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Blocks and strongly p-embedded Frobenius subgroups

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原田耕一郎氏は論文 [H] の中で次の予想を与えています。

予想 G を有限群, p を素数, B を G の p-ブロックとする. もし、$\mathrm{Irr}(B)$ の空でない部分集合 J で、$\omega=\sum_{\chi_j\in J}\chi_j(1)\chi_j$ がすべての p-singular 元上でゼロの値を取るとすると、J は $\mathrm{Irr}(B)$ と一致する.
(この予想の逆の命題は、よく知られた結果です.)

この予想は以下の場合には証明されています.
(a) B が巡回不足群を持つとき、 [H],
(b) G が p-可解であるとき、 [KO],
(c) G のシロー p-部分群が $p=2$ で dihedral, semidihedral, または quaternion であるか、$p=3$ で位数9の群であるとき、 [K],
(d) $G=\mathrm{PSL}_2(q)$ [I1],
(e) $G=\mathrm{PSp}(4,q)$ または $G_2(q)$ で、$(q,2p)=1$ となるとき、[I2]
(f) もし、B のすべての既約ブラワー指標が liftable であるとき、 [I2].

これらの結果は分解行列に関する知識を使って行われています。この論文では、別の方法を紹介しょうましょう。即ち、シロー p-部分群 P の正規化群 $N_G(P)$ が strongly p-embedded フロベニウス部分群という条件の下に、上の予想が正しいことを分解行列を使わずに証明するわけです。

定理 G を有限群とし, p を素数, P を G のシロー p-部分群とします. もし、P がアーベル群で、$N_G(P)$ が strongly p-embedded フロベニウス部分群とすると、G と p に対して予想が正しい.

定理の証明 Suppose false and let B be a block and $J\subsetneq \mathrm{Irr}(B)$ is a counterexample to the conjecture, that is, $\emptyset\neq J\neq \mathrm{Irr}(B)$ and $\omega=\sum_{\chi\in J}\chi(1)\chi$ vanishes on all p-singular elements. In order to simplify the arguments, we may assume that P is not cyclic by [H]. Set $N=N_G(P)$ and let K be a Frobenius kernel of N and H a complement of K in N, then $K=P\times C$ and $|H|<|P|$, where $C=O_{p'}(C_G(P))$.

Lemma 1. *K is a T.I.-set.*

[Proof] For $1\neq c\in C$, $C_G(c)$ contains a strongly p-embedded subgroup $N_G(P)\cap C_G(c)=P\times C_C(c)$. Therefore, $C_G(c)=O_{p'}(C_G(c))\rtimes P$. Since $\mathrm{rank}(P)\geq 2$, we have

$C_G(c) \subseteq \langle C_G(r) \mid 1 \neq r \in P\rangle \subseteq N$ for any $1 \neq c \in C$. Therefore, for any $1 \neq E \subseteq C$, $P < C_G(E)$ and so $N_G(E) = N_{N_G(E)}(P)C_G(E) \leq N$, which implies that K is a T.I.-set. □

In this paper, ρ_P denotes the regular representation of P. We note that $\mathbb{Z}\rho_P$ is an ideal of the character ring $\mathrm{ch}(P)$ of P. The assumption of the theorem implies $\omega_{|P} \in \mathbb{Z}\rho_P$. We will divide the proof into three parts.

(1) Assume that $K = P$ and H acts on $P - \{1\}$ transitively.
Then P is an elementary abelian group and $\mathrm{Irr}(P) = \{1_P, {\xi_2}^h \mid h \in H\}$ for some nontrivial linear character ξ_2 of P. Since ${\xi_2}^N - (1_P)^N$ vanishes on p-regular elements, we have

$$\langle 1_P^G - {\xi_2}^G, 1_P^G - {\xi_2}^G\rangle = \langle 1_P^N - {\xi_2}^N, 1_P^N - {\xi_2}^N\rangle = |H| + 1.$$

For any $\mu \in \mathrm{Irr}(G)$, if $\langle \mu, 1_P^G - {\xi_2}^G\rangle = 0$, then $\langle \mu_{|P}, 1_P\rangle = \langle \mu_{|P}, {\xi_2}^h\rangle$ for all $h \in H$ and so $\mu_{|P} = a\rho_P$ for some $a \in \mathbb{Z}$. In this case, μ has a trivial defect group and $\{\mu\}$ is a block. Therefore, we may assume that for any $\mu \in B$, $\langle \mu, 1_P^G - {\xi_2}^G\rangle_G = a_\mu \neq 0$. Then

$$\mu_{|P} = (a_\mu + t)1_P + t(\sum_{h\in H} {\xi_2}^h) \equiv a_\mu 1_P \pmod{\mathbb{Z}\rho_P}$$

for some $t \in \mathbb{Z}$ and so

$$\mu(1) \equiv a_\mu \pmod{|P|}.$$

Hence we have

$$\mu(1)\mu_{|P} \equiv a_\mu^2 1_P \pmod{(|P|, \rho_P)} \qquad \text{and}$$
$$0 \equiv \textstyle\sum_{\mu\in J} \mu(1)\mu_{|P} \equiv \sum_{\mu \in J} a_\mu^2 1_P \pmod{(|P|, \rho_P)},$$

where $(|P|, \rho_P)$ denotes an ideal of $\mathrm{ch}(P)$ generated by $|P|$ and ρ_P. On the other hand, since $0 \leq \sum_{\mu\in \mathrm{Irr}(B)} a_\mu^2 \leq |H| + 1 \leq |P|$ and $\mathrm{Irr}(B) - J \neq \emptyset$ and $a_\mu \neq 0$ for $\mu \in \mathrm{Irr}(B)$, we have $0 < \sum_{\mu\in J} a_\mu^2 < |P|$, a contradiction.

(2) Assume that $K = P$ and H does not act on $P - \{1\}$ transitively. Set $\mathrm{Irr}(P) = \{1_P, \phi_2^h, \ldots, \phi_r^h \mid h \in H\}$. By the theory of exceptional characters, there are $\chi_i \in \mathrm{Irr}(G)$ and $\epsilon \in \{\pm 1\}$ such that

$$(\phi_i - \phi_j)^G = \epsilon(\chi_i - \chi_j)$$

for $i, j \geq 2$. There is also a virtual character A satisfying $\langle A, \chi_i\rangle = 0$ such that

$$(1_P - \phi_2)^G = \epsilon(A - \chi_2 + s\sum_{i=2}^{r} \chi_i)$$

for some $s \in \mathbb{Z}$ since $\langle (1_P - \phi_2)^G, (\phi_i - \phi_j)^G\rangle = -\delta_{i2} + \delta_{j2}$. For $\mu \in \mathrm{Irr}(G)$, if

$$\langle \mu, (\phi_i - \phi_2)^G\rangle = 0 = \langle \mu, (1_P - \phi_2)^G\rangle$$

for all i, then $\langle \mu_{|P}, \phi_i \rangle = \langle \mu_{|P}, 1_P \rangle$ for all i and so

$$\mu_{|P} \in \mathbb{Z}\rho_P,$$

which implies that $\{\mu\}$ is a block with trivial defect. Therefore we may assume

$$\mathrm{Irr}(B) \subseteq \{\chi_i \mid i = 2, ..., r\} \cup \mathrm{Irr}(A),$$

where $\mathrm{Irr}(A) = \{\mu \in \mathrm{Irr}(G) \mid \langle \mu, A \rangle \neq 0\}$. Set $\langle \mu, A \rangle = a_\mu$. Since $(\phi_i - \phi_j)^G$ vanishes on all p-regular elements, we have $\langle \omega, (\phi_i - \phi_j)^G \rangle = 0$. Therefore, if J contains some χ_i, then J contains all χ_j. Taking J or $B - J$ as J, we may assume

$$J \subseteq \mathrm{Irr}(A).$$

For any $\mu \in \mathrm{Irr}(A)$, since $\langle \mu, (\phi_i - \phi_j)^G \rangle = 0$ and $\langle \mu, (1_P - \phi_2)^G \rangle = \epsilon a_\mu$, we have

$$\mu_{|P} \equiv \epsilon a_\mu 1_P \pmod{\rho_P}$$

and so

$$\mu(1) \equiv \epsilon a_\mu \pmod{|P|}.$$

Hence

$$0 \equiv \omega_{|P} = \sum_{\mu \in J} \mu(1)\mu_{|P} \equiv \sum a_\mu^2 1_P \pmod{(|P|, \rho_P)},$$

which contradicts to

$$0 < \sum_{\mu \in \mathrm{Irr}(A)} a_\mu^2 = \langle A, A \rangle = |H| < |P|.$$

(3) Assume $C \neq 1$. Since H acts on C fixed point freely, C is nilpotent. Set $\mathrm{Irr}(P) = \{1_P = \phi_1, \phi_2, \ldots, \phi_{|P|}\}$ and $\mathrm{Irr}(C) = \{1_C = \xi_1, {\xi_2}^h, \ldots, {\xi_s}^h \mid h \in H\}$, where $\deg(\xi_2) = 1$. Then $\mathrm{Irr}(K) = \{\phi_i \otimes \xi_1, \phi_i \otimes {\xi_2}^h, \ldots, \phi_i \otimes {\xi_s}^h \mid h \in H, i = 1, ..., |P|\}$ and $(\phi_i \otimes \xi_j)^N$ are irreducible for $(i, j) \neq (1, 1)$ since N is a Frobenius group with the kernel K.

By the theory of exceptional characters [S], there are $\chi_{i,j} \in \mathrm{Irr}(G)$ for $(i, j) \neq (1, 1)$ and $\epsilon_j \in \{\pm 1\}$ for j such that

$$(\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_k)^G = \epsilon_j(\chi_{i,j} - \chi_{h,k})$$

for $(i, j), (h, k) \neq (1, 1)$ and $\deg(\xi_j) = \deg(\xi_k)$. We note that since $\deg(\xi_1) = \deg(\xi_2) = 1$, $\chi_{i,j}$ are all well-defined for $(i, j) \neq (1, 1)$. We also note that since $\langle (\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_j)^G, (\phi_a \otimes \xi_h)^G - (\phi_b \otimes \xi_h)^G \rangle = 0$ for $j \neq h$, we have $\chi_{i,j} \neq \chi_{h,k}$ for $(i, j) \neq (h, k)$ except $j = 1 = k$ and ϕ_i is H-conjugate to ϕ_h. Since $\langle \phi_1 \otimes \xi_1 - \phi_2 \otimes \xi_1, \phi_i \otimes \xi_2 - \phi_h \otimes \xi_2 \rangle = 0$ and $\langle \phi_1 \otimes \xi_1 - \phi_2 \otimes \xi_1, \phi_2 \otimes \xi_1 - \phi_h \otimes \xi_2 \rangle = -1$ for $h \neq 2$, we also have a virtual character A of G satisfying $\langle A, \chi_{2,1} \rangle = 0 = \langle A, \chi_{i,2} \rangle$ for $i = 1, ..., |P|$ and $r \in \mathbb{Z}$ such that

$$(\phi_1 \otimes \xi_1)^G - (\phi_2 \otimes \xi_1)^G = \epsilon(A + (r - 1)\chi_{2,1} + r(\sum_{i=1}^{|P|} \chi_{i,2})).$$

However, since $\langle(\phi_1 \otimes \xi_1)^G - (\phi_h \otimes \xi_1)^G, (\phi_1 \otimes \xi_1)^G - (\phi_h \otimes \xi_1)^G\rangle = 1 + |H|$ and the number of $\chi_{i,2}$ is greater than $|H|$, we have $r = 0$ and $\langle A, A\rangle = |H|$. Moreover, since $\langle(\phi_1 \otimes \xi_1)^G - (\phi_1 \otimes \xi_2)^G, (\phi_i \otimes \xi_h)^G - (\phi_j \otimes \xi_h)^G\rangle = 0$ for $h \geq 2$, if $\chi_{i,h} \in \mathrm{Irr}(A)$, then $\mathrm{Irr}(A)$ contains all $\{\chi_{i,h} \mid i = 1, ..., |P|\}$, which contradicts to $\langle A, A\rangle = |H| < |P|$. Therefore $\langle A, \chi_{i,h}\rangle = 0$ for $h \geq 2$.

If $\mu \in \mathrm{Irr}(G)$ satisfies $\langle \mu, (\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_j)^G\rangle = 0$ for all $j, (h, k)$, then there are $\lambda_j \in \mathbb{Z}$ such that $\mu_{|K} = \sum_j \lambda_j (\sum_{a\in H} \sum_i (\phi_i \otimes \xi_j)^a)$, which implies $\mu_{|P} \in \mathbb{Z}\rho_P$ and $\{\mu\}$ is a block as we did in the first part. Therefore, we may assume

$$\mathrm{Irr}(B) \subseteq \{\chi_{i,j} \mid i, j\} \cup \mathrm{Irr}(A).$$

We also have:

Lemma 2. *For* $(s, t) \neq (1, 1)$,

$$(\chi_{s,t})_{|K} \equiv \epsilon \sum_{h\in H} (\phi_s \otimes \xi_t)^h \quad (\mathrm{mod}\ \rho_P \otimes \mathrm{ch}(K)).$$

[Proof] For $t \neq j$, since

$$\langle \chi_{s,t}, \chi_{i,j} - \chi_{h,j}\rangle = \langle (\chi_{s,t})_{|K}, \epsilon_j(\phi_i \otimes \xi_j - \phi_h \otimes \xi_j)\rangle = 0,$$

$\langle (\chi_{s,t})_{|K}, \phi_i \otimes \xi_j\rangle$ does not depend on the choice of i, say a_j. Since $\delta_{s,i} - \delta_{s,h} = \langle \chi_{s,t}, \chi_{i,t} - \chi_{h,t}\rangle = \langle (\chi_{s,t})_{|K}, \epsilon_t(\phi_i \otimes \xi_t - \phi_h \otimes \xi_t)\rangle$, $\langle (\chi_{s,t})_{|K}, \phi_i \otimes \xi_t\rangle$ does not depend on the choice of $i \neq s$, say a_t. Therefore $(\chi_{s,t})_{|K} = \sum_j a_j \sum_{h\in H} (\rho_P \otimes \xi_j)^h + \epsilon \sum_{h\in H} (\phi_s \otimes \xi_t)^h$. □

Lemma 3. *$\chi_{i,j}$ and $\chi_{h,k}$ belong to the same block if and only if $j = k$. In particular, $\{\chi_{i,k} \mid i = 1, ..., |P|\}$ is a p-block of G for $k \neq 1$.*

[Proof] Since $\phi_i \otimes \xi_s - \phi_j \otimes \xi_s$ vanishes on all p-regular elements, so does $\chi_{i,s} - \chi_{j,s}$. Hence $\chi_{i,s}$ and $\chi_{j,s}$ belong to the same block. Let G^0 and N^0 denote the set of all p-regular elements of G and N, respectively. Since $G - G^0$ is a disjoint union of $\{(N - N^0)^g \mid g \in G/N\}$ and we have $N - N^0 = \{(g, c) \mid 1 \neq g \in P, c \in C\}$, if $j \neq k$, then we have:

$$\begin{aligned}
\langle \chi_{i,j}, \chi_{h,k}\rangle_{G^0} &= -\langle \chi_{i,j}, \chi_{h,k}\rangle_{G-G^0} \\
&= -\langle (\chi_{i,j})_{|N}, (\chi_{h,k})_{|N}\rangle_{N-N^0} \\
&= -\frac{1}{|N|} \sum_{1\neq g\in P, c\in C} \chi_{i,j}(gc)\overline{\chi_{h,k}(gc)} \\
&= -\frac{1}{|N|} \sum_{1\neq g\in P} \sum_{c\in C} \chi_{i,j}(gc)\overline{\chi_{h,k}(gc)} \\
&= -\frac{1}{|N|} \sum_{1\neq g\in P} \sum_{c\in C} \sum_{a\in H} \phi_i{}^a(g)\xi_j{}^a(c) \sum_{b\in H} \overline{\phi_h{}^b(g)\xi_k{}^b(c)} \\
&= -\frac{1}{|N|} \sum_{1\neq g\in P} \sum_{a,b\in H} \phi_i{}^a(g)\overline{\phi_h{}^b(g)} \sum_{c\in C} \xi_j{}^a(c)\overline{\xi_k{}^b(c)} = 0,
\end{aligned}$$

since $\sum_{c\in C} \xi_j{}^a(c)\overline{\xi_k{}^b(c)} = 0$ for any $a, b \in H$. Therefore, $\chi_{i,j}$ and $\chi_{h,k}$ don't belong to the same block. □

If B is a block $\{\chi_{i,j} \mid i = 1, \ldots, |P|\}$ for some $j \neq 1$, then since $\langle \omega, \chi_{i,j} - \chi_{k,j} \rangle = 0$, J contains all $\chi_{k,j}$ and so $J = B$. Therefore, we may assume

$$\mathrm{Irr}(B) \subseteq \mathrm{Irr}(A) \cup \{\chi_{i,1} \mid i = 1, \ldots, |P|\}.$$

Since $\langle \omega, \chi_{i,1} - \chi_{j,1} \rangle = 0$, taking J or $\mathrm{Irr}(B) - J$ as J, we may assume that $J \subsetneq \mathrm{Irr}(A)$ and $J \cap \{\chi_{i,1} \mid i = 1, \ldots, |P|\} = \emptyset$.

Set $a_\mu = \langle \mu, A \rangle$. For $\mu \in J$, since $a_\mu = \epsilon\langle \mu, (\phi_1 \otimes \xi_1)^G - (\phi_h \otimes \xi_1)^G \rangle$ and $\langle \mu, (\phi_i \otimes \xi_j)^G - (\phi_h \otimes \xi_j)^G \rangle = 0$ for $(i, j), (h, j) \neq (1, 1)$, we have

$$\mu_{|P} \equiv a_\mu 1_P \pmod{\rho_P} \quad \text{and}$$
$$\mu(1) \equiv a_\mu \pmod{|P|}$$

and so

$$0 \equiv \omega_{|P} = \sum_{\mu \in J} \mu(1)\mu_{|P} \equiv \sum_{\mu \in J} a_\mu^2 1_P \pmod{(|P|, \rho_P)}.$$

However, since

$$0 < \sum_{\mu \in J} a_\mu^2 \leq \sum_{\text{all } \mu} a_\mu^2 = |H| < |P|,$$

we have a contradiction.

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