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An initial-boundary value problem for motion of incompressible inhomogeneous fluid-like bodies *

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Abstract

An initial-boudary value problem for the system of equations governing the flow of inhomogeneous incompressible fluid-like bodies is studied. This model equation arises from the study of incompressible flows of granular materials. Rewriting this problem by Lagrangian coordinates, we prove its solvability in anisotropic Sobolev-Slobodetskiĭ spaces.

1 Introduction

Here we are concerned with the motion of inhomogeneous incompressible fluid-like bodies. The body under consideration is a sort of granular materials including sand, powder and so on. Granular bodies respond in a fluid-like manner. Taking this character into account, we introduce a continuum model of motion of granular materials. The model studied in this paper is derived by Málek & Rajagopal [10].

The motion of inhomogeneous incompressible fluid-like bodies in a bounded domain $Q_T = \Omega\,(\subset \mathbb{R}^3) \times (0,T)$ is described by the system of equations for the velocity field $\mathbf{v} = (v_1, v_2, v_3)(x,t)$, the pressure $p = p(x,t)$ and the density $\varrho = \varrho(x,t)$:

$$\begin{cases} \dfrac{\mathrm{D}\varrho}{\mathrm{D}t} = 0, \qquad \nabla\cdot\mathbf{v} = 0 \qquad \text{in } Q_T, \\ \varrho\dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \nabla\cdot\mathbb{T} + \varrho\mathbf{b} \qquad \text{in } Q_T, \\ \quad \text{with } \mathbb{T} = -p\,\mathbb{I} + 2\nu(\varrho)\mathbb{D} - \beta_1\left(\nabla\varrho\otimes\nabla\varrho - \dfrac{1}{3}|\nabla\varrho|^2\,\mathbb{I}\right). \end{cases} \tag{1.1}$$

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*This study is a joint work with Atusi Tani (Keio University).

Here $\frac{\mathrm{D}}{\mathrm{D}t}$ is the Lagrangian derivative; $\mathbb{T}$ is the Cauchy stress tensor; $\mathbf{b} = (b_1, b_2, b_3)(x, t)$ is the external body forces; $\mathbb{D} = \frac{1}{2}(\nabla\mathbf{v} + [\nabla\mathbf{v}]^{\mathrm{T}})$ is the symmetric part of the velocity gradient; $\nu(\varrho) = \nu(\varrho(x,t))$ is the viscosity; β_1 is a positive constant; T is a positive finite number.

A thermodynamic framework that has been recently put into place to describe the dissipative response of materials is used to develop a model for the response of inhomogenous incompressible fluid-like bodies whose stored energy depends on the gradient of the density [14]. We also emphasize that dependence of the stress on the gradient of the density in this model is the consequence of the inhomogeneity of the body. And in fact, granular materials are naturally inhomogeneous, we shall therefore consider the inhomogeneous models.

Bodies under consideration in this model are incompressible. Naturally, granular materials are invariably compressible due to the interstitial spaces that exist between the grains. As the grain size becomes smaller, however, they behave as though they are incompressible due to the interlocking conditon of the grains. Such models are but relatively crude approximations of real bodies, and in this sense the spirit of the approximation is no different than that used to develop models for fluids. Here, we regard a material as incompressible when its compressibility is insignificant and more importantly, this compressibility has insignificant consequences concerning the response of the body.

The viscosity ν may be either a constant, or a function of the density ϱ, $\mathbb{D}$ specifically through $|\mathbb{D}|^2 (= \sum_{i,j=1}^{3} D_{ij}^2)$ or the pressure p. The form $\nu = \nu(p, \varrho, |\mathbb{D}|^2)$ is the most general case of the viscosity within this setting (see [9, 10, 11] for details). In this study we shall consider the special case $\nu = \nu(\varrho)$ below.

For the system mentioned above, we need to assign appropriate boundary conditions. One can consider an adherence condition or other boundary conditions such as "slip" conditions. In case of considering behaviour of granular materials, one should adopt boundary conditions which include the slip condition.

For example, Navier [12] derived a slip condition which can be duly generalized to the condition

$$\mathbf{v} \cdot \boldsymbol{\tau} = -K\mathbb{T}\mathbf{n} \cdot \boldsymbol{\tau}, \quad K \geq 0,$$

where $\boldsymbol{\tau}$ and $\mathbf{n}$ are the unit tangential and the unit outward normal vectors to the surface, respectively, and K is usually assumed to be a constant but it could, however, be assumed to be a function of the normal stresses and the shear rate, i.e.,

$$K = K(\mathbb{T}\mathbf{n} \cdot \mathbf{n}, |\mathbb{D}|^2).$$

A. Tani, S. Ito and N. Tanaka [21] studied the Navier-Stokes equations with the above boundary conditions in the case $K = K(x,t)$.

Another boundary condition is Stokes' slip as the "threshold-slip" condition that is sometimes used, especially when dealing with non-Newtonian fluids. This takes the form

$$\begin{cases} \text{if} \quad |\mathbb{T}\mathbf{n}\cdot\boldsymbol{\tau}| \le \alpha|\mathbb{T}\mathbf{n}\cdot\mathbf{n}| \quad \text{then} \quad \mathbf{v}\cdot\boldsymbol{\tau} = 0, \\ \text{if} \quad |\mathbb{T}\mathbf{n}\cdot\boldsymbol{\tau}| > \alpha|\mathbb{T}\mathbf{n}\cdot\mathbf{n}| \quad \text{then} \quad \mathbf{v}\cdot\boldsymbol{\tau} \ne 0 \quad \text{and} \quad \mathbb{T}\mathbf{n}\cdot\boldsymbol{\tau} = -\gamma\dfrac{\mathbf{v}\cdot\mathbf{n}}{|\mathbf{v}\cdot\mathbf{n}|}, \end{cases}$$

where $\gamma = \gamma(\mathbf{v}\cdot\boldsymbol{\tau}, \mathbb{T}\mathbf{n}\cdot\mathbf{n})$. The above condition implies that the fluid will not slip until the ratio of the magnitude of the shear stress and that of the normal stress exceeds a critical value. When it does exceed that value, it slips with the velocity depending on both the shear and normal stresses. It may happen that γ depends on $|\mathbb{D}|^2$ (see [9] for details).

In this study, instead of the slip boundary conditions mentioned above, we shall impose, just for the sake of simplicity, that

$$\mathbf{v} = 0 \quad \text{on } G_T (= \Gamma \times [0,T]), \tag{1.2}$$

where Γ is the boundary of Ω.

The initial conditions are also assigned

$$\varrho(x,0) = \varrho_0(x) \qquad \text{and} \qquad \mathbf{v}(x,0) = \mathbf{v}_0(x) \quad \text{in } \Omega, \tag{1.3}$$

where ϱ_0 and $\mathbf{v}_0$ are given functions defined in Ω.

We shall consider the problem (1.1) with (1.2) and (1.3) in the following section.

2 Mathematical issues and Main results

2.1 Setting up the problem

In this section we are concerned with the initial-boundary value problem describing the motion discussed above. The problem (1.1)-(1.3) can be rewritten in Lagrangian coordinates y. Let $\mathbf{u}(y,t)$ and $q(y,t)$ be the velocity field and pressure expressed as functions of the Lagrangian coordinates. The relationship between Lagrangian and Eulerian coordinates are given by

$$x = y + \int_0^t \mathbf{u}(y,\tau)\, d\tau \equiv X_{\mathbf{u}}(y,t), \quad \mathbf{u}(y,t) = \mathbf{v}(X_{\mathbf{u}}(y,t),t). \tag{2.1}$$

From $(1.1)_1$ it is easy to derive

$$\frac{\partial \hat{\varrho}}{\partial t}(y,t) = 0 \tag{2.2}$$

for $\hat{\varrho}(y,t) := \varrho(X_{\mathbf{u}}(y,t),t)$. Then, (2.2) has a solution

$$\hat{\varrho}(y,t) = \hat{\varrho}(y,0) = \varrho(X_{\mathbf{u}}(y,0),0) = \varrho(y,0) = \varrho_0(y), \tag{2.3}$$

i.e., one can find that the density of a fixed particle does not change, while the density can change from point to point in the initial state of the body.

The Jacobian matrix of the transformation $X_{\mathbf{u}}$ is denoted by $A = (a_{ij}(y,t))_{i,j=1,2,3}$ with the elements $a_{ij}(y,t) = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial y_j}(y,\tau)d\tau$ and the Jacobian determinant $J_{\mathbf{u}}(y,t) = \det A(y,t)$ is the solution of the Cauchy problem

$$\begin{aligned}\frac{\partial J_{\mathbf{u}}(y,t)}{\partial t} &= \sum_{i,j=1}^{3} \frac{\partial a_{ij}}{\partial t} A_{ij} = \sum_{i,j=1}^{3} A_{ij} \sum_{k=1}^{3} \frac{\partial v_i}{\partial x_k}(X_{\mathbf{u}}(y,t),t) a_{kj} \\ &= J_{\mathbf{u}}(y,t) \nabla \cdot \mathbf{v}(x,t)\big|_{x=X_{\mathbf{u}}(y,t)}, \\ J_{\mathbf{u}}(y,0) &= 1.\end{aligned}$$

According to $(1.1)_2$, we have $J_{\mathbf{u}}(y,t) \equiv 1$.

In general,

$$\nabla_y\{F(X_{\mathbf{u}}(y,t),t)\} = A^{\mathrm{T}} \nabla_x F(x,t),$$

so that

$$\nabla_x F(x,t) = \nabla_{\mathbf{u}} \hat{F}(y,t),$$
$$\nabla_{\mathbf{u}} := A^{-\mathrm{T}} \nabla_y, \quad \hat{F}(y,t) := F(X_{\mathbf{u}}(y,t),t),$$

where $A^{-\mathrm{T}}$ is the inverse matrix of A^{T}. And note that $A^{-1} = J_{\mathbf{u}}^{-1} \mathscr{A} = \mathscr{A}$; $\mathscr{A}$ is the adjugate matrix of A.

In the same way as (2.3), we have $\mathbf{u}(y,0) = \mathbf{v}_0(y)$, thus problem (1.1)-(1.3) becomes

$$\begin{cases} \varrho_0 \mathbf{u}_t = \nabla_{\mathbf{u}} \cdot \widehat{\mathbb{T}} + \varrho_0 \widehat{\mathbf{b}}^{(u)}, \quad \nabla_{\mathbf{u}} \cdot \mathbf{u} = 0 \quad \text{in } Q_T, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \quad \mathbf{u}|_{\Gamma} = 0 \quad \text{on } G_T. \end{cases} \tag{2.4}$$

Here

$$\widehat{\mathbb{T}} = -q\mathbf{I} + 2\nu(\varrho_0)\widehat{\mathbb{D}}^{(u)} - \beta_1\Big(\nabla_{\mathbf{u}}\varrho_0 \otimes \nabla_{\mathbf{u}}\varrho_0 - \tfrac{1}{3}|\nabla_{\mathbf{u}}\varrho_0|^2\mathbf{I}\Big),$$
$$\widehat{\mathbb{D}}^{(u)} = \tfrac{1}{2}(\nabla_{\mathbf{u}}\mathbf{u} + (\nabla_{\mathbf{u}}\mathbf{u})^T), \quad \widehat{\mathbf{b}}^{(u)}(y,t) = \mathbf{b}(X_{\mathbf{u}}(y,t),t).$$

The aim of this paper is to prove a theorem on local in time solvability of problem (2.4) in Sobolev-Slobodetskiĭ spaces.

Furthermore, we consider the following linear problem

$$\begin{cases} \varrho_0(y)\,\mathbf{u}_t = -\nabla q + \nu(y)\nabla^2\mathbf{u} + \varrho_0(y)\mathbf{f}, \quad \nabla\cdot\mathbf{u} = g \quad \text{in } Q_T, \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \quad \mathbf{u}|_\Gamma = \mathbf{d} \quad \text{on } \Gamma_T, \end{cases} \tag{2.5}$$

where $\nabla^2 = \nabla\cdot\nabla$, $\nu(y)$ a given positive function defined in Ω, f and g given functions defined in Q_T and $\mathbf{d}$ a given function on Γ_T.

2.2 Function spaces

In this subsection we introduce the function spaces used in this paper. Let $\mathscr{G}$ be a domain in $\mathbb{R}^n$ and r is a non-negative number. By $W_2^r(\mathscr{G})$ we denote the space of functions equipped with the standard norm

$$\|u\|^2_{W_2^r(\mathscr{G})} = \sum_{|\alpha|<r} \|D^\alpha u\|^2_{L_2(\mathscr{G})} + \|u\|^2_{\dot W_2^r(\mathscr{G})}, \tag{2.6}$$

where

$$\|u\|^2_{\dot W_2^r(\mathscr{G})} = \sum_{|\alpha|=r} \|D^\alpha u\|^2_{L_2(\mathscr{G})}$$

if r is an integer, and

$$\|u\|^2_{\dot W_2^r(\mathscr{G})} = \sum_{|\alpha|=[r]} \int_{\mathscr{G}}\int_{\mathscr{G}} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\{r\}}}\,dxdy$$

if r is not an integer. Here $[r]$ is the integral part and $\{r\}$ the fractional part of r, respectively. $\|f\|_{L_2(\mathscr{G})} = (\int_{\mathscr{G}} |f(x)|^2 dx)^{\frac{1}{2}}$ is the norm in $L_2(\mathscr{G})$, $D^\alpha f = \partial^{|\alpha|} f/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}$ is the generalized derivative of the function f in the distribution sense of order $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ being a multi-index.

The anisotropic space $W_2^{r,r/2}(\mathfrak{G}_T)$ in the cylindrical domain $\mathfrak{G}_T = \mathscr{G}\times(0,T)$ is defined by $L_2(0,T; W_2^r(\mathscr{G})) \cap L_2(\mathscr{G}; W_2^{r/2}(0,T))$, whose norm is introduced by the formula

$$\|u\|^2_{W_2^{r,r/2}(\mathfrak{G}_T)} = \int_0^T \|u\|^2_{W_2^r(\mathscr{G})} dt + \int_{\mathscr{G}} \|u\|^2_{W_2^{r/2}(0,T)} dx$$
$$\equiv \|u\|^2_{W_2^{r,0}(\mathfrak{G}_T)} + \|u\|^2_{W_2^{0,r/2}(\mathfrak{G}_T)},$$

where $W_2^{r,0}(\mathfrak{G}_T) = L_2(0,T; W_2^r(\mathscr{G}))$ and $W_2^{0,r/2}(\mathfrak{G}_T) = L_2(\mathscr{G}; W_2^{r/2}(0,T))$. Similarly, the norm in $W_2^{r/2}(0,T)$ (for nonintegral $r/2$) is defined by

$$\|u\|^2_{W_2^{r/2}(0,T)} = \sum_{j=0}^{[r/2]} \left\| \frac{d^j u}{dt^j} \right\|^2_{L_2(0,T)}$$

$$+\int_0^T dt\int_0^t \left|\frac{d^{[r/2]}u(t)}{dt^{[r/2]}}-\frac{d^{[r/2]}u(t-\tau)}{dt^{[r/2]}}\right|^2\frac{d\tau}{\tau^{1+2\{r/2\}}}.$$

Other equivalent norms of these spaces are possible. For $l\in(0,1)$ we set

$$\|f\|_{\mathfrak{O}_T}^{(l,l/2)}=\left\{\|f\|^2_{W_2^{l,l/2}(\mathfrak{O}_T)}+\frac{1}{T^l}\|f\|^2_{L_2(\mathfrak{O}_T)}\right\}^{1/2},$$

$$\|f\|_{\mathfrak{O}_T}^{(2+l,1+l/2)}=\left\{\|f\|^2_{W_2^{2+l,1+l/2}(\mathfrak{O}_T)}+\left(\|f_t\|_{\mathfrak{O}_T}^{(l,l/2)}\right)^2\right.$$

$$\left.+\sum_{|\alpha|=2}\left(\|D_x^\alpha f\|_{\mathfrak{O}_T}^{(l,l/2)}\right)^2+\sup_{t\in(0,T)}\|f\|^2_{W_2^{1+l}(\mathscr{G})}\right\}^{1/2}.$$

For any finite $T>0$ these norms are equivalent to the norms in the spaces $W_2^{l,l/2}(\mathfrak{O}_T)$ and $W_2^{2+l,1+l/2}(\mathfrak{O}_T)$, respectively. Let also

$$\|f\|_{\mathfrak{O}_T}^{(0,l/2)}=\left\{\|f\|^2_{W_2^{0,l/2}(\mathfrak{O}_T)}+\frac{1}{T^l}\|f\|^2_{L_2(\mathfrak{O}_T)}\right\}^{1/2}.$$

If $\mathscr{G}$ is a smooth manifold (in this paper the boundary of a domain in $\mathbb{R}^3$ may play this role), then the norm in $W_2^r(\mathscr{G})$ is defined by means of local charts, i.e., a partition of $\mathscr{G}$ into subsets each of which is mapped into a domain of Euclidean space where the norms of W_2^r are defined by formula (2.6). After this the spaces $W_2^{r,r/2}(\mathfrak{O}_T)$ on $\mathfrak{O}_T(=\mathscr{G}\times(0,T))$ are introduced as indicated above.

The same symbols $W_2^r(\mathscr{G})$, $W_2^{r,r/2}(\mathfrak{O}_T)$ are used for the spaces of vector fields $\mathbf{f}=(f_1,f_2,\dots,f_n)$ etc. Their norms are introduced in standard form; for example,

$$\|\mathbf{f}\|^2_{W_2^r(\mathscr{G})}=\sum_{i=1}^n\|f_i\|^2_{W_2^r(\mathscr{G})}.$$

We introduce several propositions that concern the well-known inequalities of norms in Sobolev-Slobodetskiĭ spaces (see Lemma 4.1 of [18]).

Lemma 2.1 For any $f\in W_2^l(\Omega)$, $g,\ h\in W_2^{1+l}(\Omega)$, $\Omega\subset\mathbb{R}^3$, $l\in(1/2,1)$

$$\|fg\|_{W_2^l(\mathscr{G})}\le c\|f\|_{W_2^l(\mathscr{G})}\|g\|_{W_2^{1+l}(\mathscr{G})},\tag{2.7}$$

$$\|gh\|_{W_2^{1+l}(\mathscr{G})}\le c\|g\|_{W_2^{1+l}(\mathscr{G})}\|h\|_{W_2^{1+l}(\mathscr{G})}.\tag{2.8}$$

These estimates also hold in the case $n=2$, when the index l may be replaced by $l-1/2$.

For functions f, g depending also on $t \in (0,T)$ we obtain the inequalities

$$\|fg\|_{W_2^{l,0}(\mathfrak{G}_T)} \le c \sup_{t\le T} \|g\|_{W_2^{1+l}(\mathscr{G})} \|f\|_{W_2^{l,0}(\mathfrak{G}_T)}, \tag{2.9}$$

$$\|fg\|_{W_2^{l,0}(\mathfrak{G}_T)} \le c \sup_{t\le T} \|f\|_{W_2^{l}(\mathscr{G})} \|g\|_{W_2^{1+l,0}(\mathfrak{G}_T)}, \tag{2.10}$$

$$\|gh\|_{W_2^{1+l,0}(\mathfrak{G}_T)} \le c \sup_{t\le T} \|g\|_{W_2^{1+l}(\mathscr{G})} \|h\|_{W_2^{1+l,0}(\mathfrak{G}_T)}. \tag{2.11}$$

And also for $f \in W_2^{l,l/2}(\mathfrak{G}_T)$ and $g \in W_2^{1+l}(\mathscr{G})$

$$\|fg\|_{\mathfrak{G}_T}^{(l,l/2)} \le c\|f\|_{\mathfrak{G}_T}^{(l,l/2)} \|g\|_{W_2^{1+l}(\mathscr{G})} \tag{2.12}$$

holds.

2.3 Main Results

Let us now describe the results in this paper. First of all, we consider the problem (2.5) in the spaces $W_2^{2+l,1+l/2}(Q_T)$ and $W_2^{l,l/2}(Q_T)$.

Theorem 2.1 *Let Ω be a bounded domain, $\Gamma \in W^{3/2+l}$, $l \in (1/2,1)$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(y) \ge R_0 > 0$, $\nu \in W_2^{2+l}(\Omega)$ and $\nu > 0$. For arbitrary $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$, $\mathbf{f} \in W_2^{l,l/2}(Q_T)$, $g \in W_2^{1+l,1/2+l/2}(Q_T)$, $g = \nabla\cdot\mathbf{G}$, $\mathbf{G} \in W_2^{0,1+l/2}(Q_T)$ and $\mathbf{d} \in W_2^{3/2+l,3/4+l/2}(\Gamma_T)$ satisfying the compatibility conditions*

$$\nabla\cdot\mathbf{v}_0 = g(\,\cdot\,,0) \text{ in } \Omega, \quad \mathbf{v}_0 = \mathbf{d}(\cdot,0) \text{ on } \Gamma, \quad \int_\Gamma \mathbf{G}\cdot\mathbf{n}\,dS = \int_\Gamma \mathbf{d}\cdot\mathbf{n}\,dS,$$

the problem (2.5) *has a unique solution $(\mathbf{u},\nabla q)$ in $W_2^{2+l,1+l/2}(Q_T)\times W_2^{l,l/2}(Q_T)$ and*

$$\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla q\|_{Q_T}^{(l,l/2)} \le c(T)\Big(\|\mathbf{f}\|_{Q_T}^{(l,l/2)} + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}$$
$$+\|g\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\mathbf{G}_t\|_{Q_T}^{(0,l/2)} + \|\mathbf{d}\|_{W_2^{3/2+l,3/4+l/2}(\Gamma_T)}\Big), \tag{2.13}$$

where $c(T)$ is a non-decreasing function of T.

Theorem 2.1 can be proved by the same procedure used in [20, 21], thus we leave out the proof in this paper.

Finally, we consider the problem (2.4), and the following theorem on time-local solvability is proved in § 4.

Theorem 2.2 *Let Ω be a bounded domain, $\Gamma \in W^{3/2+l}$, $l \in (1/2,1)$, $\varrho_0 \in W_2^{2+l}(\Omega)$, $\varrho_0(y) \ge R_0 > 0$, $\nu \in C^3(\overline{\mathbb{R}}_+)$, $\nu > 0$, and assume that $\mathbf{b}$ has continuous derivatives of order one and two, and that $\mathbf{b}$, $\mathbf{b}_{x_k}$ satisfy the Lipschitz condition in x and the*

Hölder condition with the exponent $\beta \geq 1/2$ in t, and assume that $\mathbf{v}_0 \in W_2^{1+l}(\Omega)$ satisfying the compatibility conditions

$$\nabla \cdot \mathbf{v}_0 = 0 \text{ in } \Omega, \quad \mathbf{v}_0 = 0 \text{ on } \Gamma.$$

Then the problem (2.4) has a unique solution $(\mathbf{u}, \nabla q) \in W_2^{2+l,1+l/2}(Q_{T'}) \times W_2^{l,l/2}(Q_{T'})$ on a finite interval $(0, T')$ whose magnitude T' depends on the data, i.e., on the norms of $\mathbf{b}$ and ϱ_0 (see the condition (4.7) below).

3 Auxiliary estimates

Before proving Theorem 2.2, we begin with auxiliary propositions.

We assume below that $\mathbf{u} \in W_2^{2+l,1+l/2}(Q_T)$ and

$$T^{1/2}\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} \leq \delta \tag{3.1}$$

is satisfied with sufficiently small $\delta > 0$.

The problem (2.4) is rewritten in the form

$$\begin{cases} \varrho_0\mathbf{u}_t - \nu(\varrho_0)\nabla^2\mathbf{u} + \nabla q = \mathbf{l}_1^{(\mathbf{u})}(\mathbf{u}, q) + 2\nu'(\varrho_0)\widehat{\mathbb{D}}^{(u)}\nabla_{\mathbf{u}}\varrho_0 \\ \qquad\qquad -\dfrac{\beta_1}{3}(\nabla_{\mathbf{u}}^{(j)}\nabla_{\mathbf{u}}^{(i)}\varrho_0)\nabla_{\mathbf{u}}\varrho_0 - \beta_1\nabla_{\mathbf{u}}^2\varrho_0\nabla_{\mathbf{u}}\varrho_0 + \varrho_0\widehat{\mathbf{b}}^{(u)}, \\ \nabla \cdot \mathbf{u} = \mathbf{l}_2^{(\mathbf{u})}(\mathbf{u}), \quad \mathbf{u}|_{t=0} = \mathbf{v}_0, \quad \mathbf{u}|_\Gamma = 0, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \mathbf{l}_1^{(u)}(\mathbf{w}, s) &= \nu(\varrho_0)(\nabla_{\mathbf{u}}^2 - \nabla^2)\mathbf{w} - (\nabla_{\mathbf{u}} - \nabla)s, \\ \mathbf{l}_2^{(u)}(\mathbf{w}) &= (\nabla - \nabla_{\mathbf{u}}) \cdot \mathbf{w} = \nabla \cdot \mathscr{L}^{(\mathbf{u})}(\mathbf{w}). \end{aligned} \tag{3.3}$$

Hereafter we estimate the right-hand side of (3.2), which is neccesary to prove the solvability of the problem (2.4). Let us introduce the following notation:

$$a_{ij} = \delta_{ij} + b_{ij}, \quad b_{ij} = \int_0^t \frac{\partial u_i}{\partial y_j} d\tau, \quad A_{ij} = \delta_{ij} + B_{ij},$$

where $\mathscr{A} = (A_{ij})$ (see p. 4). Since

$$A_{ii} = a_{jj}a_{kk} - a_{jk}a_{kj}, \quad A_{ij} = a_{ki}a_{jk} - a_{ji}a_{kk},$$

where $i \neq j$, $j \neq k$, $k \neq i$, it follows that

$$B_{ii} = b_{jj} + b_{kk} + b_{jj}b_{kk} - b_{jk}b_{kj}, \quad B_{ij} = -b_{ji} + b_{ki}b_{jk} - b_{ji}b_{kk}. \tag{3.4}$$

We denote by a'_{ij}, b'_{ij}, A'_{ij}, B'_{ij} the same functions corresponding to another vector field $\mathbf{u}'(y,t)$, and set $\tilde{b}_{ij} = b_{ij} - b'_{ij}$, $\widetilde{B}_{ij} = B_{ij} - B'_{ij}$, etc. We have

$$\begin{aligned} \widetilde{B}_{ii} &= \tilde{b}_{jj}(1+b_{kk}) + \tilde{b}_{kk}(1+b'_{jj}) - b_{kj}\tilde{b}_{jk} - b'_{jk}\tilde{b}_{kj}, \\ \widetilde{B}_{ij} &= -\tilde{b}_{ji}(1+b_{kk}) - \tilde{b}_{kk}b'_{ji} + \tilde{b}_{jk}b_{ki} + b'_{jk}\tilde{b}_{ki}. \end{aligned} \tag{3.5}$$

Finally, set that

$$D\mathbf{u} = \left\{\frac{\partial u_i}{\partial y_j}\right\}_{i,j=1,2,3}, \qquad D^2\mathbf{u} = \left\{\frac{\partial^2 u_i}{\partial y_j \partial y_k}\right\}_{i,j,k=1,2,3},$$

$$|D\mathbf{u}|_\Omega = \max_{i,j}\sup_{y\in\Omega}\left|\frac{\partial u_i}{\partial y_j}\right|, \qquad |D^2\mathbf{u}|_\Omega = \max_{i,j,k} sup_{y\in\Omega}\left|\frac{\partial^2 u_i}{\partial y_j \partial y_k}\right|,$$

$$\|D\mathbf{u}\|_{W_2^r(\Omega)} = \left(\sum_{j=1}^{3}\left\|\frac{\partial \mathbf{u}}{\partial y_j}\right\|_{W_2^r(\Omega)}\right)^{1/2},$$

etc.

We proceed to estimates of the functions (3.4) and (3.5). All lemmata stated below were proved mainly in [19].

Lemma 3.1 If $\mathbf{u}$, $\mathbf{u}' \in W_2^{2+l,1+l/2}(Q_T)$, then

$$|\widetilde{B}_{ij}(y,t)| \le 2\int_0^t |D(\mathbf{u}-\mathbf{u}')|d\tau \left(1 + \int_0^t |D\mathbf{u}|_\Omega d\tau + \int_0^t |D\mathbf{u}'|_\Omega d\tau\right), \tag{3.6}$$

$$\begin{aligned} \|\widetilde{B}_{ij}(\cdot,t)\|_{W_2^{1+l}(\Omega)} &\le c\int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{1+l}(\Omega)} d\tau \\ &\quad \times \left(1 + \int_0^t \|D\mathbf{u}\|_{W_2^{1+l}(\Omega)} d\tau \int_0^t \|D\mathbf{u}'\|_{W_2^{1+l}(\Omega)} d\tau\right), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\|\widetilde{B}_{ij}(\cdot,t) - \widetilde{B}_{ij}(\cdot,t-\tau)\|_{L_q(\Omega)} \\ &\le 2\int_{t-\tau}^t \|D(\mathbf{u}-\mathbf{u}')\|_{L_q(\Omega)} d\tau \left(1 + \int_0^t |D\mathbf{u}|_\Omega d\tau + \int_0^t |D\mathbf{u}'|_\Omega d\tau\right) \\ &+2\int_0^t |D(\mathbf{u}-\mathbf{u}')|_\Omega d\tau \int_{t-\tau}^t (\|D\mathbf{u}\|_{L_q(\Omega)} + \|D\mathbf{u}'\|_{L_q(\Omega)}) d\tau, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\|\nabla\widetilde{B}_{ij}(\cdot,t) - \nabla\widetilde{B}_{ij}(\cdot,t-\tau)\|_{L_2(\Omega)} \\ &\le 2\int_{t-\tau}^t \|D^2(\mathbf{u}-\mathbf{u}')\|_{L_2(\Omega)} d\tau' \left(1 + \int_0^t |D\mathbf{u}|_\Omega d\tau' + \int_0^t |D\mathbf{u}'|_\Omega d\tau'\right) \end{aligned}$$

$$+2\int_0^t \|D^2(\mathbf{u}-\mathbf{u}')\|_{L_3(\Omega)}d\tau' \int_{t-\tau}^t (\|D\mathbf{u}\|_{L_6(\Omega)} + \|D\mathbf{u}'\|_{L_6(\Omega)})d\tau''$$

$$+2\int_{t-\tau}^t \|D(\mathbf{u}-\mathbf{u}')\|_{L_6(\Omega)}d\tau' \int_0^t (\|D^2\mathbf{u}\|_{L_3(\Omega)} + \|D^2\mathbf{u}'\|_{L_3(\Omega)})d\tau''$$

$$+2\int_0^t |D(\mathbf{u}-\mathbf{u}')|_\Omega d\tau' \int_{t-\tau}^t (\|D^2\mathbf{u}\|_{L_2(\Omega)} + \|D^2\mathbf{u}'\|_{L_2(\Omega)})d\tau'', \tag{3.9}$$

where $\tau \in (0,t)$. Such estimates (with $\mathbf{u}' = 0$ on the right hand side) also hold for the functions B_{ij}.

Inequalities (3.6)–(3.9) can easily be obtained directly from formulae (3.5). In the proof of (3.9) we used the Hölder inequality

$$\|fg\|_{L_2(\Omega)} \leq \|f\|_{L_3(\Omega)}\|g\|_{L_6(\Omega)}.$$

We note that

$$\int_0^t \|D\mathbf{u}\|_{W_2^{1+l}(\Omega)}d\tau \leq \sqrt{t}\ \|\mathbf{u}\|_{W_2^{2+l,0}(Q_T)} \leq \delta, \tag{3.10}$$

$$\int_0^t \|D\mathbf{u}'\|_{W_2^{1+l}(\Omega)}d\tau \leq \sqrt{t}\,\|\mathbf{u}'\|_{W_2^{2+l,0}(Q_T)} \leq \delta, \tag{3.11}$$

$$\int_0^t \|D\mathbf{u}\|_{W_2^1(\Omega)}\frac{d\tau}{(t-\tau)^{1/2}} \leq \frac{t^{1/2-l/2}}{\sqrt{1-l}}\left(\int_0^t \|D\mathbf{u}\|^2_{W_2^1(\Omega)}d\tau\right)^{1/2}$$

$$\leq \frac{T^{1/2}}{\sqrt{1-l}}\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} \leq \frac{\delta}{\sqrt{1-l}}, \tag{3.12}$$

hold.

Lemma 3.2 If $\mathbf{u}, \mathbf{u}' \in W_2^{2+l,1+l/2}(Q_T)$ satisfy condition (3.1), then for $t \leq T$

$$\|\widetilde{B}_{ij}\|_{W_2^{1+l}(\Omega)} \leq c\int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{1+l}(\Omega)}d\tau, \tag{3.13}$$

$$\left(\int_0^t \|\widetilde{B}_{ij}(\cdot,t) - \widetilde{B}_{ij}(\cdot,t-\tau)\|^2_{W_2^1(\Omega)}\frac{d\tau}{\tau^{1+l}}\right)^{1/2}$$

$$\leq c\left(\int_0^t \|D(\mathbf{u}-\mathbf{u}')\|_{W_2^{1+l}(\Omega)}d\tau + \int_0^t \frac{\|D(\mathbf{u}-\mathbf{u}')\|_{W_2^1(\Omega)}}{(t-\tau)^{l/2}}d\tau\right). \tag{3.14}$$

Such inequalities (with $\mathbf{u}' = 0$ on the right side) hold also for B_{ij}.

To derive (3.14) the fact that $W_2^{1+l}(\Omega)$ is embedded in $C(\bar{\Omega})$ (and also in $L_6(\Omega)$) and $W_2^l(\Omega)$ is embedded in $L_3(\Omega)$ is used.

Lemma 3.3 If $\mathbf{u}$, $\mathbf{u}' \in W_2^{2+l,1+l/2}(Q_T)$ satisfy condition (3.1), then for any $f \in W_2^{l,l/2}(Q_T)$ and $h \in W_2^{1+l,1/2+l/2}(Q_T)$

$$\left\|\widetilde{B}_{ij} f\right\|_{Q_T}^{(l,l/2)} \le c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)} \|f\|_{Q_T}^{(l,l/2)}, \tag{3.15}$$

$$\|\widetilde{B}_{ij} h\|_{W_2^{1+l,1/2+l/2}(Q_T)} \le c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}$$
$$\times(\|h\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\nabla h\|_{Q_T}^{(0,l/2)} + \|h\|_{Q_T}^{(0,l/2)}). \tag{3.16}$$

Setting $\mathbf{u}' = 0$ in (3.15) and (3.16) and noting (3.10) and (3.12), we arrive at the following proposition.

Lemma 3.4 If $\mathbf{u}$ satisfies (3.1), then

$$\|B_{ij} f\|_{Q_T}^{(l,l/2)} \le c\delta\, \|f\|_{Q_T}^{(l,l/2)}, \tag{3.17}$$

$$\|B_{ij} h\|_{W_2^{1+l,1/2+l/2}(Q_T)} \le c\delta(\|h\|_{W_2^{1+l,1/2+l/2}(Q_T)} + \|\nabla h\|_{Q_T}^{(0,l/2)} + \|h\|_{Q_T}^{(0,l/2)}). \tag{3.18}$$

Lemma 3.5 Let $\mathbf{u} \in W_2^{2+l,1+l/2}(Q_T)$, $T_0 > 0$, then for any $0 < T \le T_0$

$$\|D\mathbf{u}\|_{Q_T}^{(l,l/2)} \le c(T_0)\left(T^{1/2}\|\mathbf{u}\|_{Q_T}^{(2+l,1+l/2)} + T^{1/2-l/2}\|\mathbf{u}(\cdot,0)\|_{W_2^l(\Omega)}\right). \tag{3.19}$$

(3.19) is derived from the interpolation inequality

$$\|Df\|_{L_2(\Omega)} \le c(\varepsilon\|D^2 f\|_{L_2(\Omega)} + \varepsilon^{-1}\|f\|_{L_2(\Omega)}).$$

We proceed to estimates of $\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w},s) - \mathbf{l}_1^{(\mathbf{u}')}(\mathbf{w},s)$, $\mathbf{l}_2^{(\mathbf{u})}(\mathbf{w}) - \mathbf{l}_2^{(\mathbf{u}')}(\mathbf{w})$ and $\mathscr{L}^{(\mathbf{u})}(\mathbf{w}) - \mathscr{L}^{(\mathbf{u}')}(\mathbf{w})$, where $\mathbf{l}_1^{(\mathbf{u})}$, $\mathbf{l}_1^{(\mathbf{u}')}$, etc. are determined by formulae (3.3) on the basis of the vector fields $\mathbf{u}$ and $\mathbf{u}'$.

From (2.12) we have

$$\|\nu(\varrho_0) f\|_{Q_T}^{(l,l/2)} \le c\|\nu(\varrho_0)\|_{W_2^{1+l}(\Omega)} \|f\|_{Q_T}^{(l,l/2)} \le c(\varrho_0)\, \|f\|_{Q_T}^{(l,l/2)}, \tag{3.20}$$

where

$$c(\varrho_0) = c\left\{\sup_{\varrho}|\nu(\varrho)||\Omega|^{\frac{1}{2}} + \left(\sup_{\varrho}|\nu'(\varrho)| + \|\nabla\varrho_0\|_{W_2^l(\Omega)}\right)\|\nabla\varrho_0\|_{W_2^l(\Omega)}\right\}.$$

Then we obtain the following estimates:

Lemma 3.6 Let $\mathbf{u}$ and $\mathbf{u}'$ satisfy condition (3.1). For arbitrary $\mathbf{w} \in W_2^{2+l,1+l/2}(Q_T)$, $\nabla s \in W_2^{l,l/2}(Q_T)$ it holds

$$\left\|\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w},s) - \mathbf{l}_1^{(\mathbf{u}')}(\mathbf{w},s)\right\|_{Q_T}^{(l,l/2)}$$

$$\leq c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}(\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla s\|_{Q_T}^{(l,l/2)}), \tag{3.21}$$

$$\|\mathbf{l}_2^{(\mathbf{u})}(\mathbf{w}) - \mathbf{l}_2^{(\mathbf{u}')}(\mathbf{w})\|_{W_2^{1+l,1/2+l/2}(Q_T)} \leq c\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}, \tag{3.22}$$

$$\left\|\frac{\partial}{\partial t}(\mathscr{L}^{(\mathbf{u})}(\mathbf{w}) - \mathscr{L}^{(\mathbf{u}')}(\mathbf{w}))\right\|_{Q_T}^{(0,l/2)} \leq c\Big(\sqrt{T}\|\mathbf{u}-\mathbf{u}'\|_{Q_T}^{(2+l,1+l/2)}$$

$$+T^{1/2-l/2}\|\mathbf{u}(\cdot,0)-\mathbf{u}'(\cdot,0)\|_{W_2^l(\Omega)}\Big)\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}. \tag{3.23}$$

If $\mathbf{w}|_{t=0}= 0$, then (3.23) is valid also without the second term in the parenthesis of the right hand side.

Setting $\mathbf{u}' = 0$ in (3.21)–(3.23), we obtain that

Lemma 3.7 If $\mathbf{u}$ satisfies condition (3.1), then

$$\left\|\mathbf{l}_1^{(\mathbf{u})}(\mathbf{w},s)\right\|_{Q_T}^{(l,l/2)} \leq c\delta\left(\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)} + \|\nabla s\|_{Q_T}^{(l,l/2)}\right), \tag{3.24}$$

$$\|\mathbf{l}_2^{(\mathbf{u})}(\mathbf{w})\|_{W_2^{1+l,1/2+l/2}(Q_T)} \leq c\delta\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}, \tag{3.25}$$

$$\left\|\frac{\partial}{\partial t}\mathscr{L}^{(\mathbf{u})}(\mathbf{w})\right\|_{Q_T}^{(0,l/2)} \leq c\Big(\delta + T^{1/2-l/2}\|\mathbf{u}(\cdot,0)\|_{W_2^l(\Omega)}\Big)\|\mathbf{w}\|_{Q_T}^{(2+l,1+l/2)}. \tag{3.26}$$

In the case $\mathbf{w}|_{t=0}= 0$ the second term in the parenthesis of the right hand side of (3.26) can be dropped.

The next auxiliary proposition concerns the difference

$$\widehat{\mathbf{b}}^{(\mathbf{u})}(y,t) - \widehat{\mathbf{b}}^{(\mathbf{u}')}(y,t) = \mathbf{b}(X_{\mathbf{u}},t) - \mathbf{b}(X_{\mathbf{u}'},t)$$

$$= \sum_{k=1}^{3}\int_0^1 \mathbf{b}_{x_k}(X_{\mathbf{u}_\theta},t)d\theta\int_0^t (u_k - u_k')d\tau, \tag{3.27}$$

where $\mathbf{u}-\mathbf{u}'=\tilde{\mathbf{u}}$, $\mathbf{u}_\theta=\mathbf{u}'+\theta\tilde{\mathbf{u}}$ $(\theta\in(0,1))$, $X_{\mathbf{u}}=y+\int_0^t \mathbf{u}d\tau$, $X_{\mathbf{u}'}=y+\int_0^t \mathbf{u}'d\tau$ and $X_{\mathbf{u}_\theta}=y+\int_0^t \mathbf{u}_\theta d\tau$.

Lemma 3.8 If $\mathbf{b}$ satisfies the conditions of Theorem 2.2 and condition (3.1) is satisfied, then

$$\left\|\widehat{\mathbf{b}}^{(\mathbf{u})}-\widehat{\mathbf{b}}^{(\mathbf{u}')}\right\|_{Q_T}^{(l,l/2)}\le c(T)\int_0^T \|\mathbf{u}-\mathbf{u}'\|_{W_2^1(\Omega)}dt, \tag{3.28}$$

where $c(T)$ is a nondecreasing (power) function of T.

Finally, we remark that by elementary calculation it holds

$$\|\varrho_0^{-1}f\|_{Q_T}^{(l,l/2)}\le c(1+\frac{1}{R_0}+\|\varrho_0\|_{W_2^{2+l}(\Omega)}^3)\,\|f\|_{Q_T}^{(l,l/2)}. \tag{3.29}$$

4 Proof of Theorem 2.2

Proof of Theorem 2.2. We shall solve the problem (3.2) by the method of successive approximations, setting $\mathbf{u}^{(0)}=\mathbf{v}_0$, $q^{(0)}=0$ and determining $(\mathbf{u}^{(m+1)},q^{(m+1)})$ $(m=0,1,2,\dots)$ as a solution of the problem

$$\begin{cases} \varrho_0\mathbf{u}_t^{(m+1)}-\nu(\varrho_0)\nabla^2\mathbf{u}^{(m+1)}+\nabla q^{(m+1)} \\ \qquad =\mathbf{l}_1^{(m)}(\mathbf{u}^{(m)},q^{(m)})+2\nu'(\varrho_0)\widehat{\mathbb{D}}^{(m)}\nabla_m\varrho_0 \\ \qquad -\frac{\beta_1}{3}(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0-\beta_1\nabla_m^2\varrho_0\nabla_m\varrho_0+\varrho_0\widehat{\mathbf{b}}^{(m)}, \\ \nabla\cdot\mathbf{u}^{(m+1)}=\mathbf{l}_2^{(m)}(\mathbf{u}^{(m)}),\quad \mathbf{u}^{(m+1)}|_{t=0}=\mathbf{v}_0,\quad \mathbf{u}^{(m+1)}|_\Gamma=0. \end{cases} \tag{4.1}$$

Here $\nabla_m=\nabla_{\mathbf{u}^{(m)}}$, $\mathbf{l}_j^{(m)}=\mathbf{l}_j^{(\mathbf{u}^{(m)})}$ $(j=1,2)$, $\widehat{\mathbb{D}}^{(m)}=\widehat{\mathbb{D}}^{(\mathbf{u}^{(m)})}$, $\widehat{\mathbf{b}}^{(m)}=\widehat{\mathbf{b}}^{(\mathbf{u}^{(m)})}$. From Theorem 2.1 it follows that $(\mathbf{u}^{(m+1)},\ \nabla q^{(m+1)})$ are uniquely determined, and $(\mathbf{u}^{(1)},q^{(1)})$ is a solution of problem (4.1) i.e .,

$$\begin{cases} \mathbf{u}_t^{(1)}-\frac{\nu(\varrho_0)}{\varrho_0}\nabla^2\mathbf{u}^{(1)}+\frac{1}{\varrho_0}\nabla q^{(1)} \\ \qquad =-\frac{\beta_1}{3\varrho_0}(\nabla^{(j)}\nabla^{(i)}\varrho_0)\nabla\varrho_0-\frac{\beta_1}{\varrho_0}\nabla^2\varrho_0\nabla\varrho_0+\mathbf{b}, \\ \nabla\cdot\mathbf{u}^{(1)}=0,\quad \mathbf{u}^{(1)}|_{t=0}=\mathbf{v}_0,\quad \mathbf{u}^{(1)}|_\Gamma=0 \end{cases} \tag{4.2}$$

with the estimates

$$N[\mathbf{u}^{(1)},q^{(1)}]:=\|\mathbf{u}^{(1)}\|_{Q_T}^{(2+l,1+l/2)}+\|\nabla q^{(1)}\|_{Q_T}^{(l,l/2)}$$

$$\leq c\left(\frac{|\beta_1|}{3}\left\|\frac{1}{\varrho_0}(\nabla^{(j)}\nabla^{(i)}\varrho_0)\nabla\varrho_0\right\|_{Q_T}^{(l,l/2)}\right.$$

$$\left.+|\beta_1|\left\|\frac{1}{\varrho_0}\nabla^2\varrho_0\nabla\varrho_0\right\|_{Q_T}^{(l,l/2)}+\|\mathbf{b}\|_{Q_T}^{(l,l/2)}+\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right)$$

$$\leq c_1\left((T^{1/2}+T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)}+\|\mathbf{b}\|_{Q_T}^{(l,l/2)}+\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right), \qquad (4.3)$$

where c_1 is a nondecreasing function of T.

For the differences $\mathbf{Z}^{(m+1)} := \mathbf{u}^{(m+1)}-\mathbf{u}^{(m)}$, $\mathbf{P}^{(m+1)} := q^{(m+1)}-q^{(m)}$ ($m = 1,2,3,\ldots$), we have

$$\begin{cases} \varrho_0\mathbf{Z}_t^{(m+1)}-\nu(\varrho_0)\nabla^2\mathbf{Z}^{(m+1)}+\nabla\mathbf{P}^{(m+1)} \\ \quad =\mathbf{l}_1^{(m)}(\mathbf{Z}^{(m)},\mathbf{P}^{(m)})+\mathbf{l}_1^{(m)}(\mathbf{u}^{(m-1)},q^{(m-1)})-\mathbf{l}_1^{(m-1)}(\mathbf{u}^{(m-1)},q^{(m-1)}) \\ \qquad +2\nu'(\varrho_0)\left(\widehat{\mathbb{D}}^{(m)}\nabla_m\varrho_0-\widehat{\mathbb{D}}^{(m-1)}\nabla_{m-1}\varrho_0\right) \\ \qquad -\frac{\beta_1}{3}\left\{(\nabla_m^{(j)}\nabla_m^{(i)}\varrho_0)\nabla_m\varrho_0-(\nabla_{m-1}^{(j)}\nabla_{m-1}^{(i)}\varrho_0)\nabla_{m-1}\varrho_0\right\} \\ \qquad -\beta_1\left(\nabla_m^2\varrho_0\nabla_m\varrho_0-\nabla_{m-1}^2\varrho_0\nabla_{m-1}\varrho_0\right)+\varrho_0(\widehat{\mathbf{b}}^{(m)}-\widehat{\mathbf{b}}^{(m-1)}), \\ \nabla\cdot\mathbf{Z}^{(m+1)}=\mathbf{l}_2^{(m)}(\mathbf{Z}^{(m)})+\mathbf{l}_2^{(m)}(\mathbf{u}^{(m-1)})-\mathbf{l}_2^{(m-1)}(\mathbf{u}^{(m-1)}), \\ \mathbf{Z}^{(m+1)}|_{t=0}=0, \quad \mathbf{Z}^{(m+1)}|_{\Gamma}=0, \end{cases}$$

We suppose that the condition (3.1) is satisfied for $\mathbf{u}^{(n)}$ ($n\leq m$).

Lemmata in § 3 yield

$$\left\|\mathbf{l}_1^{(n)}(\mathbf{Z}^{(n)},\mathbf{P}^{(n)})\right\|_{Q_T}^{(l,l/2)}+\left\|\mathbf{l}_1^{(n)}(\mathbf{u}^{(n-1)},q^{(n-1)})-\mathbf{l}_1^{(n-1)}(\mathbf{u}^{(n-1)},q^{(n-1)})\right\|_{Q_T}^{(l,l/2)}$$

$$\leq c\delta\left(\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)}+\|\nabla\mathbf{P}^{(n)}\|_{Q_T}^{(l,l/2)}\right),$$

$$\left\|\widehat{\mathbb{D}}^{(n)}\nabla_n\varrho_0-\widehat{\mathbb{D}}^{(n-1)}\nabla_{n-1}\varrho_0\right\|_{Q_T}^{(l,l/2)}$$

$$\leq c\|\varrho_0\|_{W_2^{2+l}(\Omega)}\left(1+T^{1/2-l/2}\|\mathbf{v}_0\|_{W_2^l(\Omega)}\right)T^{1/2}\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)},$$

$$\left\|(\nabla_n^{(j)}\nabla_n^{(i)}\varrho_0)\nabla_n\varrho_0-(\nabla_{n-1}^{(j)}\nabla_{n-1}^{(i)}\varrho_0)\nabla_{n-1}\varrho_0\right\|_{Q_T}^{(l,l/2)}$$

$$\leq c\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}(T^{1/2}+T^{1/2-l/2})T^{1/2}\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)},$$

$$\left\|\nabla_m^2\varrho_0\nabla_m\varrho_0-\nabla_{n-1}^2\varrho_0\nabla_{n-1}\varrho_0\right\|_{Q_T}^{(l,l/2)}$$

$$\leq c\|\varrho_0\|^2_{W_2^{2+l}(\Omega)}(T^{1/2}+T^{1/2-l/2})T^{1/2}\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|\widehat{\mathbf{b}}^{(n)}-\widehat{\mathbf{b}}^{(n-1)}\|_{Q_T}^{(l,l/2)}\leq cT^{1/2}\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)},$$

$$\|\mathbf{l}_2^{(n)}(\mathbf{Z}^{(n)})\|_{W_2^{1+l,1/2+l/2}(Q_T)}+\|\mathbf{l}_2^{(n)}(\mathbf{u}^{(n-1)})-\mathbf{l}_2^{(n-1)}(\mathbf{u}^{(n-1)})\|_{W_2^{1+l,1/2+l/2}(Q_T)}$$

$$\leq c\delta\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)},$$

$$\left\|\frac{\partial}{\partial t}\mathscr{L}^{(n)}(\mathbf{Z}^{(n)})\right\|_{Q_T}^{(0,l/2)}+\left\|\frac{\partial}{\partial t}\left(\mathscr{L}^{(n)}(\mathbf{u}^{(n-1)})-\mathscr{L}^{(n-1)}(\mathbf{u}^{(n-1)})\right)\right\|_{Q_T}^{(0,l/2)}$$

$$\leq c\delta\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)}.$$

Then, we obtain that

$$N[\mathbf{Z}^{(n+1)},\mathbf{P}^{(n+1)}]\equiv\|\mathbf{Z}^{(n+1)}\|_{Q_T}^{(2+l,1+l/2)}+\|\nabla\mathbf{P}^{(n+1)}\|_{Q_T}^{(l,l/2)}$$

$$\leq C\left(\delta N[\mathbf{Z}^{(n)},\mathbf{P}^{(n)}]+T^{1/2}\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)}\right),\qquad(4.4)$$

where $C=C(T;\mathbf{v}_0,\varrho_0)$ is a nondecreasing function with respect to T. If we choose δ satisfying $C\delta<\frac{1}{4}$, we obtain

$$N[\mathbf{Z}^{(n+1)},\mathbf{P}^{(n+1)}]\leq\tfrac{1}{4}N[\mathbf{Z}^{(n)},\mathbf{P}^{(n)}]+CT^{1/2}\|\mathbf{Z}^{(n)}\|_{Q_T}^{(2+l,1+l/2)}$$

$$\leq(\tfrac{1}{4}+CT^{1/2})N[\mathbf{Z}^{(n)},\mathbf{P}^{(n)}]\leq\cdots\leq(\tfrac{1}{4}+CT^{1/2})^nN[\mathbf{Z}^{(1)},\mathbf{P}^{(1)}].\qquad(4.5)$$

We sum (4.5) in n from 0 to m and set $\Sigma_{m+1}=\sum_{n=0}^m N[\mathbf{Z}^{(n+1)},\mathbf{P}^{(n+1)}]$. Since

$$\sum\nolimits_{m+1}=\sum_{n=0}^m N[\mathbf{Z}^{(n+1)},\mathbf{P}^{(n+1)}]\leq N[\mathbf{u}^{(1)},q^{(1)}]\sum_{n=0}^m(\frac{1}{4}+CT^{1/2})^n$$

$$\leq c_1\left((T^{1/2}+T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)}+\|\mathbf{b}\|_{Q_T}^{(l,l/2)}+\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right)\sum_{n=0}^m(\frac{1}{4}+CT^{1/2})^n,$$

we obtain

$$N[\mathbf{u}^{(m+1)},q^{(m+1)}]\leq\sum\nolimits_{m+1}+N[\mathbf{u}^{(1)},q^{(1)}]$$

$$\leq c_1\Big((T^{1/2}+T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)}$$

$$+\|\mathbf{b}\|_{Q_T}^{(l,l/2)}+\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\Big)\times\left(1+\sum_{n=0}^m(\frac{1}{4}+CT^{1/2})^n\right).\qquad(4.6)$$

Note that c_1 and C are nondecreasing functions of T, then condition (3.1) for $\mathbf{u}^{(m+1)}$ is satisfied if $CT^{1/2} \le \frac{1}{4}$ and

$$3T^{1/2}c_1\left((T^{1/2}+T^{1/2-l/2})\|\varrho_0\|^3_{W_2^{2+l}(\Omega)}+\|\mathbf{b}\|^{(l,l/2)}_{Q_T}+\|\mathbf{v}_0\|_{W_2^{1+l}(\Omega)}\right)\le\delta. \tag{4.7}$$

The left side does not depend on m. Thus, $N[\mathbf{u}^{(m)},q^{(m)}]$ is uniformly bounded, the sequence $\{\mathbf{u}^{(m)},q^{(m)}\}$ converges in the norm $N[\,\cdot\,,\,\cdot\,]$, and the limit is a solution of the problem (3.2). The solution obtained is unique, since the difference of two solutions $\mathbf{w}=\mathbf{u}-\mathbf{u}'$, $s=q-q'$ satisfies the relations

$$\begin{cases} \varrho_0\mathbf{w}_t-\nu(\varrho_0)\nabla^2\mathbf{w}+\nabla s \\ \quad =\mathbf{l}_1^{(\mathbf{u})}(\mathbf{u},q)-\mathbf{l}_1^{(\mathbf{u}')}(\mathbf{u}',q')+2\nu'(\varrho_0)\left(\widehat{\mathbf{D}}^{(\mathbf{u})}\nabla_{\mathbf{u}}\varrho_0-\widehat{\mathbf{D}}^{(\mathbf{u}')}\nabla_{\mathbf{u}'}\varrho_0\right) \\ \quad\quad -\frac{\beta_1}{3}\left\{(\nabla^{(j)}_{\mathbf{u}}\nabla^{(i)}_{\mathbf{u}}\varrho_0)\nabla_{\mathbf{u}}\varrho_0-(\nabla^{(j)}_{\mathbf{u}'}\nabla^{(i)}_{\mathbf{u}'}\varrho_0)\nabla_{\mathbf{u}'}\varrho_0\right\} \\ \quad\quad -\beta_1\left(\nabla^2_{\mathbf{u}}\varrho_0\nabla_{\mathbf{u}}\varrho_0-\nabla^2_{\mathbf{u}'}\varrho_0\nabla_{\mathbf{u}'}\varrho_0\right)+\varrho_0(\widehat{\mathbf{b}}^{(\mathbf{u})}-\widehat{\mathbf{b}}^{(\mathbf{u}')}), \\ \nabla\cdot\mathbf{w}=\mathbf{l}_2^{(\mathbf{u})}(\mathbf{w})+\mathbf{l}_2^{(\mathbf{u})}(\mathbf{u}')-\mathbf{l}_2^{(\mathbf{u}')}(\mathbf{u}'), \\ \mathbf{Z}^{(m+1)}|_{t=0}=0, \quad \mathbf{Z}^{(m+1)}|_{\Gamma}=0. \end{cases}$$

Applying to this problem the estimate (2.13) and repeating the arguments carried out, we arrive at the inequality

$$N[\mathbf{w},s]\le c(\delta+T^{1/2})N[\mathbf{w},s].$$

This implies $(\mathbf{w},s)=0$, and Theorem 2.2 is proved.

5 Concluding remarks

We mentioned that ν can take the form

$$\nu=\nu(p,\varrho,|\mathbf{D}|^2) \tag{5.1}$$

in the most general case (see §1), however, there are several difficulties in considering the problem (1.1)-(1.3) with (5.1) unlike the problem (1.1)-(1.3) with $\nu=\nu(\varrho)$. In short the same method as we used to prove Theorem 2.2 is not valid for the problem (1.1)-(1.3) with (5.1). We shall give some remarks of the difficulties of it.

i. The pressure p is determined within an arbitrary function depending on t, because only the pressure gradient appears in the equations. In the case that

ν depends on p, the arbitrary function needs to be fixed by some additional condition, for example,

$$\int_\Omega p(x,t)dx = 0 \quad \text{a.e. in } (0,T).$$

Under this condition we can apply Poincaré's inequarity to p, however, the difficulty about the regularity of p with respect to t still remains.

ii. If ν depends only on ϱ, as we mentioned in § 2, we may just consider $\nu(\varrho_0(y))$, which is a known function independent of t, in the transformed problem (2.4) written in Lagrangian coordinates. On the otherhand, in the case of ν dependent on p, ϱ and $|\mathbb{D}|^2$, the transformed viscosity $\nu(q, \varrho_0, |\widehat{\mathbb{D}}^{(u)}|^2)$ is still an unknown coefficient of the equations even though we consider the transformed problem.

iii. $\nu(q, \varrho_0, |\widehat{\mathbb{D}}^{(u)}|^2)$ has, at most, the same regularity as that of q or $|\widehat{\mathbb{D}}^{(u)}|^2$. While we can assume the regularity of ϱ_0 as much as we need, the regularity of q and $|\widehat{\mathbb{D}}^{(u)}|^2$ are determined by the function spaces of solutions under consideration. This implies the problem (2.5) cannot be a linearized problem of the problem (2.4) with $\nu = \nu(q, \varrho_0, |\widehat{\mathbb{D}}^{(u)}|^2)$. Hence, we have to consider the different method for this problem.

Despite these points at issue we have already observed that we can overcome the difficulties by considering the appropriate function spaces for the solution if $\nu = \nu(\varrho, |\mathbb{D}|^2)$. In this case the problem (2.5) is also a linearized problem, thus we can use the strategy similar to that we used in this study. We strongly believe that we can prove the existence theorem for the problem (1.1)-(1.3) with $\nu = \nu(\varrho, |\mathbb{D}|^2)$ in a forthcoming study.

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