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On regularity of suitable weak solutions to the Navier-Stokes equations in unbounded domains

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1 Introduction

Let us consider the Navier-Stokes equations in $\Omega\times(0,T)$ with $0<T<\infty$, where Ω is a general domain with uniformly C^2-boundary $\partial\Omega\neq\emptyset$ in $\mathbf{R}^3$. In particular, we are interested in the problem in *unbounded* domains with *non-compact* boundary:

$$\begin{cases} \partial_t u-\Delta u+u\cdot\nabla u+\nabla p=0 & \text{in } \Omega\times(0,T), \\ \operatorname{div} u=0 & \text{in } \Omega\times(0,T), \end{cases} \tag{1.1}$$

$$\begin{cases} u=0 & \text{on } \partial\Omega\times(0,T), \\ u|_{t=0}=u_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $u=u(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))$ and $p=p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at the point $(x,t)\in\Omega\times(0,T)$, respectively, while $u_0=u_0(x)=(u_{0,1}(x),u_{0,2}(x),u_{0,3}(x))$ is the given initial velocity vector.

For $u_0\in L^2$, it is known that there exists a global weak solutions to (1.1)-(1.2), so-called Leray-Hopf weak solution. Although uniqueness and regularity of weak solutions are still open problems, we have the partial result by Caffarelli-Kohn-Nirenberg [1]. Introducing the notion of suitable weak solutions, they showed that the one dimensional Hausdorff measure of the singular set of such solutions is zero. The existence of a suitable weak solution for $u_0\in L^2$ is known in the whole space, half spaces, bounded and exterior domains, see e.g. Taniuchi [12]. F.-H. Lin [4] proved the same result in a much simpler way with a slightly different definition. Seregin [6] developed the partial regularity theory near the boundary. The partial regularity can be used to prove the regularity for large $|x|$. Indeed, Caffarelli-Kohn-Nirenberg [1] proved that the suitable weak solutions are regular for large $|x|$ in $\mathbf{R}^3$. The same result was shown in exterior domains by Sohr-von Wahl [10]. The most important point for their results is to show that the pressure is small for large $|x|$.

It is known that the standard approach to the Stokes equations in L^q, $1<q<\infty$, cannot be extended to general unbounded domains except $q=2$; the Helmholtz decomposition in L^q holds for some special q in a certain unbounded domain, see Maslennikova-Bogovskii [5]. However, Farwig-Kozono-Sohr [2] show that L^q theories of the Stokes equations remain true in any uniformly C^2-domains if we replace L^q by L^2+L^q for $1<q<2$ and by $L^2\cap L^q$ for $2<q<\infty$, respectively. As a by-product, they prove the existence of a suitable weak solution for $u_0\in L^2$ in such domains.

Our purpose is to prove the regularity of suitable weak solutions for large $|x|$ in general unbounded domains. For the proof, the so-called ε-regularity theorem for suitable weak solutions plays a crucial role. Although such theorems are well-known by [1, 4, 6], it seems impossible to apply it directly to our situation. The reason is that their characterization of the ε-regularity theorem includes integrals of the pressure $p(x,t)$, while it generally seems very difficult to determine the class of the pressure $p(x,t)$ in general domains with non-compact boundary. Therefore, we need to modify the known ε-regularity theorem not by means of the integral of the pressure $p(x,t)$ itself but by means of that of the pressure gradient $\nabla p(x,t)$. Applying the maximal regularity theorem in L^2+L^q with $1<q<2$ for the Stokes equations [2], we show that the pressure gradient satisfies $\nabla p \in L^{5/4}(\delta,T;L^2+L^{5/4})$ for arbitrary $\delta>0$. Our ε-regularity theorem up to the boundary enables us to obtain a compact subset $K_\delta \subset \Omega$ depending only on $\delta>0$ such that every suitable weak solution $u(x,t)$ is Hölder continuous for $(x,t)\in(\overline{\Omega}\setminus K_\delta)\times(\delta,T)$. Simultaneously, our result shows that there is no singularity near the boundary $\partial\Omega$ for large $|x|$. Therefore, we may regard the main theorem below as regularity theorem up to the boundary for large $|x|$.

2 Main Theorem

Before stating our result, we introduce some notations. Let $B(x_0,R)$ and $B(x_0',R')$ be the open balls with radius $R>0$ centered at $x_0\in\mathbf{R}^3$ and $x_0'\in\mathbf{R}^2$, respectively. For $z_0=(x_0,t_0)$, $Q(z_0,R)=\{(x,t);x\in B(x_0,R),t\in(t_0-R^2,t_0)\}$ is the standard parabolic cylinder. For simplicity, we abbreviate $B(0,R)$ and $B(0,1)$ to $B(R)$ and B, respectively. $L^q(\Omega)$ stands for the usual (vector-valued) L^q-space with norm $\|\cdot\|_{q,\Omega}$; $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and the duality pairing between $L^q(\Omega)$ and $L^{q'}(\Omega)$, where $\frac{1}{q}+\frac{1}{q'}=1$. We denote by $C_{0,\sigma}^\infty(\Omega)$ the set of all C^∞ functions ψ with compact support in Ω such that $\operatorname{div}\psi=0$. The space $L^q_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^q-norm $\|\cdot\|_{q,\Omega}$ for $1<q<\infty$.

Throughout this paper, we use the following assumption.

Assumption Let s,q and q_* be positive numbers satisfying the following relations:

$$\frac{2}{s}+\frac{3}{q}=4 \quad \text{for} \quad 1<s<2 \quad \text{and} \quad 1<q<\frac{3}{2}, \quad \frac{1}{q_*}=\frac{1}{q}-\frac{1}{3}.$$

Our definition of a weak solution is as follows.

Definition 2.1 *Let $u_0\in L^2_\sigma(\Omega)$. A function u is called a weak solution of* (1.1)-(1.2) *in $\Omega\times(0,T)$ if*

(i) $\quad u\in L^\infty(0,T;L^2_\sigma(\Omega))\cap L^2(0,T;W^{1,2}_{0,\sigma}(\Omega))$,

(ii)

$$-\int_0^T(u,\phi)h'\,dt+\int_0^T(\nabla u,\nabla\phi)h\,dt+\int_0^T(u\cdot\nabla u,\phi)h\,dt=(u_0,\phi)h(0)$$

for all $h\in C_0^\infty([0,T)),\phi\in C_{0,\sigma}^\infty(\Omega)$.

We give definitions of *interior* and *boundary* suitable weak solutions.

Definition 2.2 *The pair $(u,\nabla p)$ is called an interior suitable weak solution of the Navier-Stokes equations* (1.1) *in $\Omega\times(0,T)$ if the following conditions are satisfied:*

(i) $u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))$, $\nabla p\in L^s_{loc}((0,T);L^q_{loc}(\overline{\Omega}))$.

(ii) $(u, \nabla p)$ *satisfies* (1.1) *in the sense of distribution in* $\Omega \times (0,T)$.

(iii) (*generalized energy inequality*) *There holds*

$$\begin{aligned}\int_\Omega |u(y,t)|^2\phi(y,t)\,dy + 2\int_0^t\int_\Omega |\nabla u|^2\phi\,dyd\tau \\ \leq \int_0^t\int_\Omega \left\{|u|^2(\phi_\tau + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi\right\}dyd\tau\end{aligned}$$

for all $t \in (0,T)$ *and all nonnegative functions* $\phi \in C_0^\infty(\Omega \times (0,T))$.

Definition 2.3 *Let* Γ *be a relatively open subset of* $\partial\Omega$. *The pair* $(u, \nabla p)$ *is called a boundary suitable weak solution of the Navier-Stokes equations* (1.1) *near* $\Gamma \times (0,T)$ *if the following conditions are satisfied:*

(i) $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$, $\nabla^2 u, \nabla p \in L^s_{loc}((0,T);L^q_{loc}(\overline{\Omega}))$.

(ii) $(u, \nabla p)$ *satisfies* (1.1) *in the sense of distribution in* $\Omega \times (0,T)$ *and*

$$u = 0 \quad \text{on} \quad \Gamma \times (0,T).$$

(iii) (*generalized energy inequality*) *There holds*

$$\begin{aligned}\int_\Omega |u(y,t)|^2\phi(y,t)\,dy + 2\int_0^t\int_\Omega |\nabla u|^2\phi\,dyd\tau \\ \leq \int_0^t\int_\Omega \left\{|u|^2(\phi_\tau + \Delta\phi) + (|u|^2 + 2p)u\cdot\nabla\phi\right\}dyd\tau\end{aligned}$$

for all $t \in (0,T)$ *and all nonnegative functions* $\phi \in C_0^\infty(\mathbf{R}^3 \times (0,T))$ *vanishing in a neighborhood of the set* $(\partial\Omega\backslash\Gamma) \times (0,T)$.

If $\Gamma = \partial\Omega$, *then* $\partial\Omega \setminus \Gamma = \emptyset$ *and this inequality holds for all* $t \in (0,T)$ *and all nonnegative functions* $\phi \in C_0^\infty(\mathbf{R}^3 \times (0,T))$, *see* [8, p.340].

Remark 2.4 In the corresponding definitions of [7] and [8], it holds the stronger global condition $\nabla^2 u, \nabla p \in L^s(0,T;L^q(\Omega))$ with $q = \frac{9}{8}, s = \frac{3}{2}, \frac{2}{s} + \frac{3}{q} = 4$. The weaker conditions on $\nabla^2 u$ and ∇p in Definitions 2.2 and 2.3 are useful in particular in order to admit initial values $u_0 \in L^2_\sigma(\Omega)$; see the existence result in Theorem 2.6 where $q = s = \frac{5}{4}$ and where $\varepsilon = 0$ is possible under a stronger condition on u_0.

We give a precise definition of uniformly C^2-domains, see [9].

Definition 2.5 *We call* Ω *uniformly* C^2*-domain if and only if there exist positive constants* $\alpha, \beta, K > 0$ *with the following properties: for each* $x_0 \in \partial\Omega$ *there exist a Cartesian coordinate system* $y = (y', y_3) = (y_1, y_2, y_3)$ *with the origin* x_0 *and* C^2*-function* $h_{x_0}(y'), |y'| \leq \alpha$ *with* $\|h_{x_0}\|_{C^2(\overline{B}'(\alpha))} \leq K$ *such that the neighborhood*

$$U_\alpha(x_0) = U_{\alpha,\beta,h_{x_0}}(x_0) := \{(y', y); h_{x_0}(y') - \beta < y_3 < h_{x_0}(y') + \beta, |y'| \leq \alpha\}$$

satisfies

$$U^+_\alpha(x_0) = U^+_{\alpha,\beta,h_{x_0}}(x_0) := \{(y',y); h_{x_0}(y') < y_3 < h_{x_0}(y') + \beta, |y'| \le \alpha\}$$
$$= \Omega \cap U_\alpha(x_0)$$

and

$$\partial\Omega \cap U_\alpha(x_0) = \{(y',y); y_3 = h_{x_0}(y'), |y'| \le \alpha\}.$$

We recall the existence of a suitable weak solution in general domains.

Theorem 2.6 [2] *Let $\Omega \subseteq \mathbf{R}^3$ be a uniformly C^2-domain and let $u_0 \in L^2_\sigma(\Omega)$. Then there exists a suitable weak solution $u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2_{loc}([0,T);W^{1,2}_{0,\sigma}(\Omega))$ in the sense of Definitions 2.2 and 2.3 with $\Gamma = \partial\Omega$ and $s = q = \frac{5}{4}$ satisfying the following regularity properties:*

$$u_t,\ u,\ \nabla u,\ \nabla^2 u,\ \nabla p \in L^{5/4}(\varepsilon,T;L^2 + L^{5/4}) \quad \textit{for all } 0 < \varepsilon < T, \tag{2.1}$$

Remark 2.7 (i) Although it is not mentioned specifically, we can see that the suitable weak solution constructed in [2] is actually *interior* and *boundary* suitable weak solution in the sense of Definition 2 and 3 with $\Gamma = \partial\Omega$.

(ii) Since L^2 and $L^{5/4}$ are reflexive, for u and p satisfying (2.1) there exist $u^{(1)}, u^{(2)}, p^{(1)}$ and $p^{(2)}$ such that

$$u = u^{(1)} + u^{(2)},\ p = p^{(1)} + p^{(2)},$$
$$u^{(1)}_t, u^{(1)}, \nabla u^{(1)}, \nabla^2 u^{(1)}, \nabla p^{(1)} \in L^{5/4}(\varepsilon,T;L^2) \quad \textit{for all } 0 < \varepsilon < T,$$
$$u^{(2)}_t, u^{(2)}, \nabla u^{(2)}, \nabla^2 u^{(2)}, \nabla p^{(2)} \in L^{5/4}(\varepsilon,T;L^{5/4}) \quad \textit{for all } 0 < \varepsilon < T$$

and

$$\begin{aligned}\|u_t\|_Y + \|u\|_Y + \|\nabla u\|_Y + \|\nabla p\|_Y \\ &= \|u^{(1)}_t\|_{Y^{(1)}} + \|u^{(1)}\|_{Y^{(1)}} + \|\nabla^2 u^{(1)}\|_{Y^{(1)}} + \|\nabla p^{(1)}\|_{Y^{(1)}} \\ &+ \|u^{(2)}_t\|_{Y^{(2)}} + \|u^{(2)}\|_{Y^{(2)}} + \|\nabla^2 u^{(2)}\|_{Y^{(2)}} + \|\nabla p^{(2)}\|_{Y^{(2)}}\end{aligned}$$

where the spaces $Y, Y^{(1)}$ and $Y^{(2)}$ are defined by $Y = L^{5/4}(\varepsilon,T;L^2+L^{5/4})$, $Y^{(1)} = L^{5/4}(\varepsilon,T;L^2)$ and $Y^{(2)} = L^{5/4}(\varepsilon,T;L^{5/4})$. For details, see [2, Remark 2.8].

Our main result in this paper now reads:

Theorem 2.8 *Let Ω be a uniformly C^2-domain and let $u_0 \in L^2_\sigma(\Omega)$. Suppose that $(u, \nabla p)$ is any suitable weak solution of (1.1), (1.2) in the sense of Definitions 2.2 and 2.3 with $\Gamma = \partial\Omega$. Then for any $0 < \delta < T$ there exists a positive constant K such that u is Hölder continuous on $\{x \in \overline{\Omega}; |x| \ge K\} \times (\delta, T)$.*

Remark 2.9 The regularity of suitable weak solutions for large $|x|$ has been proved in the whole space $\mathbf{R}^3$ [1] and exterior domains [10]. In both cases, since there is no boundary outside a sufficiently large ball, it sufficies to apply the *interior* ε-regularity theorem in [1, 4, 3] to the proof of the smoothness of $u(x,t)$ for large $|x|$. In general domains with non-compact boundary, it is necessary to consider the smoothness not only in the interior of Ω but also near the boundary. The notion of *boundary* suitable weak solutions makes it possible to prove regularity up to the boundary. All the previous ε-regularity theorems [1, 4, 3, 8] are characterized by the integral of the pressure $p(x,t)$. However, non-compactness of the boundary prevents us from obtaining behavior of $p(x,t)$ by means of the information on $\nabla p(x,t)$. Therefore, we modify these previous results in terms of the integral of the pressure gradient $\nabla p(x,t)$. Although it is generally known that the singularity may occur near the boundary, our theorem makes it clear that, in the same way as in $\mathbf{R}^3$ and exterior domains, we can prove the smoothness of the solution for sufficiently large $|x|$ even in general unbounded domains.

3 Interior partial regularity

Let $z_0 \in \Omega$ and let $R > 0$. For (u,p), we denote the integral average by the slash

$$
\begin{aligned}
(u)_{z_0,R} &:= \fint\!\!\fint_{Q(z_0,R)} u(z)\,dz = \frac{1}{|Q(z_0,R)|}\iint_{Q(z_0,R)} u(z)\,dz,\\
[p]_{x_0,R} &:= \fint_{B(x_0,R)} p(y,t)\,dy = \frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} p(y,t)\,dy.
\end{aligned}
$$

We introduce $Y_1(u;Q(z_0,R)), Y_2(u;Q(z_0,R)), Y_3(p;Q(z_0,R))$ defined by

$$
\begin{aligned}
Y_1(u;Q(z_0,R)) &= \Big(\fint\!\!\fint_{Q(z_0,R)} |u-(u)_{z_0,R}|^3\Big)^{1/3},\\
Y_2(u;Q(z_0,R)) &= \Big(\fint_{t_0-R^2}^{t_0}\Big(\fint_{B(x_0,R)} |u-(u)_{z_0,R}|^{q'_*}\Big)^{s'/q'_*}\Big)^{1/s'},\\
Y_3(p;Q(z_0,R)) &= R^2\Big(\fint_{t_0-R^2}^{t_0}\Big(\fint_{B(x_0,R)} |\nabla p|^q\Big)^{s/q}\Big)^{1/s}.
\end{aligned}
$$

Furthermore, we define $Y(u,p;Q(z_0,R))$ and $Z(u,p;Q(z_0,R))$ by

$$
\begin{aligned}
Y(u,p;Q(z_0,R)) &= Y_1(u;Q(z_0,R)) + Y_2(u;Q(z_0,R)) + Y_3(p;Q(z_0,R)),\\
Z(u,p;Q(z_0,R)) &= Y_1(u;Q(z_0,R)) + Y_3(p;Q(z_0,R)).
\end{aligned}
$$

In order to prove our main theorem, we need the following version of the ε-regularity theorem, which is different from that of [4].

Theorem 3.1 *There exists an absolute constant $\varepsilon_\sharp > 0$ such that if any interior suitable weak solution $(u,\nabla p)$ in $Q = Q(0,1)$ satisfies one of the following conditions:*

(i) *for $1 < s < \frac{3}{2}$,*

$$
A(Q) := \iint_Q |u|^3 + \int_{-1}^0\Big(\int_B |u|^{q'_*}\Big)^{s'/q'_*} + \int_{-1}^0\Big(\int_B |\nabla p|^q\Big)^{s/q} < \varepsilon_\sharp, \tag{3.1}
$$

(ii) *for* $\frac{3}{2} \le s < 2$,

$$B(Q) := \iint_Q |u|^3 + \int_{-1}^{0} \left(\int_B |\nabla p|^q \right)^{s/q} < \varepsilon_\sharp, \tag{3.2}$$

then u *is Hölder continuous on* $\overline{Q}(\frac{1}{2}) = \overline{Q}(0, \frac{1}{2})$.

Remark 3.2 The hypotheses (3.1) and (3.2) include only the pressure gradient, while the ε-regularity theorem in the previous results [1, 3, 4] requires the assumption on the pressure itself. In the whole space and exterior domains, it is possible to obtain regularity of the pressure by means of that of the pressure gradient. However, since the boundary $\partial\Omega$ is non-compact in our case, we can hardly expect to obtain global regularity of the pressure itself.

The proof is based on the standard blow-up argument with some modifications of [3]. For details, see [11].

Lemma 3.3 *Let* $M > 3$. *For* $0 < \theta_0 < \frac{1}{2}$, *there exist positive constants* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that if any interior suitable weak solution* $(u, \nabla p)$ *of the Navier-Stokes equations* (1.1) *in* Q *satisfies*

$$\begin{cases} |(u)_{,1}| < M, \quad Y(u,p;Q) < \varepsilon_0 & if \quad 1 < s < \frac{3}{2}, \\ |(u)_{,1}| < M, \quad Z(u,p;Q) < \varepsilon_0 & if \quad \frac{3}{2} \le s < 2, \end{cases}$$

then there holds

$$\begin{cases} Y(u,p;Q(\theta_0)) \le C_0\, \theta_0^{2/s'}\, Y(u,p;Q) & if \quad 1 < s < \frac{3}{2}, \\ Z(u,p;Q(\theta_0)) \le C_0\, \theta_0^{2/s'}\, Z(u,p;Q) & if \quad \frac{3}{2} \le s < 2. \end{cases}$$

By the successive procedure of Lemma 3.3 and the scaling transformation

$$u_R(y,s) = Ru(x_0 + Ry, t_0 + R^2 s), \quad p_R(y,s) = R^2 p(x_0 + Ry, t_0 + R^2 s),$$

we obtain the following general result.

Lemma 3.4 *Let* $M > 3$ *and let* $0 \le \beta < \frac{2}{s'}$. *Suppose that* $0 < \theta < \frac{1}{2}$ *is a constant such that*

$$C_0 \theta_0^{\frac{2-s'\beta}{2s'}} \le 1,$$

where C_0 *is the constant in Lemma 3.3. Suppose that any interior suitable weak solution* $(u, \nabla p)$ *in* $Q(z_0, R)$ *satisfies*

$$\begin{cases} R|(u)_{,z_0}| < M, \quad RY(u,p;Q(z_0,R)) < \bar{\varepsilon}_0 & if \ 1 < s < \frac{3}{2}, \\ R|(u)_{,z_0}| < M, \quad RY(u,p;Q(z_0,R)) < \bar{\varepsilon}_0 & if \ \frac{3}{2} \le s < 2, \end{cases}$$

where $\bar{\varepsilon}_0 = \min\{\varepsilon_0, \frac{1}{4}\theta_0^5 M\}$. *Then there exists a positive constant* C *such that*

$$\begin{cases} Y(u,p;Q(z_0,\rho)) \le C\left(\frac{\rho}{R}\right)^{\frac{2+s'\beta}{2s'}} Y(u,p;Q(z_0,R)) & if\ 1 < s < \frac{3}{2}, \\ Z(u,p;Q(z_0,\rho)) \le C\left(\frac{\rho}{R}\right)^{\frac{2+s'\beta}{2s'}} Z(u,p;Q(z_0,R)) & if\ \frac{3}{2} \le s < 2, \end{cases}$$

for all $\rho \in (0,R]$.

Proof of Theorem 3.1. We take M sufficiently large and $\beta = \frac{1}{s'}$. For $z_0 \in \overline{Q}(\frac{3}{4})$, there holds

$$Q\Big(z_0, \frac{1}{4}\Big) \subset Q \quad \text{and} \quad \frac{1}{4}|(u)_{z_0,\frac{1}{4}}| \le C A^{\frac{1}{3}}(Q).$$

and

$$\begin{cases} \frac{1}{4}Y(u,p;Q(z_0,\frac{1}{4})) \le C\,(A^{1/3}(Q) + A^{1/s'}(Q) + A^{1/s}(Q)) & \text{if } 1 < s < \frac{3}{2}, \\ \frac{1}{4}Z(u,p;Q(z_0,\frac{1}{4})) \le C\,(B^{1/3}(Q) + B^{1/s}(Q)) & \text{if } \frac{3}{2} \le s < 2. \end{cases}$$

Let $\varepsilon_\sharp$ be such that

$$\begin{cases} C\varepsilon_\sharp^{1/3} < M \ \text{ and } \ C\big(\varepsilon_\sharp^{1/3} + \varepsilon_\sharp^{1/s'} + \varepsilon_\sharp^{1/s}\big) < \bar{\varepsilon}_0 & \text{if } 1 < s < \frac{3}{2}, \\ C\varepsilon_\sharp^{1/3} < M \ \text{ and } \ C\big(\varepsilon_\sharp^{1/3} + \varepsilon_\sharp^{1/s}\big) < \bar{\varepsilon}_0 & \text{if } \frac{3}{2} \le s < 2. \end{cases}$$

It follows from Lemma 3.4 with $\beta = 0$ that

$$\begin{cases} Y(u,p;Q(z_0,\rho)) \le \rho^{1/s'} Y(u,p;Q(z_0,\frac{1}{4})) \le \rho^{1/s'}\bar{\varepsilon}_0 & \text{if } 1 < s < \frac{3}{2}, \\ Z(u,p;Q(z_0,\rho)) \le \rho^{1/s'} Z(u,p;Q(z_0,\frac{1}{4})) \le \rho^{1/s'}\bar{\varepsilon}_0 & \text{if } \frac{3}{2} \le s < 2, \end{cases}$$

for all $z_0 \in \overline{Q}(\frac{3}{4})$ and $0 < \rho < \frac{1}{4}$. It follows from the Campanato embedding theorem of parabolic type that u is Hölder continuous on $\overline{Q}(\frac{2}{3})$ with exponent $1/s'$. This completes the proof of Theorem 3.1.

4 Boundary partial regularity

Let $Q^+(z_0,R) = \{(x,t) \in B(x_0,R) \times (t_0 - R^2, t_0); x_{03} > 0\}$ be the half cylinder. We introduce $Y_1^+(u;Q^+(z_0,R))$ and $Y_2^+(u;Q^+(z_0,R))$ defined by

$$Y_1^+(u;Q^+(z_0,R)) = \Big(⨍\!\!⨍_{Q^+(z_0,R)} |u|^3\Big)^{1/3},$$

$$Y_2^+(u;Q^+(z_0,R)) = \Big(⨍_{t_0-R^2}^{t_0} \Big(⨍_{B^+(x_0,R)} |u|^{q'_*}\Big)^{s'/q'_*}\Big)^{1/s'}.$$

Furthermore, we define $Y^+(u,p;Q^+(z_0,R)), Z^+(u,p;Q^+(z_0,R))$ by

$$\begin{aligned} Y^+(u,p;Q^+(z_0,R)) &= Y_1^+(u;Q^+(z_0,R)) + Y_2^+(u;Q^+(z_0,R)) \\ &\quad + Y_3(p;Q^+(z_0,R)), \\ Z^+(u,p;Q^+(z_0,R)) &= Y_1^+(u;Q^+(z_0,R)) + Y_3(p;Q^+(z_0,R)). \end{aligned}$$

We shall prove the following boundary ε-regularity theorem, see [7, 8].

Theorem 4.1 *Let Ω be a uniformly C^2-domain and let Γ be an open subset of the boundary $\partial\Omega$. There exist an absolute constant $\varepsilon_* > 0$ and $R_* > 0$ such that if any boundary suitable weak solution $(u, \nabla p)$ of the Navier-Stokes equation (1.1) near $\Gamma \times (0,T)$ and $z_0 = (x_0, t_0)$ with $x_0 \in \Gamma$, $0 < t_0 \le T$ and $t_0 - R_* > 0$, satisfy one of the following conditions:*

(i) *for* $1 < s < \frac{3}{2}$,

$$\frac{1}{R_*^2}\int_{t_0-R_*^2}^{t_0}\int_{U_{R_*}^+(x_0)} |u|^3 + \frac{1}{R_*^{s'}}\int_{t_0-R_*^2}^{t_0}\left(\int_{U_{R_*}^+(x_0)} |u|^{q'_*}\right)^{s'/q'_*} + \frac{1}{R_*^s}\int_{t_0-R_*^2}^{t_0}\left(\int_{U_{R_*}^+(x_0)} |\nabla p|^q\right)^{s/q} < \varepsilon_*, \tag{4.1}$$

(ii) *for* $\frac{3}{2} \le s < 2$,

$$\frac{1}{R_*^2}\int_{t_0-R_*^2}^{t_0}\int_{U_{R_*}^+(x_0)} |u|^3 + \frac{1}{R_*^s}\int_{t_0-R_*^2}^{t_0}\left(\int_{U_{R_*}^+(x_0)} |\nabla p|^q\right)^{s/q} < \varepsilon_*, \tag{4.2}$$

then u is Hölder continuous on $\overline{U^+_{\frac{R_*}{4}}}(x_0) \times [t_0 - \frac{R_*^2}{16}, t_0]$. Here, $U^+_{R_*}(x_0)$ is the set defined in Definition 2.5.

We straighten the boundary by the relation

$$x = \widehat{h}(y) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 - h(y_1, y_2) \end{pmatrix}, \tag{4.3}$$

where $h \in C^2(\overline{B'}(\alpha))$ satisfies

$$h(0,0) = 0, \quad \nabla' h(0,0) = 0, \quad \|h\|_{C^2} \le K, \quad \|\nabla' h\|_\infty \le M \tag{4.4}$$

for arbitrary $M > 0$. Then the Navier-Stokes equations (1.1) turn into the form

$$\partial_t \widehat{u} - \widehat{\Delta}_h \widehat{u} + \widehat{u} \cdot \widehat{\nabla}_h \widehat{u} + \widehat{\nabla}_h \widehat{p} = 0, \quad \widehat{\nabla}_h \cdot \widehat{u} = 0, \quad \widehat{u}|_{x_3=0} = 0,$$

where $\widehat{u} = u \circ \widehat{h}, \widehat{p} = p \circ \widehat{h}$ and $\widehat{\nabla}_h$ and $\widehat{\Delta}_h$ are defined by the formulas

$$\widehat{\nabla}_h = \left(\frac{\partial}{\partial x_1} - \frac{\partial h}{\partial x_1}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} - \frac{\partial h}{\partial x_2}\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_3}\right)$$

and

$$\widehat{\Delta}_h = \sum_{i,j=1}^3 a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^3 b_i(x)\frac{\partial}{\partial x_i},$$

where

$$\Big(a_{ij}(x)\Big)_{1\le i,j\le 3} = \begin{pmatrix} 1 & 0 & -\frac{\partial h}{\partial x_1} \\ 0 & 1 & -\frac{\partial h}{\partial x_2} \\ -\frac{\partial h}{\partial x_1} & -\frac{\partial h}{\partial x_2} & 1 + |\nabla' h|^2 \end{pmatrix},$$

$$\left(b_i(x)\right)_{1\le i\le 3} = \begin{pmatrix} 0 \\ 0 \\ -\Delta' h \end{pmatrix}.$$

The following global estimate plays an essential role to prove Theorem 4.1.

Proposition 4.2 [8] *Let $1 < q, s < \infty$ and $h \in C^2(\mathbf{R}^2)$. Then there exists an absolute constant $K_* > 0$ such that if h satisfies* (4.4) *for $K \le K_*$, then there exists a unique solution (u,p) of the perturbed Stokes equations*

$$\partial_t u - \widehat{\Delta}_h u + \widehat{\nabla}_h p = f,\ \ \widehat{\nabla}_h \cdot u = 0 \ \text{ in } \Pi_1^+, \quad u|_{x_3=0} = 0, \ \ u|_{t=-1} = 0,$$

where $\Pi_1^+ = \mathbf{R}^3_+ \times (-1,0)$. Moreover, it holds that

$$\|u_t\|_{q,s,\Pi_1^+} + \|\nabla^2 u\|_{q,s,\Pi_1^+} + \|\nabla p\|_{q,s,\Pi_1^+} \le C\|f\|_{q,s,\Pi_1^+}.$$

Let us consider the perturbed Navier-Stokes equations

$$\partial_t u - \widehat{\Delta}_h u + u\cdot\widehat{\nabla}_h u + \widehat{\nabla}_h p = 0,\ \widehat{\nabla}_h\cdot u = 0 \text{ in } Q^+ = Q^+(0,1),\ u|_{x_3=0} = 0. \tag{4.5}$$

The notion of suitable weak solutions for the perturbed Navier-Stokes equations can be defined by the same way as in Definition 2.3.

Definition 4.3 *The pair $(u, \nabla p, h)$ is called a boundary suitable weak solution of the perturbed Navier-Stokes equations* (4.5) *in Q^+ if the following conditions are satisfied:*

(i) $u \in L^\infty(-1,0;L^2(B^+)) \cap L^2(-1,0;W^{1,2}(B^+))$,
$\nabla^2 u, \nabla p \in L^s(-1,0;L^q(B^+))$.

(ii) $(u,\nabla p)$ *satisfies* (1.1) *in the sense of distribution in Q^+ and*

$$u = 0 \quad \text{on} \quad \{x \in \overline{B}^+; x_3 = 0\} \times (-1,0).$$

(iii) (*generalized energy inequality*) *There holds*

$$\int_{B^+} |u(y,t)|^2\phi(y,t)\,dy + 2\int_{-1}^t\int_{B^+} |\widehat{\nabla}_h u|^2\phi\,dyd\tau$$
$$\le \int_{-1}^t\int_{B^+}\left\{|u|^2(\phi_\tau + \widehat{\Delta}_h\phi) + (|u|^2 + 2p)u\cdot\widehat{\nabla}_h\phi\right\}dyd\tau$$

for all $t \in (-1,0)$ and all nonnegative functions $\phi \in C_0^\infty(Q)$.

Theorem 4.1 can be deduced from the following result.

Proposition 4.4 *Assume that $h \in C^2(\overline{B'})$ satisfies* (4.4) *with $K \le K_*$, where K_* is the constant as in Proposition 4.2. Then there exists an absolute constant $\varepsilon_* > 0$ such that if any boundary suitable weak solution $(u,\nabla p, h)$ of the perturbed Navier-Stokes equations* (4.5) *in Q^+ satisfies*

$$\begin{cases} Y^+(u,p;Q^+) < \bar{\varepsilon}_* & \textit{for } 1 < s < \frac{3}{2}, \\ Z^+(u,p;Q^+) < \bar{\varepsilon}_* & \textit{for } \frac{3}{2} \le s < 2, \end{cases} \tag{4.6}$$

then u is Hölder continuous on $\overline{Q}^+(\frac{1}{2}) = \overline{Q}^+(0,\frac{1}{2})$.

We give the proof of Theorem 4.1 assuming Proposition 4.4.

Proof of Theorem 4.1. Let $R = \frac{2}{3}R_*$. If R_* is small enough, it holds

$$U^+_{\frac{R_*}{2}}(x_0) \subset V(x_0, R) \subset U^+_{R_*}(x_0),$$

where $V(x_0, R) = \widehat{h}^{-1}(B^+(x_0, R))$. Set $\varepsilon_* = (\frac{2}{3})^{s'}\overline{\varepsilon}_*$. Then we have that

$$\begin{cases} Y^+(u,p; V(x_0,R) \times (t_0 - R^2, t_0)) < \overline{\varepsilon}_* & \text{if } 1 < s < \frac{3}{2}, \\ Z^+(u,p; V(x_0,R) \times (t_0 - R^2, t_0)) < \overline{\varepsilon}_* & \text{if } \frac{3}{2} \le s < 2, \end{cases}$$

By the transformation (4.3), we see that the functions $(u, \nabla p, h)$ are a boundary suitable weak solution of the perturbed Navier-Stokes equations (4.5) in $Q^+(z_0, R)$ satisfying

$$\begin{cases} Y^+(u,p; Q^+(z_0,R)) < \overline{\varepsilon}_* & \text{if } 1 < s < \frac{3}{2}, \\ Z^+(u,p; Q^+(z_0,R)) < \overline{\varepsilon}_* & \text{if } \frac{3}{2} \le s < 2. \end{cases}$$

Therefore, by the scaling transformation

$$\begin{aligned} &u_R(y,s) = Ru(x_0 + Ry, t_0 + R^2 s), \quad p_R(y,s) = R^2 p(x_0 + Ry, t_0 + R^2 s), \\ &h_R(y_1, y_2) = \frac{1}{R}h(Ry_1, Ry_2), \end{aligned} \tag{4.7}$$

the new functions (u_R, p_R, h_R) are a boundary suitable weak solution of the perturbed Navier-Stokes equations (4.5) in Q^+ satisfying

$$\begin{cases} Y^+(u_R, p_R; Q^+) < \varepsilon_* & \text{if } 1 < s < \frac{3}{2}, \\ Z^+(u_R, p_R; Q^+) < \varepsilon_* & \text{if } \frac{3}{2} \le s < 2 \end{cases}$$

and

$$\|h_R\|_{C^2(\overline{B'})} \le R\|h\|_{C^2(\overline{B'}(R))} \le KR.$$

By putting $R_* = \frac{K_*}{K}$, all the conditions of Proposition 4.4 are satisfied. Hence, we conclude that u_R is Hölder continuous on $\overline{Q}^+(\frac{1}{2})$. Taking (4.7) into consideration, we see that u is Hölder continuous on $\overline{V}(x_0, \frac{R}{2}) \times [t_0 - \frac{R^2}{4}, t_0]$. Since $U^+_{\frac{R_*}{4}}(x_0) \subset V(x_0, \frac{R}{2})$, it completes the proof of Theorem 4.1.

In order to prove Proposition 4.4, we use the several steps similar to the interior case. The first step is the following result.

Lemma 4.5 *For* $0 < \theta_1 < \frac{1}{2}$, *there exist positive constants* $\varepsilon_1 > 0$ *and* $C_1 > 0$ *such that if any boundary suitable weak solution* $(u, \nabla p, h)$ *of the perturbed Navier-Stokes equation* (4.5) *in* Q^+ *satisfies*

$$\begin{cases} Y^+(u,p; Q^+) < \varepsilon_1 & \text{if } 1 < s < \frac{3}{2}, \\ Z^+(u,p; Q^+) < \varepsilon_1 & \text{if } \frac{3}{2} \le s < 2, \end{cases}$$

then there holds

$$\begin{cases} Y^+(u,p; Q^+(\theta_1)) \le C_1 \theta_1^{1/s'} Y^+(u,p; Q^+) & \text{if } 1 < s < \frac{3}{2}, \\ Z^+(u,p; Q^+(\theta_1)) \le C_1 \theta_1^{1/s'} Z^+(u,p; Q^+) & \text{if } \frac{3}{2} \le s < 2. \end{cases}$$

As in Proposition 3.3, the proof is based on the blow-up argument with some modifications of [8]. With the same procedure as in [6], we can show the following general result.

Lemma 4.6 *If any boundary suitable weak solution* $(u, \nabla p, h)$ *of the perturbed Navier-Stokes equations* (4.5) *in* Q^+ *and* $z_0 = (x_0, t_0) \in \overline{B}^+ \times (-1, 0)$ *satisfy*

$$(B(x_0,R) \cap \partial B^+) \subset \{x \in \overline{B}^+; x_3 = 0\},\ 0 < R < R_0,\ t_0 - R^2 > -1,$$

$$\begin{cases} Y^+(u,p;\omega(z_0,R)) < \bar{\varepsilon}_1 & \textit{if}\ 1 < s < \frac{3}{2}, \\ Z^+(u,p;\omega(z_0,R)) < \bar{\varepsilon}_1 & \textit{if}\ \frac{3}{2} \le s < 2, \end{cases}$$

then there holds

$$\begin{cases} Y(u,p;\omega(z_0,\rho)) \le C_{58} \left(\frac{\rho}{R}\right)^{1/s'} & \textit{if}\ 1 < s < \frac{3}{2}, \\ Z(u,p;\omega(z_0,\rho)) \le C_{58} \left(\frac{\rho}{R}\right)^{1/s'} & \textit{if}\ \frac{3}{2} \le s < 2, \end{cases}$$

for all $\rho \in (0, R)$. *Here,* $\omega(z_0, R) := \big(B(x_0, R) \cap B^+\big) \times (t_0 - R^2, t_0)$.

We are now in a position to prove Proposition 4.4.
Proof of Proposition 4.4. It follows from Lemma 3.4 with $\beta = 0$ that if

$$\begin{cases} Y(u,p;Q) < \bar{\varepsilon}_0 & \text{if}\ 1 < s < \frac{3}{2}, \\ Z(u,p;Q) < \bar{\varepsilon}_0 & \text{if}\ \frac{3}{2} \le s < 2, \end{cases}$$

then there holds

$$\begin{cases} Y(u,p;Q(z_0,\rho)) \le \rho^{1/s'}\bar{\varepsilon}_0 & \text{if}\ 1 < s < \frac{3}{2}, \\ Z(u,p;Q(z_0,\rho)) \le \rho^{1/s'}\bar{\varepsilon}_0 & \text{if}\ \frac{3}{2} \le s < 2 \end{cases}$$

for all $z_0 \in Q^+(\frac{3}{4})$ and $0 < \rho < \frac{1}{4}$. By Lemma 4.6, the same assertion holds for all z_0 belonging to the flat part of the lateral boundary with $\bar{\varepsilon}_0$ replaced by $\bar{\varepsilon}_1$. Combining the interior and boundary estimates as in [6, Lemma 5.2], we obtain

$$\begin{cases} Y(u,p;Q(z_0,\rho)) \le C\rho^{1/s'} & \text{if}\ 1 < s < \frac{3}{2}, \\ Z(u,p;Q(z_0,\rho)) \le C\rho^{1/s'} & \text{if}\ \frac{3}{2} \le s < 2 \end{cases}$$

for all $z_0 \in Q^+(\frac{2}{3})$ and $0 < \rho < \frac{1}{16}$. Therefore, it follows from the Campanato embedding theorem that u is Hölder continuous on $\overline{Q}^+(\frac{1}{2})$ with exponent $1/s'$.

5 Proof of Main theorem

Step 1. (Interior regularity) We shall prove the following:

For $0 < \sigma_1 < T$, there exists $K_1 > 0$ such that $u(x,t)$ is Hölder continuous for $(x,t) \in \{(x,t) \in \Omega \times (\sigma_1, T); |x| \ge K_1, \operatorname{dist}(x, \partial\Omega) > \sqrt{\sigma_1}\}$.

Since $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$, we see that

$$\|u\|_{L^3(\Omega\times(0,T))} \le C\|u\|^{1/2}_{L^\infty(0,T;L^2(\Omega))}\|\nabla u\|^{1/2}_{L^2(0,T;L^2(\Omega))}.$$

By Remark 2.7, there exist $\nabla p_2^{(1)}, \nabla p_2^{(2)}$ such that

$$\nabla p_2^{(1)} \in L^{5/4}(0,T;L^2(\Omega)),\quad \nabla p_2^{(2)} \in L^{5/4}(0,T;L^{5/4}(\Omega)),$$
$$\nabla p_2 = \nabla p_2^{(1)} + \nabla p_2^{(2)}.$$

Therefore, for $\sigma_1 < t < T$ we can choose $K_1' > 0$ so large that

$$\begin{aligned}&\frac{1}{\sigma_1^2}\int_{t/2}^T\int_{|y|>K_1'}|u|^3 + \frac{1}{\sigma_1}\sup_{t/2<s<T}\int_{|y|>K_1'}|u|^2 + \sigma_1\int_{t/2}^T\int_{|y|>K_1'}|\nabla p_1|^2\\ &+\frac{1}{\sigma_1^{1/8}}\int_{t/2}^T\Big(\int_{|y|>K_1'}|\nabla p_2^{(1)}|^2\Big)^{5/8} + \frac{1}{\sigma_1^{5/4}}\int_{t/2}^T\int_{|y|>K_1'}|\nabla p_2^{(2)}|^2 < \varepsilon_\sharp,\end{aligned}\tag{5.1}$$

where $\varepsilon_\sharp$ is the constant as in Theorem 3.1. Hence, we obtain

$$\begin{aligned}&\frac{1}{\sigma_1^2}\int_{t_0-\sigma_1^2}^{t_0}\int_{B_{\sigma_1}(x_0)}|u|^3 + \frac{1}{\sigma_1^5}\int_{t_0-\sigma_1^2}^{t_0}\Big(\int_{B_{\sigma_1}(x_0)}|u|^{15/8}\Big)^{8/3}\\ &\qquad + \frac{1}{\sigma_1^{5/4}}\int_{t_0-\sigma_1^2}^{t_0}\int_{B_{\sigma_1}(x_0)}|\nabla p|^{5/4} < \varepsilon_\sharp\end{aligned}$$

for all $(x_0,t_0) \in \{x \in \Omega; |x| \ge K_1' + \sigma_1, \operatorname{dist}(x,\partial\Omega) > \sqrt{\sigma_1}\} \times (\sigma_1,T)$. It follows from Theorem 3.1 with $s = q = \frac{5}{4}$ that $u(x,t)$ is Hölder continuous for $(x,t) \in \{x \in \Omega; |x| \ge K_1' + \frac{\sigma_1}{2}, \operatorname{dist}(x,\partial\Omega) > \sqrt{\sigma_1}\} \times (\sigma_1,T)$. The assertion of the interior regularity is proved.

Step 2. (Boundary regularity) We shall prove the following:

For $0 < \sigma_2 < T$, there exists $K_2 > 0$ such that $u(x,t)$ is Hölder continuous for $(x,t) \in \{(x,t) \in \overline{\Omega} \times (\sigma_2,T); |x| \ge K_2, \operatorname{dist}(x,\partial\Omega) \le \frac{R_*}{8}\}$.

In the same way as in Step 1, choose $K_2' > 0$ so large that

$$\begin{aligned}&\frac{1}{R_*^2}\int_{R_*/2}^T\int_{|y|>K_2'}|u|^3 + \frac{1}{R_*}\sup_{R_*/2<s<T}\int_{|y|>K_2'}|u|^2 + R_*\int_{R_*/2}^T\int_{|y|>K_2'}|\nabla p_1|^2\\ &+\frac{1}{R_*^{1/8}}\int_{R_*/2}^T\Big(\int_{|y|>K_2'}|\nabla p_2^{(1)}|^2\Big)^{5/8} + \frac{1}{R_*^{5/4}}\int_{R_*/2}^T\int_{|y|>K_2'}|\nabla p_2^{(2)}|^{5/4} < \varepsilon_*,\end{aligned}\tag{5.2}$$

where ε_* and R_* are constant as in Theorem 4.1. Hence, we obtain

$$\begin{aligned}&\frac{1}{R_*^2}\int_{t_0-R_*^2}^{t_0}\int_{U^+_{R_*}(x_0)}|u|^3 + \frac{1}{R_*^5}\int_{t_0-R_*^2}^{t_0}\Big(\int_{U^+_{R_*}(x_0)}|u|^{15/8}\Big)^{8/3}\\ &\qquad + \frac{1}{R_*^{5/4}}\int_{t_0-R_*^2}^{t_0}\int_{U^+_{R_*}(x_0)}|\nabla p|^{5/4} < \varepsilon_*\end{aligned}$$

for all $(x_0, t_0) \in \{x \in \overline{\Omega}; |x| \geq K_2' + R_*\} \times (\sigma_2, T)$. It follows from Theorem 4.1 with $s = q = \frac{5}{4}$ that $u(x,t)$ is Hölder continuous on $(x,t) \in \{x \in \overline{\Omega}; |x| \geq K_2 + \frac{\sqrt{R_*}}{8}, \operatorname{dist}(x, \partial\Omega) \leq \frac{R_*}{8}\} \times (\sigma_2, T)$. The assertion of the boundary regularity is proved.

Step 3. As a direct consequence of Step 1 and Step 2, we can prove our main theorem. Indeed, it follows from Step 2 that $u(x,t)$ is regular for sufficiently large $|x|$ near the boundary. Moreover, $u(x,t)$ is smooth for such $|x|$ with $\operatorname{dist}(x, \partial\Omega) \geq \frac{R_*}{8}$ by Step 1. This completes the proof of our main theorem.

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