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2-Homogeneity and Completely Regular Strongly Regular Subgraphs

— Variations of a theorem of K. Nomura by his results —

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1 Introduction

In 1993, K. Nomura introduced a notion of t-homogeneous property of graphs and gave a complete classification of 2-homogeneous bipartite distance-regular graphs. Soon after Nomura's classification, N. Yamazaki showed that bipartite distance-regular graphs of valency k is 2-homogeneous if the multiplicity of an eigenvalue is equal to k. About the same time P. Terwilliger and his then students G. Dickie and B. Curtin were studying antipodal Q-polynomial distance-regular graphs and a special class of bipartite Q-polynomial distance-regular graphs, and Nomura's result turned out to cover a special case. In particular the following is known. See [3, 4, 8, 15].

Theorem 1 *Let Γ be a distance-regular graph of diameter at least two. Then the following are equivalent.*

(i) Γ *is bipartite and* 2*-homogeneous.*

(ii) Γ *is bipartite and the multiplicity of an eigenvalue is equal to its valency.*

(iii) Γ *is a bipartite, antipodal Q-polynomial distance-regular graph.*

(iv) Γ *is bipartite* 2*-thin with exactly one irreducible module of endpoint* 2.

A few years later P. Terwilliger and G. Dickie completed a classification of antipodal Q-polynomial distance-regular graphs, and J. Caughman is now very close to complete a classification of bipartite Q-polynomial distance-regular graphs. Recently, A. Jurišić, J. H. Koolen and Š. Miklavič and others developed a study of triangle-free distance-regular graphs such that the multiplicity of an eigenvalue is equal to its valency [5]. Hence Nomura's result gave, in some sense, a starting point of all that followed.

But I have other thoughts on Nomura's classification of 2-homogeneous bipartite distance-regular graphs. Most of the known classes of distance-regular graphs with unbounded diameter are Q-polynomial. A distance-regular graph is Q-polynomial, or a Q-distance-regular graph if Γ satisfies the balanced condition defined by Terwilliger. The balanced condition is roughly speaking a 'nice' embedding of Γ on a sphere, which is closely related to combinatorial regularity. Hence Q-distance-regular graphs, or infinite series of known distance-regular graphs have several types of combinatorial regularity. Therefore it is natural to investigate the following problem.

Classify distance-regular graphs with additional combinatorial regularities.

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However there are not many successful classifications of this type besides the classification of 2-homogeneous bipartite distance-regular graphs. This is a motivation to review Nomura's result.

In the following we will discuss possible generalizations of Nomura's result and its connection to distance-regular graphs with strongly regular completely regular codes. At the end we remark that a special case of the results can be drastically improved if we apply an old result of Nomura and indicate a direction of possible generalizations.

2 Distance-Regular Graphs

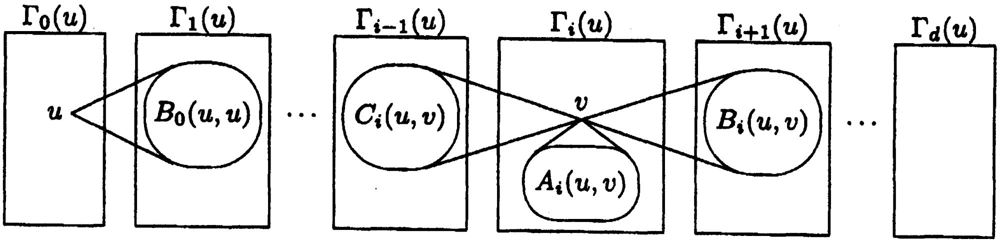
Let $\Gamma = (X, R)$ be a connected graph with vertex set X and edge set R. Let $\partial(x, y)$ denote the distance between x and y, which is the length of a shortest path connecting x and y, and the diameter d of Γ is defined by $d = \max\{\partial(x, y) \mid x, y \in X\}$. For $u \in X$ and $j \in \{0, 1, \ldots, d\}$, let

$$\Gamma_j(u) = \{x \in X \mid \partial(u, x) = j\} \text{ and } \Gamma(u) = \Gamma_1(u).$$

For $u \in X$ and $S \subset X$, $e(u, S) = |\Gamma(u) \cap S|$.

For $u, v \in X$ with $\partial(u, v) = j$ let

$$\begin{aligned} C(u,v) &= C_j(u,v) = \Gamma_{j-1}(u) \cap \Gamma(v), \\ A(u,v) &= A_j(u,v) = \Gamma_j(u) \cap \Gamma(v), \text{ and} \\ B(u,v) &= B_j(u,v) = \Gamma_{j+1}(u) \cap \Gamma(v). \end{aligned}$$



Definition 1 A connected graph $\Gamma = (X, R)$ is said to be *distance regular*, or a *distance-regular graph* (*DRG*), if the number

$$p^h_{i,j} = |\Gamma_i(u) \cap \Gamma_j(v)|, \; h, i, j \in \{0, 1, \ldots, d\}$$

depends only on i, j and $h = \partial(u, v)$.

Let $\Gamma = (X, R)$ be a DRG of diameter d. Then the numbers $c_j = |C(u, v)|$, $a_j = |A(u, v)|$ and $b_j = |B(u, v)|$ depend only on $j = \partial(u, v)$ and they determine all $p^h_{i,j}$'s.

$$\iota(\Gamma) = \{b_0, b_2, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

is called the *intersection array* of Γ.

A subset $C \subset X$ is said to be *strongly closed* if

$$C(u, v) \cup A(u, v) \subset C \text{ for all } u, v \in C.$$

For each $i = 0, 1, \ldots, d$, let $A_i \in \mathrm{Mat}_X(C)$ be the *i-th adjacency matrix* defined by

$$(A_i)_{x,y} = \begin{cases} 1 & \partial(x, y) = i \\ 0 & \text{otherwise.} \end{cases}$$

We set $A = A_1$. A is called the *adjacency matrix* of Γ. The eigenvalues of A is called the eigenvalues of Γ. Let $k_i = |\Gamma_i(x)|$ and $v_i(\lambda)$'s are polynomials of degree i in λ defined by $v_0 = 1, v_1(\lambda) = \lambda$ and $\lambda v_i(\lambda) = b_{i-1}v_{i-1}(\lambda) + a_i v_i(\lambda) + c_{i+1}v_{i+1}(\lambda)$. Then $A_i = v_i(A)$.

For the general theory of distance-regular graphs we refer the reader to [1].

3 t-Homogeneity

Let $\Gamma = (X, R)$ be a DRG of diameter d. For $x,\ y \in X$ with $\partial(x,y) = h$, set

$$D^i_j = D^i_j(x,y) = \Gamma_i(x) \cap \Gamma_j(y),\ 0 \le i,j \le d.$$

A *Nomura diagram*[2] (*intersection diagram*) of rank h is the collection $\{D^i_j\}_{i,j}$ with lines between D^i_j's and D^s_t's. We draw a line

$$D^i_j \text{ ——— } D^s_t$$

if there is possibility of existence of edges.

We sometimes write

$$D^i_j \overset{p}{\text{———}} D^s_t$$

in order to indicate that $e(x, D^s_t) = p$ for every $x \in D^i_j$.

We should note that the number $e(x, D^s_t)$ depends on $x \in D^i_j$ in general.

Definition 2 [Nomura [7]] Let $\Gamma = (X, R)$ be a DRG. Then Γ is *t-homogeneous* if

$$x \in D^r_s(u,v),\ x' \in D^r_s(u',v') \Rightarrow e(x, D^i_j(u,v)) = e(x', D^i_j(u',v'))$$

for all r, s, i, j and $u, v, u', v' \in X$ with $\partial(u,v) = \partial(u',v') = t$.

Hence Γ is t-homogeneous if and only if the number $e(x, D^i_j(u,v))$ is independent on $x \in D^r_s(u,v)$ and on the choice of u, v. In particular, all numbers on the lines in the Nomura diagram of Γ of rank t are determined.

As for the 2-homogeneity, the following two lemmas are of fundamental importance.

Lemma 2 (Nomura [9]) *Let $\Gamma = (X, R)$ be a DRG of diameter d. Then the following are equivalent.*

(H1) *There are integers $\delta_2, \ldots, \delta_d$ such that $e(x, D^{r-1}_{r-1}(u,v)) = \delta_r$ for all $u, v \in X$ with $\partial(u,v) = 2$ and $x \in D^r_r(u,v)$ and $r = 2, \ldots, d$.*

(H2) *There are integers $\gamma_1, \ldots, \gamma_d$ such that $|\Gamma(u) \cap \Gamma(v) \cap \Gamma_{r-1}(x)| = \gamma_r$ for every $x \in X$ and $u, v \in \Gamma_r(x)$ with $\partial(u,v) = 2$ and $r = 1, 2, \ldots, d$.*

Lemma 3 (Nomura [8]) *Let $\Gamma = (X, R)$ be a bipartite or almost bipartite ($a_1 = \cdots = a_{d-1} = 0$) DRG of diameter d. Then Γ is 2-homogeneous if and only if Γ satisfies* (H1).

4 A Classification by Nomura

K. Nomura classified 2-homogeneous bipartite DRGs completely and he extended its proof to cover almost bipartite case, i.e., $a_1 = \cdots = a_{d-1} = 0$, where d is the diameter.

Theorem 4 (Nomura [8, 9]) *Let Γ be a bipartite or almost bipartite 2-homogeneous DRG of diameter $d > 0$ and valency k. Then Γ has one of the following intersection arrays:*

(i) $\{k; 1\}$, $k > 0$. *(K_{k+1}.)*

[2] Nomura is not the first one to use this diagram but he is the one who applied it extensively to varieties of problems and made efficient use of the diagrams of various ranks. The definition of t-homogeneity is a very natural one if one uses this diagram of various ranks.

(ii) $\{k, k-1; 1, k\}$, $k > 1$. *($K_{k,k}$.)*

(iii) $\{k, k-1; 1, c\}$, $k = \gamma(\gamma^2 + 3\gamma + 1)$. $c = \gamma(\gamma + 1)$, $\gamma > 0$[3].

(iv) $\{k, k-1, 1; 1, k-1, k\}$, $k > 1$. *(Complement of $2 \times (k+1)$-grid.)*

(v) $\{4\gamma, 4\gamma - 1, 2\gamma, 1; 1, 2\gamma, 4\gamma - 1, 4\gamma\}$, $\gamma > 0$. *(Hadamard graph of valency $k = 4\gamma$.)*

(vi) $\{k, k-1, k-c, c, 1; 1, c, k-c, k-1, k\}$, $k = \gamma(\gamma^2 + 3\gamma + 1)$. $c = \gamma(\gamma+1)$, $\gamma > 0$. *(Antipodal double cover of (iii).)*

(vii) $\{2, 1, \cdots, 1; 1, \cdots, 1, 2\}$, $d > 1$. *(Cycle of length $2d$.)*

(viii) $\{2, 1, \cdots, 1; 1, \cdots, 1, 1\}$, $d > 1$. *(Cycle of length $2d + 1$.)*

(ix) $\{k, k-1, k-2, k-3, \cdots, 1; 1, 2, 3, \cdots, k-1, k\}$, $k = d$. *(d-dimensional hypercube.)*

(x) $\{2d+1, 2d, 2d-1, 2d-2, \cdots, d+2; 1, 2, 3, \cdots, d-1, d\}$, $d > 1$. *(Folded graph of $(2d+1)$-dimensional hypercube.)*

Besides the trivial cases (i) and (viii), (x) is the only nonbipartite case.

Outline of the Proof: By Lemma 2 and Lemma 3, we set

$$\gamma_i = |\Gamma_{i-1}(u) \cap \Gamma(x) \cap \Gamma(y)|$$

where $\partial(x, y) = 2$ and $u \in D_i^i(x, y)$.

The following lemma provides basic equations on parameters. The proof of it is given by counting arguments.

Lemma 5 (Nomura [8, 9]) *The following hold.*

(i) $(k-2)(\gamma_2 - 1) = (c_2 - 1)(c_2 - 2)$.

(ii) $\gamma_i(c_{i+1} - 1) = c_i(c_2 - 1)$, $(0 < i < d)$.

(iii) $(c_2 - 1)(\gamma_i - 1) = (c_i - 1)(\gamma_2 - 1)$, $(0 < i < d)$.

Now we explain how the proof of Theorem 4 goes.

1. We may assume $d \geq 3$, $k \geq 3$ and $\gamma_2 \geq 1$.
2. Suppose $\gamma_2 > 1$. Then by (i) and (ii),
$$k = \frac{(c_2 - 1)(c_2 - 2)}{\gamma_2 - 1}, \quad c_3 = \frac{c_2(c_2 - 1)}{\gamma_2} + 1,$$
and $2c_3 > k \geq c_3 + b_3$. Hence $d \leq 5$, as $c_3 > b_3$.
3. Suppose $\gamma_2 = 1$. Then $c_2 = 1$ or $c_2 = 2$ by (i).
4. If $\gamma_2 = 1$ and $c_2 = 1$, then by (ii) $c_{i+1} = 1$ or $\gamma_i = 0$ and $b_{i-1} = 1$.
5. Suppose $\gamma_2 = 1$ and $c_2 = 1 + q > 1$. Then $\gamma_i = 1$ by (iii). Now by (ii), $c_{i+1} - 1 = c_i(c_2 - 1) = c_i \cdot q$. Hence $c_i = 1 + q + \cdots + q^{i-1}$. In particular $c_i = i$ if $c_2 = 2 = 1 + q$.

Note that in (i) $c_2 = 1$ and 2 are special, while (ii) and (iii) describes equations which hold for much larger class of DRGs.

[3]Antipodal quotient of 5-dimensional hypercube when $\gamma = 1$, Higman-Sims graph when $\gamma = 2$, the existence of graphs is unknown when $\gamma > 2$.

Bipartite	$\subset$	Regular Near $2d$-gon
$\cap$		$\cap$
Bipartite or Almost Bipartite	$\subset$	Regular Near Polygon
$\cap$		$\cap$
Triangle Free		Parallelogram Free
$\cap$	$\subset$	$\cap$
Order (s,t)	$\supset$	Kite Free

Table 1: DRGs of Order (s,t)

5 DRGs of Order (s,t)

To seek generalizations, we review classes of DRGs which include bipartite DRGs as a subclass.

Let $\Gamma = (X, R)$ be a DRG of valency k and diameter d. Γ is said to be *triangle-free* if $a_1 = 0$.

A *kite* of length i is a 4-tuple $xyzw$ such that $\partial(x,y) = \partial(x,z) = \partial(y,z) = 1$, $\partial(x,w) = i$, and $\partial(y,w) = \partial(z,w) = i-1$. Γ is said to be *kite-free* if there is no kite of any length.

A *parallelogram* of length i is a 4-tuple $xyzw$ such that $\partial(x,y) = \partial(z,w) = 1$, $\partial(x,z) = i$, and $\partial(x,w) = \partial(y,w) = \partial(y,z) = i-1$. Γ is said to be *parallelogram-free* if there is no parallelogram of any length. The parallelogram-free condition is closely related to the existence of strongly closed subgraphs. See [10, 14].

$K_{2,1,1}$ is a graph with four vertices and five edges, and it is nothing but a kite of length 2 and a parallelogram of length 2. If Γ does not contain $K_{2,1,1}$, then there exist positive integers s, t such that every maximal clique is of size $s+1$ and $\Gamma(x)$ is isomorphic to a disjoint union of $t+1$ cliques of size s. Γ is said to be of *order* (s,t) if there is no $K_{2,1,1}$, $a_1 = s-1$ and $k = s(t+1)$.

A DRG $\Gamma = (X, R)$ of diameter d is said to be a *regular near polygon* (RNP) if it is of order (s,t) for some integers s and t, and for every maximal clique L and a vertex $x \in X$ with $\partial(x, L) = i < d$, $|\Gamma_i(x) \cap L| = 1$. If there is no maximal clique L such that $L \subset \Gamma_d(x)$ for any $x \in X$, it is called a *regular near $2d$-gon* (RN $2d$-gon). A regular near 4-gon is called a *generalized quadrangle.*

6 A Modification of 2-Homogeneity

In order to include larger class of DRGs having a certain rank 2 regularity, we weaken the 2-homogeneity a little and present a generalization of Theorem 4.

Definition 3 Let $\Gamma = (X, R)$ be a triangle-free-DRG of diameter d.
Then Γ is said to be *2-homogeneous* [resp. *almost 2-homogeneous*] if γ_i exists for $i = 2, 3, \ldots, d$, [resp. $i = 2, 3, \ldots, d-2$],

$$|\Gamma(u) \cap \Gamma(v) \cap \Gamma_{i-1}(w)| = \gamma_i$$

for every $w \in X$, $u, v \in \Gamma_i(w)$ with $\partial(u,v) = 2$.

Theorem 6 *Let $\Gamma = (X, R)$ is a triangle-free almost 2-homogeneous DRG of diameter $d \geq 3$ and valency $k \geq 3$. If $d = 3$, in addition assume γ_2 exists. Then Γ is isomorphic to one of the following.*

(i) *2-homogeneous bipartite:*

(ii) *The folded graph of a binary Hamming graph* $H(2d+1,2)$.

(iii) *The folded graph of a binary Hamming graph* $H(2d,2)$.

(iv) *The Coxeter graph with intersection array:* $\{3,2,2,1;1,1,1,2\}$.

(v) *The dodecahedron with intersection array:* $\{3,2,1,1,1;1,1,1,2,3\}$.

(vi) *The Biggs-Smith graph with intersection array:* $\{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$.

(vii) *The coset graph of the extended binary Golay code with intersection array:* $\{24,23,22,21;1,2,3,24\}$[4].

(viii) *A DRG with parameters* $c_{d-1}=1$, $a_1=\cdots=a_{d-1}=0$ *and* $d\leq 6$.

(ix) *A DRG with parameters* $c_i=i$ *for* $i=1,2,\ldots,d-1$, $a_1=\cdots=a_{d-3}=0$, $a_{d-2}=d-1<a_{d-1}$.

(x) *A DRG of valency* $k\geq 4$ *with parameters* $c_{d-1}=1$, $a_1=\cdots=a_{d-4}=0$, $a_{d-3}\leq 1$, $a_{d-2}\leq 1$ *and* $a_{d-1}>0$.

Moreover, if γ_{d-1} *exits,* (iii), (vii), (viii), (ix) *are excluded and in* (x), $c_d=1$, $a_{d-3}=0$, *and* $a_{d-2}=a_{d-1}=1$.

7 Completely Regular Codes

In Theorem 4 and Theorem 6, the case $c_2=2$ is essential and the existence of γ_i's is closely related to the condition that every quadrangle is completely regular. Before we state a result that states their relation, we give definitions of necessary terminologies.

Let $\Gamma=(S,R)$ be a DRG, and C a nonempty subset of X. Then the *width* of C is defined by $w(C)=\max\{\partial(x,y)\mid x,y\in C\}$. For $x\in X$, $\partial(x,C)=\min\{\partial(x,y)\mid y\in C\}$, and $t(C)=\max\{\partial(x,C)\mid x\in X\}$ is called the *covering radius* of C.

A nonempty subset C of a vertex set X is said to be *completely regular*, or a *completely regular code*, if the following number

$$\pi_{i,j}=|\Gamma_j(x)\cap C|$$

depends only on $i=\partial(x,C)$ and j for all $i,j\in\{0,1,2,\ldots,n\}$.

Let $\Gamma=(X,R)$ be a DRG of diameter d, $C=\{x,y,z,w\}$ a quadrangle, $x\sim y\sim z\sim w\sim x$, $x\not\sim z$ and $y\not\sim w$. Let $u\in\Gamma_{i+2}(x)\cap\Gamma_i(z)$ with $i+2\leq d$. Then $|\Gamma_i(x)\cap C|=1$, and the following proposition is straightforward.

Proposition 7 *Let* $\Gamma=(X,R)$ *be a DRG of diameter* $d\geq 3$. *Then the following are equivalent.*

(i) *There is a quadrangle, and for every quadrangle* C *and a vertex* $u\in X$ *with* $\partial(u,C)=i\leq d-2$, $\gamma_i(u,C)=|\Gamma_i(u)\cap C|=1$.

(ii) *For all vertices* $x,y\in X$ *with* $\partial(x,y)=2$ *and a vertex* $u\in X$ *with* $\partial(x,u)=\partial(y,u)=i\leq d-2$, $\gamma(x,y;u)=|\Gamma(x)\cap\Gamma(y)\cap\Gamma_{i-1}(u)|=1$, $a_1=a_2=\cdots=a_{d-2}=0$ *and* $c_2=2$.

[4]This is an almost 2-homogeneous bipartite graph. Such graphs were classified by B. Curtin [4] and A. Jurišić, J. H. Koolen and Š. Miklavič [5]. But both of the results do not contain this graph.

Since the case (ii) above is a special case of Theorem 6, we have the following as a corollary.

Corollary 8 *Let $\Gamma = (X, R)$ be a DRG of diameter $d \geq 4$. Suppose there is a quadrangle, and every quadrangle is a completely regular code. Then one of the following holds.*

(i) *The binary Hamming graph $H(d, 2)$.*

(ii) *The folded graph of a binary Hamming graph $H(2d + 1, 2)$.*

(iii) *The folded graph of a binary Hamming graph $H(2d, 2)$.*

(iv) *The coset graph of the extended binary Golay code with intersection array:* $\{24, 23, 22, 21; 1, 2, 3, 24\}$.

Since a quadrangle is a strongly regular graph, it is natural to ask about the DRGs having completely regular strongly regular subgraphs. Here by strongly regular graphs we mean DRGs of diameter two, hence connected.

Theorem 9 *Let $\Gamma = (X, R)$ be a parallelogram-free-DRG of diameter $d \geq 4$ such that $b_1 > b_2$ and $a_2 \neq 0$. Suppose every strongly closed subgraph C of diameter 2 is completely regular. Then for every pair of vertices x, y at distance $d - 1$, Γ has a strongly closed subgraph Y of diameter $d-1$ containing x and y and Y is isomorphic to a Hamming graph $H(d-1, m)$ for some $m > 2$ or a dual polar graph.*

In general the covering radius satisfies $t(C) \geq d - w(C)$. But if equality is attained, we have an algebraic characterization of completely regular codes.

Theorem 10 ([2, 11]) *Let $\Gamma = (X, R)$ be a DRG of diameter d, and C a nonempty subset of X. Let*

$$\rho_C(\lambda) = \frac{1}{|X|} \sum_{i=0}^{d} \eta_i \frac{v_i(\lambda)}{k_i} \in \boldsymbol{R}[\lambda],$$

where $\eta_i = |\{(x, y) \in C \times C \mid \partial(x, y) = i\}|$. If $\rho_C(\lambda)$ has $w(C)$ roots among the eigenvalues of Γ, then C is completely regular of covering radius $d - w(C)$.

H. Tanaka [12] and others classified completely regular codes of covering radius $d - w(C)$ under an extra condition, i.e., $w + w^* = d$, for several classes of DRGs. When Γ is a Hamming graph or a dual polar graph, such completely regular codes are strongly closed. As an application to the previous theorem, we obtain a kind of the converse of their results.

Theorem 11 *Let $\Gamma = (X, R)$ be a parallelogram-free-DRG of order (s, t) and diameter $d \geq 4$. Suppose $b_1 > b_2$ and $a_2 \neq 0$. Let $q = c_2 - 1$. Then the following are equivalent.*

(i) *There is a completely regular code C of covering radius $d-2$ such that the induced subgraph on C is a strongly regular graph.*

(ii) *Every strongly closed subgraph of diameter 2 is completely regular with covering radius $d - 2$ and that it is a generalized quadrangle.*

(iii) *$q \neq 0$ and Γ has eigenvalues $-t - 1$ and $s - t/q$.*

(iv) *Γ is isomorphic to a Hamming graph $H(d, m)$ for some $m > 2$ or a dual polar graph.*

8 Remarks

For the case $q = 1$, much stronger result stated in Corollary 14 below holds that is a direct consequence of Corollary 13 based on Theorem 12 of K. Nomura.

Theorem 12 (Nomura [6]) *Let Γ be a DRG of order (s,t), $s > 1$, diameter $d \geq 3$ with parameters $c_i = i$ for $i = 1, 2, \ldots, e$ and $a_i = i(s-1)$ for $i = 1, 2, \ldots, e-1$ for some e with $3 \leq e \leq d$. Then there is a covering $\theta : H(t+1, s+1) \to \Gamma$ with the following properties. For a vertex u of Γ, $C = \theta^{-1}(u)$ is an e-error correcting completely regular code with covering radius d. Moreover if $e = d$, or $e = d-1$, then $d = t+1$ and Γ is isomorphic to $H(d, s+1)$.*

Although $s = 3$ is not included in the original paper, we can follow the same line of the proof if Γ is known to be of order (s,t). In the case $e = d-1$, C is a uniformly packed code. By the classification result of H. van Tilborg in [13], we obtain the assertion as $e \geq 3$.

Corollary 13 *Let Γ be a parallelogram-free-DRG of order (s,t) with $c_2 = 2$, $a_2 = 2(s-1)$ and $c_3 = 3$ with $s > 1$. If the diameter $d \geq 3$, then Γ is isomorphic to the Hamming graph $H(d,q)$.*

Corollary 14 *Let Γ be a parallelogram-free-DRG of order (s,t) with $c_2 = 2$. Suppose Γ contains a strongly regular subgraph with parameters (κ, λ, μ). If $\kappa \neq \mu$ and $\pi_{i,j} = |\Gamma_j(x) \cap C|$ depends only on $i = \partial(x, C)$ and j whenever $(i,j) = (1,1), (1,2)$ or $(2,1)$. Then Γ is isomorphic to the Hamming graph $H(d,q)$.*

Finally we list possible further generalizations of the results in this article.

Problems I. (Generalizations)

1. Characterize bipartite Q-polynomial DRGs by rank 2-structure.
2. Characterize Q-polynomial regular near polygons by rank 2-structure.
3. Characterize DRGs with $r = r(\Gamma) = \ell(c_1, a_1, b_1) > 2$ by rank $(r+1)$-structure.
4. Let C be a strongly closed completely regular of width at least two. Find a condition that
$$\gamma_j = |\Gamma_j(u) \cap C| = 1, \text{ whenever } \partial(u, C) = j.$$
5. Classify regular near polygons of order (s,t) with $s > 1$ such that $c_2 = 1 + q > 1$ and $c_3 = 1 + q + q^2$.

Problems II. (Feasible Arrays) We list three types of intersection arrays of interest. Type (A) has to do with Problem I-5 and Theorem 9. Compare it with Theorem 12. Type (B) is the case (viii) in Theorem 6, and type (C) is a general form of the cases (vii) and (ix) in the same theorem.

Let $[i]_q = 1 + q + \cdots + q^{i-1} = \binom{i}{1}_q$

$$\text{(A)} \left\{ \begin{array}{ccccccc} * & 1 & \cdots & [i]_q & \cdots & [d-1]_q & c_d \\ 0 & s-1 & \cdots & [i]_q(s-1) & \cdots & [d-1]_q(s-1) & a_d \\ s(t+1) & st & \cdots & b_i & \cdots & b_{d-1} & * \end{array} \right\}$$

$$\text{(B)} \left\{ \begin{array}{cccccccccc} * & 1 & \cdots & i & \cdots & d-3 & d-2 & d-1 & c_d \\ 0 & 0 & \cdots & 0 & \cdots & 0 & d-1 & a_{d-1} & a_d \\ k & k-1 & \cdots & k-i & \cdots & k-d+3 & k-2s+3 & b_{d-1} & * \end{array} \right\}$$

$$
\text{(C)} \left\{ \begin{array}{cccccccccc} * & 1 & \cdots & 1 & \cdots & 1 & c_{d-2} & c_{d-1} & c_d \\ 0 & s-1 & \cdots & s-1 & \cdots & s-1 & a_{d-2} & a_{d-1} & a_d \\ s(t+1) & st & \cdots & st & \cdots & st & b_{d-2} & b_{d-1} & * \end{array} \right\}
$$

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