,t)$ converges to a steady state $(V, U, \Theta, Z)(x)$ as time tends to infinity, which is a solution to the stationary problem corresponding to (1.11), (1.8), (1.9)

$$\begin{cases} \left(U_x,\ R\left(\dfrac{\Theta}{V}\right)_x,\ \kappa\left(\dfrac{\Theta_x}{V}\right)_x + \lambda\phi(\Theta)Z \right) = \left(0,\ -G\left(x - \dfrac{1}{2}\right),\ 0\right), \\ d\left(\dfrac{Z_x}{V^2}\right)_x - \phi(\Theta)Z = 0 \end{cases} \tag{1.14}$$

in Ω with the boundary conditions

$$\begin{cases} \left(R\dfrac{\Theta}{V},\ \Theta_x \right)\Big|_{x=0,1} = (p_e, 0), \\ Z_x|_{x=0,1} = 0. \end{cases} \tag{1.15}$$

This problem is easily solved as

$$\begin{cases} (V, U, \Theta)(x) = \left(\dfrac{R}{f(x)}\overline{\Theta},\ \overline{U},\ \overline{\Theta} \right), \\ Z(x) = 0 \end{cases} \tag{1.16}$$

with any constants $\overline{U}$ and $\overline{\Theta}$. Proposition 1 is the partial result for the complete proof of this expectation.

On the other hand, we obtain another result in case of the non-reactive gas, i.e., $d = 0$ in $(1.11)^4$ and $K = 0$ in (1.5), which leads $z(x,t) \equiv 1$ under $z_0(x) \equiv 1$ from (1.11), (1.8), (1.9).

Theorem 2 (Global Solution for the Non-Reactive Case) *Let* $\alpha \in (0,1)$ *and* $K=0$ *in* (1.5). *Assume that*

$$(v_0,\, u_0,\, \theta_0) \in C^{1+\alpha}(\Omega) \times \left(C^{2+\alpha}(\Omega)\right)^2$$

satisfies $(1.13)^1$, (1.12) *and* $v_0(x) > 0$, $\theta_0(x) > 0$ *for any* $x \in \overline{\Omega}$, *and that* $p_e > 0$. *Then there exists a unique solution* (v, u, θ) *of the initial-boundary value problem* $(1.11)^{1-3}$, $(1.8)^1$, $(1.9)^1$, (1.5) *with* $K=0$ *such that for any positive number* T

$$(v,\, v_x,\, v_t,\, u,\, \theta) \in \left(C^{\alpha,\alpha/2}_{x,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,1+\alpha/2}_{x,t}(Q_T)\right)^2$$

with

$$\begin{cases} |v,\, v_x,\, v_t|_{\alpha,\alpha/2},\ |u,\, \theta|_{2+\alpha,1+\alpha/2} \leq C, \\ v(x,t),\ \theta(x,t) \geq 1/C \quad \text{for any } (x,t) \in \overline{Q_T}, \end{cases} \tag{1.17}$$

where C *is a positive constant depending on initial data and* T. *Moreover, if*

$$c_v > R \quad \text{and} \quad p_e > \frac{G}{8}\frac{R}{c_v - R} \tag{1.18}$$

are satisfied, then C *can be taken to be independent of* T.

Simultaneously, we obtain the temporally asymptotic profile of the solution (v, u, θ).

Proposition 2 (Asymptotic Behaviour of the Non-Reactive Gas) *Assume that the initial data and* K *satisfy the hypotheses of Theorem* 2 *and that* (1.18) *holds. Then the solution* (v,u,θ) *of the initial-boundary value problem* $(1.11)^{1-3}$, $(1.8)^1$, $(1.9)^1$, (1.5) *with* $K=0$ *converges to a steady state* $(\tilde{v}, 0, \bar{\theta})$ *in* $C(\Omega) \cap H^1(\Omega)$ *as time tends to infinity, where*

$$\tilde{v} = \tilde{v}(x) := \frac{R}{f(x)}\frac{E_0}{c_v + R}, \quad \bar{\theta} := \frac{E_0}{c_v + R}$$

with $E_0 := \displaystyle\int_0^1 \left(\frac{1}{2}{u_0}^2 + c_v\theta_0 + f(x)v_0\right) \mathrm{d}x$ *and* $f(x) := p_e + \frac{1}{2}Gx(1-x)$.

Remarks. (i) Theorems 1 and 2 say that if $p_e > 0$, one can construct the solution until any (long) "fixed" time T. However, the solution may blow up as time tends to infinity. In this situation it is not sure whether the solution exists at $t = +\infty$ or not. On the other hand, in the non-reactive case, only under the situation that condition (1.18) is satisfied, one can apparently confirm that the classical solution uniquely exists even if $t = +\infty$.
(ii) The steady state $(\tilde{v}, 0, \bar{\theta})$ is a solution $(1.16)^1$ with $\overline{U} = 0$ and $\overline{\Theta} = \bar{\theta}$ to the corresponding stationary problem $(1.14)^1$, $(1.15)^1$ under $K = 0$ in (1.5). We also note

that the first hypothesis in (1.18) is equivalent to $1 \leq \gamma < 2$, where γ is the specific heat ratio satisfying $R/c_v = \gamma - 1$ for ideal gases (the Mayer's relation). Therefore, for these gases Theorem 2 says that if the external pressure is suitably large, the stationary solution of our problem is stable for any size of the initial disturbance.

We mention some related results concerning our problem. As is well-known for three-dimensional, compressible, viscous (and heat-conductive) fluid models, there exist only partial results concerning temporally global existence and asymptotic behaviour of the solution. Matsumura and Nishida solved globally in time the Cauchy problem [15] in 1980 and the initial-boundary value problem [16] in 1983 under the assumptions that a given potential force is sufficiently small and the initial data (ρ_0, v_0, θ_0) is sufficiently close to a positive constant state $(\overline{\rho}, 0, \overline{\theta})$. They also showed that the corresponding stationary problem has a unique solution $(\tilde{\rho}, 0, \tilde{\theta})$ near $(\overline{\rho}, 0, \overline{\theta})$ and the global in time solution converges to this stationary one as time tends to infinity. It is also noteworthy to point out another temporally global result due to Solonnikov and Tani in a series of papers [27–29]. They considered a free-boundary problem for a barotropic model with a surface tension on the free-boundary, and proved the existence of global in time solution and its convergence to a stationary solution in suitable Sobolev-Slobodetskiĭ spaces under some smallness assumptions on the initial data.

On the other hand, in spacially one-dimensional problems, many studies have been done including the case for large, smooth initial data (mainly under the assumption that coefficients of the viscosity and the thermal conduction are positive constants). In 1977 Kazhikhov and Shelukhin [12] firstly constructed the global in time, classical solution with arbitrary large, smooth initial data. They considered the initial-boundary value problem for a polytropic and ideal fluid flow with the Dirichlet boundary condition with respect to the velocity and without any external force fields. Kazhikhov [11] also proved that for arbitrary large, smooth initial data the solution of this problem converges to the one of the corresponding stationary problem as time tends to infinity. For them it is necessary to get a priori estimates of the solution, among which the most important one is the boundedness of the density from below by a strictly positive constant. To obtain such an estimate they derived a useful representation formula of the density in [12]. The analogous one is also deduced in our problem (see Lemma 1, section 2). After these pioneering works, Nagasawa [20] investigated the outer pressure problem, i.e., $p_e = p_e(t)$ in (1.8) without any external force fields. He constructed the global in time classical solution and clarified the temporally asymptotic state of the solution under some hypotheses on $\lim_{t\to+\infty} p_e(t)$. When his outer pressure is identically equal to a positive constant, the asymptotic state is the same as ours stated in Proposition 2 under $G = 0$. We also mention the investigation due to Kazhikhov [10], Okada [21] and Nagasawa [18, 19] on the model of polytropic and ideal gas which has a free-boundary to a surrounding vaccume state, i.e., $p_e \equiv 0$ in (1.8). Models for a reacting mixture composed of polytropic and ideal gases have been studied without any external force fields, by many authors including Poland and Kassoy [22], Bebernes and Bressan [1], Chen [3], Yanagi [39] and so on.

For the models of compressible, viscous and heat-conductive fluids with external forces and without any restriction on the size of the initial data, one can find only a few temporally global results. For the three-dimensional spherically symmetric problem depending on one space variable $r := |y|$, $y \in \mathbb{R}^3$, global in time solvability and the asymptotic behaviour of the solution were investigated by Ducomet [6], Yanagi [40], Nakamura and Nishibata [17] and so on. In [40] the gaseous motion in an annulus domain under a small potential force was discussed; in [17] the motion in an unbounded exteriour domain of a sphere was studied under any large potential force; in [6] a stellar model (see [2,13]) with the central rigid core was treated in the case of the free-bondary on the surface and the force field due to the self-gravitation of the gas. Ducomet [4,5,7] and Ducomet-Zlotnik [8,9] studied one-dimensional gaseous models rather similar to ours, i.e., reactive one with the free-boundary in the external force field. In a series of papers [4,5,7–9] they did not adopt the exact form (1.6) of the self-gravitation, but a special external force completely independent of time variable in the Lagrangian mass coordinate system. Such a special form is called the "pancakes model" relevant to some large-scale structure of the universe (see [25]). Although they established the temporally global existence of the solution and its temporal asymptotics in [8,9] for the gas not only polytropic and ideal but also "radiative", i.e., $p(\rho,\theta)$ and $e(\rho,\theta)$ are given by

$$p(\rho,\theta) = R\rho\theta + \frac{a}{3}\theta^4, \qquad e(\rho,\theta) = c_v\theta + a\frac{\theta^4}{\rho}$$

with the radiation-density constant a and the coefficient of the thermal conduction κ is dependent on ρ and θ (see [33–35] for details), their results did not cover the pure free-boundary case (1.8) but for the one with partially Dirichlet boundary condition for θ, i.e., $\theta|_{x=0}$ (or $\theta|_{x=1}$) $= \theta_\Gamma$ with a positive constant θ_Γ. [Also in [17] the boundary condition for θ equivalent to this was taken as $\theta|_{x=+\infty} = \theta_+ > 0$.] Furthermore, the proofs in [4–7] are not clear for the present authors.

It is our Theorems that obtained temporally global results through the exact argument under the physically realistic external force, the self-gravitation.

Proofs of Theorems 1 and 2 are based on the temporally local existence theorem and a priori estimates of the solution. The fundamental theorem about the existence and the uniqueness of the local in time classical solution in three-dimensional case was already established by Tani [30,31] under sufficiently general situations. For the self-gravitating fluid, Secchi [23,24] obtained the corresponding result in suitable Sobolev spaces. Since it is easy to see that their arguments are applicable without any essential modifications to our one-dimensional, reacting case (see also [26]), we omit the proof of next proposition.

Proposition 3 (Local Solution) *Let $\alpha \in (0,1)$. Assume that*

$$(v_0, u_0, \theta_0, z_0) \in C^{1+\alpha}(\Omega) \times \left(C^{2+\alpha}(\Omega)\right)^3$$

satisfies (1.13), (1.12) *and for a positive constant C_0*

$$|v_0|_{1+\alpha},\ |u_0, \theta_0, z_0|_{2+\alpha} \le C_0,$$

$$v_0(x),\ \theta_0(x) \ge 1/C_0, \quad 0 \le z_0(x) \le 1 \qquad \textit{for any } x \in \overline{\Omega}.$$

Then there exists a unique solution (v,u,θ,z) *of the initial-boundary value problem* (1.11), (1.8), (1.9) *with* (1.5) *such that for some positive nunber* $T^* = T^*(C_0)$

$$(v,\ v_x,\ v_t,\ u,\ \theta,\ z) \in \left(C^{\alpha,\alpha/2}_{x,t}(Q_{T^*})\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,t}(Q_{T^*})\right)^3.$$

Moreover there exists a positive constant $C^* = C^*(C_0, T^*)$ *such that*

$$|v,\ v_x,\ v_t|_{\alpha,\alpha/2},\ |u,\ \theta,\ z|_{2+\alpha,\,1+\alpha/2} \le C^*,$$

$$v(x,t),\ \theta(x,t) > 1/C^*, \quad 0 \le z(x,t) \le 1 \quad \text{for any } (x,t) \in \overline{Q_{T^*}}.$$

In order to prove Theorem 1 it is sufficient to establish the following a priori boundedness.

Proposition 4 (A priori Estimates) *Let* T *be an arbitrary positive number. Assume that* α, p_e *and the initial data satisfy the hypotheses of Theorem* 1 *and that the problem* (1.11), (1.8), (1.9) *with* (1.5) *has a solution* (v,u,θ,z) *such that*

$$(v,\ v_x,\ v_t,\ u,\ \theta, z) \in \left(C^{\alpha,\alpha/2}_{x,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,t}(Q_T)\right)^3.$$

Then there exists a positive constant C *depending on the initial data and* T *such that*

$$|v,\ v_x,\ v_t|_{\alpha,\alpha/2},\ |u,\ \theta,\ z|_{2+\alpha,\,1+\alpha/2} \le C,$$

$$v(x,t),\ \theta(x,t) > 1/C, \quad 0 \le z(x,t) \le 1 \quad \text{for any } (x,t) \in \overline{Q_T}.$$

On the other hand, in [32] the unique existence, global in time, of a classical solution for the system $(1.7)^{1-3}$, $(1.8)^1$, $(1.9)^1$ with (1.5) under $K=0$ was already established. Therefore, to complete the proof of Theorem 2 it is sufficient to establish the following

Proposition 5 (A priori Estimates for the Non-Reactive Case) *Let* T *be an arbitrary positive number. Assume that* α, K *and the initial data satisfy the hypotheses of Theorem* 2, (1.18) *holds, and that problem* $(1.11)^{1-3}$, $(1.8)^1$, $(1.9)^1$, (1.5) *with* $K=0$ *has a solution* (v,u,θ) *such that*

$$(v,\ v_x,\ v_t,\ u,\ \theta) \in \left(C^{\alpha,\alpha/2}_{x,t}(Q_T)\right)^3 \times \left(C^{2+\alpha,\,1+\alpha/2}_{x,t}(Q_T)\right)^2.$$

Then there exists a positive constant C *independent of* T *such that* (1.17) *holds.*

2 Key Lemmas for Proving propositions 4 and 5

In proving Propositions 4 and 5 (and also Propositions 1 and 2), we need several lemmas concerning the estimates of the solution and its derivatives. For details, see [36] for the proofs of Propositions 1 and 4, and [37] for Propositions 2 and 5.

As mentioned in section 1, Kazhikhov and Shelukhin firstly derived the useful representation formula of v. In the present case, we can obtain the similar one as follows.

Lemma 1 *The identity*

$$v(x,t) = \frac{1}{\mathrm{P}(x,t)\mathrm{Q}(x,t)} \left(v_0(x) + \frac{R}{\mu} \int_0^t \theta(x,\tau)\mathrm{P}(x,\tau)\mathrm{Q}(x,\tau)\, \mathrm{d}\tau \right)$$

holds, where

$$\mathrm{P}(x,t) := \exp\left[\frac{1}{\mu} \int_0^x \left(u_0(\xi) - u(\xi,t)\right) \mathrm{d}\xi \right], \quad \mathrm{Q}(x,t) := \exp\left(\frac{1}{\mu} f(x)\, t \right)$$

with $f(x) = p_e + \frac{1}{2}Gx(1-x)$.

This representation formula and some uniform in time bounds of the solution of energy form yield a priori boundedness of v both from above and from below.

Lemma 2 *For any* $(x,t) \in \overline{Q_T}$

$$C^{-1} \leq v(x,t) \leq C$$

with a constant C independent of T.

References

[1] Bebernes, J. and A. Bressan, Global a priori estimates for a viscous reactive gas, *Proc. Roy. Soc. Edinb.*, **101A** (1985), 321-333.

[2] Chandrasekhar, S., *An introduction to the study of stellar structure*, Dover, 1957.

[3] Chen, G. -Q., Global solution to the compressible Navier-Stokes equations for a reacting mixture, *SIAM J. Math. Anal.*, **23** (1992), 609-634.

[4] Ducomet, B., On the stability of a stellar structure in one dimension, *Math. Mod. Meth. Appl. Sci.*, **6** (1996), 365-783.

[5] Ducomet, B., On the stability of a stellar structure in one dimension II: The reactive case, *Math. Modelling Numer. Anal.*, **31** (1997), 381-407.

[6] Ducomet, B., Some asymptotics for a reactive Navier-Stokes-Posson system, *Math. Mod. Meth. Appl. Sci.*, **9** (1999), 1039-1076.

[7] Ducomet, B., Some stability results for reactive Navier-Stokes-Poisson systems, *Evol. Equ.: Existence Regularity Singularities Banach Center Publ.*, **52** (2000), 83-118.

[8] Ducomet, B. and A. Zlotnik, Lyapunov functional method for 1D radiative and reactive viscous gas dynamics, *Arch. Ration. Mech. Anal.*, **177** (2005), 185-279.

[9] Ducomet, B. and A. Zlotnik, On the large-time behavior of 1D radiative and reactive viscous flows for higher-order kinetics, *Nonlinear Anal.*, **63** (2005), 1011-1033.

[10] Kazhikhov, A. V., Sur le solubilité globale des problèmes monodimensionnels aux valeurs initiales-limitées pour les équations d'un gaz visqueux et calorifère, *C. R. Acad. Sci. Paris Sér. A.*, **284** (1977), 317-320.

[11] Kazhikhov, A. V., To the theory of boundary value problems for equations of a one-dimensional non-stationary motion of a viscous heat-conductive gas, *Din. Sploshn. Sredy*, **50** (1981), 37-62. [Russian]

[12] Kazhikhov, A. V. and V. V. Shelukhin, Unique global solution with respect to time of the initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.*, **41** (1977), 273-282.

[13] Kippenhahn, R. and A. Weingert, *Stellar structure and evolution*, Springer-Verlag, Berlin, 1994.

[14] Ladyžhenskaja, O. A., V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968.

[15] Matsumura, A. and T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, *J. Math. Kyoto Univ.*, **20** (1980), 67-104.

[16] Matsumura, A. and T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids, *Commun. Math. Phys.*, **89** (1983), 445-464.

[17] Nakamura, T. and S. Nishibata, Large-time behavior of spherically symmetric flow for viscous polytropic ideal gas, to appear in *Indiana Univ. Math. J.*.

[18] Nagasawa, T., On the one-dimensional motion of the polytropic ideal gas non-fixed on the boundary, *J. Differential Equations*, **65** (1986), 49-67.

[19] Nagasawa, T., On the asymptotic behavior of the one-dimensional motion of the polytropic ideal gas with stress-free condition, *Quart. Appl. Math.*, **46** (1988), 665-679.

[20] Nagasawa, T., On the outer pressure problem of the one-dimensional polytropic ideal gas, *Jpn. J. Appl. Math.*, **5** (1988), 53-85.

[21] Okada, M., Free-boundary value problems for equations of one-dimensional motion of compressible viscous fluids, *Jpn. J. Appl. Math.*, **4** (1987), 219-235.

[22] Poland, J. and D. Kassoy, The induction period of a thermal explosion in a gas between infinite parallel plates, *Combustion Flame*, **50** (1983), 259-274.

[23] Secchi, P., On the uniqueness of motion of viscous gaseous stars, *Math. Meth. Appl. Sci.*, **13** (1990), 391-404.

[24] Secchi, P., On the evolution equations of viscous gaseous stars, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **18** (1991), 295-318.

[25] Shandarin, S. F. and Y. B. Zel'dovichi, The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium, *Rev. Modern Phys.*, **61** (1989), 185-220.

[26] Solonnikov, V. A. and A. V. Kazhikhov, Existence theorems for the equations of motion of a compressible viscous fluid, *Annu. Rev. Fluid Mech.*, **13** (1981), 79-95.

[27] Solonnikov, V. A. and A. Tani, Free boundary problem for a viscous compressible flow with the surface tention, *Constantin Caratheodory: an international tribute, Th. M. Rassias (editor), World Sci. Publ.*, (1991), 1270-1303.

[28] Solonnikov, V. A. and A. Tani, Evolution free boundary problem for equations of motion of viscous compressible barotropic liquid, *Lecture Notes in Math.*, **1530** (1992), 30-55, Springer, Berlin.

[29] Solonnikov, V. A. and A. Tani, Equilibrium figures of slowly rotating viscous compressible barotropic capillary liquid, *Adv. Math. Sci. Appl.*, **2** (1993), 139-145.

[30] Tani, A., On the first initial-boundary value problem of compressible viscous fluid motion, *Publ. Res. Inst. Math. Sci.*, **13** (1977), 193-253.

[31] Tani, A., On the free boundary value problem for the compressible viscous fluid motion, *J. Math. Kyoto Univ.*, **21** (1981), 839-859.

[32] Umehara, M., Global solution to the one-dimensional equations for a self-gravitating viscous gaseous star, Thesis of Mastership, *Keio Univ.*, (2004).

[33] Umehara, M. and A. Tani, Global solution to the one-dimensional equations for a self-gravitating viscous radiative and reactive gas, *J. Differential Equations*, **234** (2007), 439-463.

[34] Umehara, M. and A. Tani, Temporally global solution to the equations for a spherically symmetric viscous radiative and reactive gas over the rigid core, to appear in *Anal. Appl.*.

[35] Umehara, M. and A. Tani, Global solvability of the free-boundary problem for one-dimensional motion of a self-gravitating viscous radiative and reactive gas, submitted.

[36] Umehara, M. and A. Tani, Global existence and asymptotic behaviour of the one-dimensional motion of a self-gravitating viscous and reactive gas, preprint, (2008).

[37] Umehara, M. and A. Tani, Uniform a priori estimates of the classical solution to the free-boundary problem for one-dimensional motion of a self-gravitating viscous gas, preprint, (2008).

[38] Williams, F., *Combustion theory*, Addison-Wesley, 1965.

[39] Yanagi, S., Asymptotic stability of the the solutions to a full one-dimensional system of heat-conductive, reactive, compressible viscous gas, *Jpn. J. Indust. Appl. Math.*, **15**, (1998), 423-442.

[40] Yanagi, S., Asymptotic stability of the spherically symmetric solutions for a viscous polytropic gas in a field of external forces, *Transport. Theor. Statist. Phys.*, **29** (2000), 333-353.