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Partial stationary reflection in $\mathcal{P}_{\omega_1}\omega_2$

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Abstract

For a stationary $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ and $k = 0, 1$, let $\mathsf{SR}_k(S^*)$ denote the principle that every stationary $S \subseteq S^*$ reflects to some ordinal in $\omega_2 \setminus \omega_1$ of cofinality ω_k. We show that if ZFC is consistent then ZFC together with $\exists S^*$, $\mathsf{SR}_k(S^*)$ is also consistent for both $k = 0, 1$.

1 Introduction

In this paper we consider the consistency of the following partial stationary reflection principle in $\mathcal{P}_{\omega_1}\omega_2$:

Definition 1.1. *For a stationary $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ and $k = 0, 1$ let $\mathsf{SR}_k(S^*)$ denote the following principle:*

> *For every stationary $S \subseteq S^*$ there exists an ordinal $\alpha \in \omega_2 \setminus \omega_1$ such that $\mathrm{cf}\,\alpha = \omega_k$ and $S \cap \mathcal{P}_{\omega_1}\alpha$ is stationary in $\mathcal{P}_{\omega_1}\alpha$.*

Recall that the stationary reflection principle in $\mathcal{P}_{\omega_1}\omega_2$, which is often called the weak reflection principle, states that for every stationary $S \subseteq \mathcal{P}_{\omega_1}\omega_2$ there exists $\alpha \in \omega_2 \setminus \omega_1$ with $S \cap \mathcal{P}_{\omega_1}\alpha$ stationary. Let $\mathsf{SR}(\mathcal{P}_{\omega_1}\omega_2)$ denote this principle. $\mathsf{SR}_k(\mathcal{P}_{\omega_1}\omega_2)$ is strengthening of $\mathsf{SR}(\mathcal{P}_{\omega_1}\omega_2)$, and $\mathsf{SR}_k(S^*)$ is a partial version of $\mathsf{SR}_k(\mathcal{P}_{\omega_1}\omega_2)$.

It is well known that if a weakly compact cardinal is Lévy collapsed to ω_2 then $\mathsf{SR}_1(\mathcal{P}_{\omega_1}\omega_2)$ holds. On the other hand Veličković [8] showed that if $\mathsf{SR}(\mathcal{P}_{\omega_1}\omega_2)$ holds then ω_2 is weakly compact in L. Hence both $\mathsf{SR}_1(\mathcal{P}_{\omega_1}\omega_2)$ and $\mathsf{SR}(\mathcal{P}_{\omega_1}\omega_2)$ are equiconsistent with the weakly compact cardinal axiom. It seems to be an open question whether $\mathsf{SR}(\mathcal{P}_{\omega_1}\omega_2)$ implies $\mathsf{SR}_1(\mathcal{P}_{\omega_1}\omega_2)$ or not.

As for the consistency of SR_0 two important facts are already known. First it is essentially shown in Foreman-Todorčević [2] that $\mathsf{SR}_0(\mathcal{P}_{\omega_1}\omega_2)$ is not consistent. Next it is shown in König-Larson-Yoshinobu [4] that if $2^{\omega_1} = \omega_2$ then $\mathsf{SR}_0(S^*)$ does not hold for any stationary $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$. As a corollary of the latter, König-Larson-Yoshinobu [4] obtained that $\mathsf{SR}(\mathcal{P}_{\omega_1}\omega_2)$ together with $2^{\omega_1} = \omega_2$ implies $\mathsf{SR}(\mathcal{P}_{\omega_1}\omega_2)$.

But it remains to be unknown whether the existence of a stationary $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ such that $\mathsf{SR}_0(S^*)$ holds is consistent or not. Here we give a positive answer:

Theorem 1.2. *If* ZFC *is consistent then so is* ZFC *with the existence of a stationary $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ such that $\mathsf{SR}_0(S^*)$ holds.*

In the above theorem note that we do not need any large cardinal for the consistency of $\mathsf{SR}_0(S^*)$ for some S^*. We prove that this is also the case with $\mathsf{SR}_1(S^*)$:

Theorem 1.3. *If* ZFC *is consistent then so is* ZFC *with the existence of a stationary $S^* \subseteq \mathcal{P}_{\omega_1}\omega_2$ such that $\mathsf{SR}_1(S^*)$ holds.*

This paper is devoted to the proof of the above theorems. We prove them in Section 5. In Section 2 we present our notation and basic facts used in this paper. In Section 3 and 4 we present tools, developed by Shelah, which we use in the proof of the above theorems. In Section 3 we review the iteration of T-complete forcing notions, and in Section 4 we present a lemma on stationary subsets of $\mathcal{P}_{\omega_1}\omega_2$.

2 Preliminaries

Here we present our notation and basic facts used in this paper. For those which are not presented below, consult Baumgartner [1], Jech [3] and Shelah [5].

The notion of club, stationary and nonstationary subsets of $\mathcal{P}_\kappa\lambda$ can be found in [3]. We often use the fact that $S \subseteq \mathcal{P}_{\omega_1}\lambda$ is stationary if and only if for every function $f : [\lambda]^{<\omega} \to \lambda$ there exists $x \in S$ which is closed under f.

For $S \subseteq \mathcal{P}_{\omega_1}\omega_2$ and $\alpha \in \omega_2 \setminus \omega_1$ we say that S *reflects to* α if $S \cap \mathcal{P}_{\omega_1}\alpha$ is stationary in $\mathcal{P}_{\omega_1}\alpha$.

For $k = 0, 1$ let E^2_k denotes the set of all limit ordinals $\alpha \in \omega_2$ with $\mathrm{cf}\,\alpha = \omega_k$.

In this paper we extensively use structures and their elementary submodels. If we say that $\mathcal{M}$ is a structure then it means that $\mathcal{M}$ is a structure of some countable language. We say that a structure $\mathcal{M}$ is an *expansion* of a structure $\mathcal{N}$ if $\mathcal{M}$ is obtained from $\mathcal{N}$ by adding countable many functions, predicates and constants. We often use the fact that if θ is a regular uncountable cardinal and M is an elementary submodel of $\langle \mathcal{H}_\theta, \in \rangle$ then $x \subseteq M$ for every countable $x \in M$.

A *forcing notion* denotes a partial ordering which have the greatest element and whose universe is a set.

Let $\mathbb{P}$ be a forcing notion and δ be a cardinal. We say that $\mathbb{P}$ has the δ-chain condition (δ-c.c.) if there are no antichain in $\mathbb{P}$ of cardinality δ. $\mathbb{P}$ is said to be ω-distributive if for every countable family $\mathcal{D}$ of dense open subsets of $\mathbb{P}$, $\bigcap \mathcal{D}$ is dense in $\mathbb{P}$. $\mathbb{P}$ is ω-distributive if and only if the forcing extension by $\mathbb{P}$ does not add any countable sequence of ordinals.

Next let $\mathbb{P}$ be a forcing notion and M be a nonempty set. $p \in \mathbb{P}$ is called an $(M, \mathbb{P})$-*generic condition* if $D \cap M$ is predense below p for every predense

$D \subseteq \mathbb{P}$ with $D \in M$. Moreover for a $\mathbb{P}$-generic filter over V let

$$M[G] = \{\dot{a}_G \mid \dot{a} \in V^{\mathbb{P}} \cap M\},$$

where $\dot{a}_G$ denotes the evaluation of $\dot{a}$ by G. We use the following basic fact:

Fact 2.1 (Shelah [5]). *Let $\mathbb{P}$ be a forcing notion, θ be a sufficiently large regular cardinal and M be an elementary submodel of $\langle \mathcal{H}_\theta, \in, \mathbb{P}\rangle$. Let $\dot{G}$ be the canonical name for a $\mathbb{P}$-generic filter. Then the following hold:*

(1) $\Vdash_{\mathbb{P}}$ "$M[\dot{G}] \prec \langle {\mathcal{H}_\theta}^{V[\dot{G}]}, \in\rangle$ ".

(Hence if $\dot{c}_0, \dot{c}_1, \ldots$ are $\mathbb{P}$-names in M then $\Vdash_{\mathbb{P}}$ "$M[\dot{G}] \prec \langle {\mathcal{H}_\theta}^{V[\dot{G}]}, \in, \dot{c}_0, \dot{c}_1, \ldots\rangle$ ".)

(2) If p is an (M, P)-generic condition then $p \Vdash_{\mathbb{P}}$ "$M[\dot{G}] \cap V = M$ ".

An iteration of forcing notions of length ζ will be denoted as $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \mid \xi \leq \zeta, \eta < \zeta\rangle$. Each $\mathbb{P}_\xi$ is a forcing notion and each $\dot{\mathbb{Q}}_\eta$ is a $\mathbb{P}_\eta$-name of a forcing notion. $\mathbb{P}_\xi$ consists of total functions p on ξ such that $p \restriction \eta \Vdash_{\mathbb{P}_\xi}$ "$p(\eta) \in \dot{Q}_\eta$ ".

We abbreviate $\Vdash_{\mathbb{P}_\xi}$ as $\Vdash_\xi$. We let $\dot{1}_\eta$ denote a fixed $\mathbb{P}_\eta$-name of the greatest element of $\dot{\mathbb{Q}}_\eta$. For each $p \in \mathbb{P}_\zeta$, let $\operatorname{supp} p := \{\eta < \zeta \mid p(\eta) = \dot{1}_\eta\}$.

An iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \mid \xi \leq \zeta, \eta < \zeta\rangle$ is called a *countable support iteration* if $\mathbb{P}_\xi$ is the inverse limit of $\langle \mathbb{P}_{\xi'} \mid \xi' < \xi\rangle$ for every limit ξ with $\operatorname{cf} \xi = \omega$ and is the direct limit for every ξ with $\operatorname{cf} \xi > \omega$. Note that if $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \mid \xi \leq \zeta, \eta < \zeta\rangle$ is a countable support iteration then $|\operatorname{supp} p| \leq \omega$ for every $p \in \mathbb{P}_\zeta$.

3 Iteration of T-complete forcing notions

Here we review the iteration of T-complete forcing notions, which was developed by Shelah [5]. For the completeness of this paper we give the proof of almost all lemmata.

We begin with the definition of T-completeness:

Definition 3.1 (Shelah). *Let $\mathbb{P}$ be a forcing notion and M be a countable set. We call a sequence $\langle p_n \mid n \in \omega\rangle$ with the following properties an $(M, \mathbb{P})$-generic sequence:*

(i) $\langle p_n \mid n \in \omega\rangle$ is a descending sequence in $\mathbb{P}$ with $p_n \in M$ for every $n \in \omega$.

(ii) For every dense open subset $D \in M$ of $\mathbb{P}$ there exists $n \in \omega$ with $p_n \in D$.

Here note that if p is a lower bound of some $(M, \mathbb{P})$-generic sequence then p is an $(M, \mathbb{P})$-generic condition.

Definition 3.2 (Shelah). *Let $\mathbb{P}$ be a forcing notion, λ be an ordinal $\geq \omega_1$ and T be a subset of $\mathcal{P}_{\omega_1}\lambda$. We say that $\mathbb{P}$ is T-complete if it satisfies the following:*

> *If θ is a sufficiently large regular cardinal, and M is a countable elementary submodel of $\langle \mathcal{H}_\theta, \in, \mathbb{P}, T\rangle$ with $M \cap \lambda \in T$ then every $(M, \mathbb{P})$-generic sequence has a lower bound in $\mathbb{P}$.*

Below we present basics on T-complete forcing notions. As is the case with properness, there are several slightly different definitions of T-completeness. First we give one of them. The proof of the following is similar as that for properness:

Lemma 3.3. *Let $\mathbb{P}$ be a forcing notion. Let λ be an ordinal $\geq \omega_1$ and T be a subset of $\mathcal{P}_{\omega_1}\lambda$. Then $\mathbb{P}$ is T-complete if and only if it satisfies the following:*

> *There exists a regular cardinal θ with $\mathbb{P}, T \in \mathcal{H}_\theta$ and an expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}_\theta, \in\rangle$ such that if M is a countable elementary submodel of $\mathcal{M}$ with $M \cap \lambda \in T$ then every $(M, \mathbb{P})$-generic sequence has a lower bound in $\mathbb{P}$.*

It is easy to see that if T is stationary then T-completeness implies ω-distributivity. Next we observe this:

Lemma 3.4 (Shelah). *Let λ be an ordinal $\geq \omega_1$ and T be a stationary subset of $\mathcal{P}_{\omega_1}\lambda$. Then every T-complete forcing notion is ω-distributive.*

Proof. Suppose that $\mathbb{P}$ is a T-complete forcing notions. Take an arbitrary family $\{D_n \mid n \in \omega\}$ of dense open subsets of $\mathbb{P}$ and an arbitrary $p \in \mathbb{P}$. We must find $p^* \leq p$ which belongs to $\bigcap_{n\in\omega} D_n$.

Let θ be a sufficiently large regular cardinal. Because T is stationary there exists a countable elementary submodel M of $\langle \mathcal{H}_\theta, \in, \mathbb{P}, T\rangle$ such that $\{p\} \cup \{D_n \mid n \in \omega\} \subseteq M$ and $M \cap \lambda \in T$. Then we can take an $(M, \mathbb{P})$-generic sequence $\langle p_n \mid n \in \omega\rangle$ with $p_0 \leq p$.

By T-completeness, there exists a lower bound p^* of $\langle p_n \mid n \in \omega\rangle$. Then $p^* \leq p$ and $p^* \in \bigcap_{n\in\omega} D_n$ clearly. □

T-completeness is preserved by countable support iterations:

Lemma 3.5 (Shelah). *Let λ be an ordinal and T be a subset of $\mathcal{P}_{\omega_1}\lambda$. Suppose that $\mathcal{I} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \mid \xi \leq \zeta, \eta < \zeta\rangle$, $\zeta \in$ On, is a countable support iteration of T-complete forcing notions. Then $\mathbb{P}_\zeta$ is T-complete.*

Proof. Let θ be a sufficiently large regular cardinal. Suppose that M is a countable elementary submodel of $\langle \mathcal{H}_\theta, \in, \mathcal{I}, T\rangle$ and that $\langle p_n \mid n \in \omega\rangle$ is an $(M, \mathbb{P}_\zeta)$-generic sequence. By Lemma 3.3 it suffices to show that $\langle p_n \mid n \in \omega\rangle$ has a lower bound. We use the following claim:

Claim . *Suppose that $\eta \in \zeta \cap M$. Then $\langle p_n \restriction \eta \mid n \in \omega\rangle$ is an $(M, \mathbb{P}_\eta)$-generic sequence. Moreover suppose that p^* is a lower bound of $\langle p_n \restriction \eta \mid n \in \omega\rangle$. Then p^* forces that $\langle p_n(\eta) \mid n \in \omega\rangle$ is an $(M[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)$-generic sequence, where $\dot{G}_\eta$ is the canonical name for $\mathbb{P}_\eta$-generic filter.*

Proof of Claim. First we prove the former. Clearly $\langle p_n \restriction \eta \mid n \in \omega\rangle$ is a descending sequence in $\mathbb{P}_\eta \cap M$. Take an arbitrary dense open subset $D \in M$ of $\mathbb{P}_\eta$. We must show that there exists $n \in \omega$ with $p_n \restriction \eta \in D$.

Note that the set $D' := \{p \in \mathbb{P}_\zeta \mid p \restriction \eta \in D\}$ is dense open in $\mathbb{P}_\zeta$ and belongs to M. Then by the $(M, \mathbb{P}_\zeta)$-genericity of $\langle p_n \mid n \in \omega\rangle$ there exists $n \in \omega$ with $p_n \in D'$. Then $p_n \restriction \eta \in D$ for such n.

Next we prove the latter. It suffices to show the genericity of $\langle p_n(\eta) \mid n \in \omega\rangle$. Take an arbitrary $\mathbb{P}_\eta$-name $\dot{D} \in M$ of a dense open subset of $\dot{\mathbb{Q}}_\eta$. We show that there exists $n \in \omega$ with $p^* \Vdash_\eta$ "$p_n(\eta) \in \dot{D}$".

It is easy to see that the set $D'' := \{p \in \mathbb{P}_\zeta \mid p \restriction \eta \Vdash_\eta \text{"}p(\eta) \in \dot{D}\text{"}\}$ is dense open in $\mathbb{P}_\zeta$ and belongs to M. Hence there exists $n \in \omega$ with $p_n \in D''$. Then $p^* \Vdash_\eta$ "$p_n(\eta) \in \dot{D}$" and $p^* \le p_n \restriction \eta$. □ (*Claim*)

Using the above claim we construct a lower bound p^* of $\langle p_n \mid n \in \omega\rangle$. p^* will be a function whose domain is ζ and such that $p^*(\eta)$ is a $\mathbb{P}_\eta$-name of a condition of $\dot{\mathbb{Q}}_\eta$ for each $\eta < \zeta$. By induction on $\eta < \zeta$ we choose $p^*(\eta)$. The following are the induction hypotheses:

(i) $p^* \restriction \eta \Vdash_\eta$ "$p^*(\eta)$ is a lower bound of $\langle p_n(\eta) \mid n \in \omega\rangle$".

(ii) $p^*(\eta) = \dot{1}_\eta$ for every $\eta \in \zeta \setminus M$.

(ii) assures that $\operatorname{supp} p^*$ is countable because M is countable. In general note that if $\eta \le \zeta$ and $p^*(\eta')$ has been chosen to satisfy the induction hypotheses for each $\eta' < \eta$ then $p^* \restriction \eta = \langle p^*(\eta') \mid \eta' < \eta\rangle$ is a lower bound of $\langle p_n \restriction \eta \mid n \in \omega\rangle$. Note also that $p^* \restriction \eta$ is an $(M, \mathbb{P}_\eta)$-generic condition because $\langle p_n \restriction \eta \mid n \in \omega\rangle$ is an $(M, \mathbb{P}_\eta)$-generic condition by Claim.

Now we describe the choice of $p^*(\eta)$. Suppose that $\eta < \zeta$ and $p^* \restriction \eta$ has been constructed. First suppose also that $\eta \notin M$. In this case let $p^*(\eta) = \dot{1}_\eta$. Note that $\operatorname{supp} p_n \subseteq M$ for each $n \in \omega$ because $\operatorname{supp} p_n$ is a countable set belonging to M and $M \prec \langle \mathcal{H}_\theta, \in\rangle$. Hence $p_n(\eta) = \dot{1}_\eta$ for each $n \in \omega$, and thus $p^*(\eta)$ satisfies the induction hypothesis (i).

Next suppose that $\eta \in M$. Let $\dot{G}_\eta$ be the canonical name for $\mathbb{P}_\eta$-generic filter. Then note that

$$p^* \restriction \eta \Vdash_\eta \text{ "} \langle p_n(\eta) \mid n \in \omega\rangle \text{ is an } (M[\dot{G}_\eta], \dot{\mathbb{Q}}_\eta)\text{-generic sequence"}$$

by Claim. Moreover

$$p^* \restriction \eta \Vdash_\eta \text{ "}M[\dot{G}_\eta] \prec \langle \mathcal{H}_\theta{}^{V[\dot{G}_\eta]}, \in, \dot{\mathbb{Q}}_\eta, T\rangle \ \wedge\ M[\dot{G}] \cap \lambda = M \cap \lambda \in T\text{"}$$

by Fact 2.1 and the fact that $p^* \restriction \eta$ is $(M, \mathbb{P}_\eta)$-generic. Hence $p^* \restriction \eta$ forces that $\langle p_n(\eta) \mid n \in \omega\rangle$ has a lower bound by T-completeness of $\dot{\mathbb{Q}}_\eta$. Let $p^*(\eta)$ be a $\mathbb{P}_\eta$-name of a lower bound of $\langle p_n(\eta) \mid n \in \omega\rangle$ in $\dot{\mathbb{Q}}_\eta$. Clearly the induction hypotheses are satisfied.

Now we could construct a lower bound p^* of $\langle p_n \mid n \in \omega\rangle$. This completes the proof. □

At the end of this section we present a condition for iterations to have ω_2-c.c. We use the following condition for forcing notions:

Definition 3.6. *A forcing notion $\mathbb{P}$ with the following properties is said to be good:*

(i) Every $p \in \mathbb{P}$ is a function such that $|p| = \omega$ and $\operatorname{ran} p \subseteq \omega_1$.

(ii) $p \leq q$ in $\mathbb{P}$ if and only if $p \supseteq q$.

(iii) For each $p, q \in \mathbb{P}$ if $p \restriction (\operatorname{dom} p \cap \operatorname{dom} q) = q \restriction (\operatorname{dom} p \cap \operatorname{dom} q)$ then p and q are compatible.

If $\mathbb{P}$ satisfies the following additional condition then we say that $\mathbb{P}$ is better:

(iv) If a descending sequence $\langle p_n \mid n \in \omega\rangle$ in $\mathbb{P}$ has a lower bound then $\bigcup_{n\in\omega} p_n \in \mathbb{P}$.

The standard argument using the Δ-system lemma shows that if CH holds then goodness implies the ω_2-c.c:

Lemma 3.7. *Every good forcing notion has the $(2^\omega)^+$-c.c.*

If CH holds and T is stationary then a countable support iteration of T-complete better forcing notions have the ω_2-c.c:

Lemma 3.8. *Let λ be an ordinal $\geq \omega_1$ and T be a stationary subset of $\mathcal{P}_{\omega_1}\lambda$. Suppose that $\mathcal{I} = \langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \mid \xi \leq \zeta, \eta < \zeta\rangle$, $\zeta \in \mathrm{On}$, is a countable support iteration of T-complete better forcing notions. Then $\mathbb{P}_\zeta$ has the $(2^\omega)^+$-c.c.*

Proof. We may assume that $\mathbb{P}_\eta$ forces that $\operatorname{dom} q \subseteq \mathrm{On}$ for each $q \in \dot{\mathbb{Q}}_\eta$. We may also assume that $\dot{1}_\eta = \check{\emptyset}$ for each $\eta < \zeta$. Outline of our proof is as follows: First we show that

$$D := \{p \in \mathbb{P}_\zeta \mid \forall \eta < \zeta \exists q \in V,\ p(\eta) = \check{q}\}$$

is dense in $\mathbb{P}_\zeta$. After that, we show that the forcing notion obtained by restricting $\mathbb{P}_\zeta$ to D is good. This together with Lemma 3.7 implies that $\mathbb{P}_\zeta$ has the $(2^\omega)^+$-c.c.

Now we start to show that D is dense in $\mathbb{P}_\zeta$. Take an arbitrary $p_0 \in \mathbb{P}_\zeta$. We find $p^* \leq p_0$ which is in D.

Let θ be a sufficiently large regular cardinal, and take a countable elementary submodel M of $\langle \mathcal{H}_\theta, \in, \mathcal{I}, T\rangle$ with $p_0 \in M$. We can take such M because T is stationary. Also, take an $(M, \mathbb{P}_\zeta)$-generic sequence $\langle p_n \mid n \in \omega\rangle$ below p_0. Our p^* will be a lower bound of $\langle p_n \mid n \in \omega\rangle$. The construction of p^* is based on that in the proof of Lemma 3.5.

By induction on $\eta < \zeta$ we choose a $\mathbb{P}_\eta$-name $p^*(\eta)$ of a condition of $\dot{\mathbb{Q}}_\eta$. The induction hypotheses are the same as (i) and (ii) in the proof of Lemma 3.5.

Suppose that $\eta < \zeta$ and that $p^* \restriction \eta$ has been constructed. If $\eta \notin M$ then let $p^*(\eta) = \dot{1}_\eta = \check{\emptyset}$ as in the Proof of Lemma 3.5. Then suppose that $\eta \in M$. In this case we claim the following:

Claim . *For each $n \in \omega$ there exists $q_n \in V$ such that $p^* \restriction \eta \Vdash_\eta$ "$\dot{p}_n(\eta) = \check{q}_n$ ".*

Proof of Claim. Fix $n \in \omega$. First note that $\mathbb{P}_\eta$ is ω-distributive by Lemma 3.4 and 3.5. Hence the set

$$B = \{p \in \mathbb{P}_\eta \mid \exists q \in V,\ p \Vdash_\eta \text{"} p_n(\eta) = \check{q} \text{"}\}$$

is a dense open subset of $\mathbb{P}_\eta$. Moreover $B \in M$.

Then there exists $m \in \omega$ with $p_m \in B$ by the $(M, \mathbb{P}_\eta)$-genericity of $\langle p_m \restriction \eta \mid m \in \omega \rangle$ (See Claim in the proof of Lemma 3.5). Then $p^* \restriction \eta \in B$ because p^* is a lower bound of $\langle p_m \mid m \in \omega \rangle$. Therefore there exists $q_n \in V$ such that $p^* \restriction \eta$ forces that $p_n(\eta) = \check{q}_n$. □ (*Claim*)

Let q_n be as in the above claim for each $n \in \omega$, and let q^* be $\bigcup_{n \in \omega} q_n$. Here the same argument as in the proof of Lemma 3.5 shows that $p^* \restriction \eta$ forces that $\langle p_n(\eta) \mid n \in \omega \rangle$ has a lower bound in $\dot{Q}_\eta$. Then $p^* \restriction \eta$ forces that $\check{q}^*$ is a lower bound of $\langle p_n(\eta) \mid n \in \omega \rangle$ by betterness of $\dot{\mathbb{Q}}_\eta$. Let $p^*(\eta)$ be $\check{q}^*$.

Now we have constructed p^*. It follows from the construction of p^* that $p^* \leq p_0$ and $p^* \in D$. This completes the proof of the density of D.

Below, for each $p \in D$ and each $\eta < \zeta$, we let $p(\eta)$ denote $q \in V$ such that $p(\eta) = \check{q}$. Note that $p(\eta)$ is a countable function from On to ω_1 by the ω-distributivity of $\mathbb{P}_\eta$.

For each $p \in D$ let $\hat{p}$ be the partial function from $\zeta \times \mathrm{On}$ to ω_1 such that

- $\operatorname{dom} \hat{p} = \{\langle \eta, \alpha \rangle \mid \alpha \in \operatorname{dom} p(\eta)\}$,
- $\hat{p}(\eta, \alpha) = p(\eta)(\alpha)$ for each $\langle \eta, \alpha \rangle \in \operatorname{dom} \hat{p}$.

Then let $\hat{\mathbb{P}}$ be the forcing notion $\{\hat{p} \mid p \in D\}$ ordered by reverse inclusions.

It is easy to see that $\hat{\mathbb{P}}$ is good. Hence $\hat{\mathbb{P}}$ has the $(2^\omega)^+$-c.c. It is also easy to check that $\hat{\mathbb{P}}$ is isomorphic to the forcing notion obtained by restricting $\mathbb{P}_\zeta$ to D. Therefore $\mathbb{P}_\zeta$ has the $(2^\omega)^+$-c.c. because D is dense in $\mathbb{P}_\zeta$.

This completes the proof of the lemma. □

4 Sup depending stationary set

In the proof of Theorem 1.2 and 1.3 we use the following lemma due to Shelah:

Lemma 4.1 (Shelah). *Suppose that $\langle E_i \mid i < \omega_1 \rangle$ is a sequence of stationary subsets of E^2_0. Then the set*

$$T := \{x \in \mathcal{P}_{\omega_1}\omega_2 \mid x \cap \omega_1 \in \omega_1 \ \wedge\ \sup x \notin x \ \wedge\ \sup x \in E_{x \cap \omega_1}\}$$

is stationary in $\mathcal{P}_{\omega_1}\omega_2$.

Variants of this lemma are used in Shelah [6] and Shelah-Shioya [7] to obtain consequences of the stationary reflection principle. Here we present the proof of the above lemma for the completeness of this paper. We use a two players' game of length ω.

For $f : [\omega_2]^{<\omega} \to \omega_2$ and $i \in \omega_1$ let $\Game(f, i)$ be the following two players' game of length ω:

BAD	α_0	α_1	α_2	$\cdots$	α_n	$\cdots$
GOOD	β_0	β_1	β_2	$\cdots$	β_n	$\cdots$

In the n-th stage, first BAD chooses $\alpha_n < \omega_2$ and then GOOD chooses β_n with $\alpha_n \leq \beta_n < \omega_2$. GOOD wins if

$$\mathrm{cl}_f(i \cup \{\beta_n \mid n \in \omega\}) \cap \omega_1 = i ,$$

where $\mathrm{cl}_f(x)$ denotes the closure of x under f. Otherwise BAD wins.

Note that $\Game(f, i)$ is an open game for BAD and thus it is determined. We claim the following:

Lemma 4.2. *For every $f : [\omega_2]^{<\omega} \to \omega_2$ there exists $i \in \omega_1$ such that* GOOD *has a winning strategy in $\Game(f, i)$.*

Proof. On the contrary, assume that f is a function from $[\omega_2]^{<\omega}$ to ω_2 and that there are no $i \in \omega_1$ such that GOOD has a winning strategy in $\Game(f, i)$. Then there exists a winning strategy σ_i for BAD in $\Game(f, i)$ for every $i \in \omega_1$. Let $\vec{\sigma} := \langle \sigma_i \mid i \in \omega_1 \rangle$.

Let θ be a sufficiently large regular cardinal, and let M be a countable elementary submodel of $\langle \mathcal{H}_\theta, \in, f, \vec{\sigma} \rangle$. Note that $i^* := M \cap \omega_1 \in \omega_1$.

By induction on $n \in \omega$ we take $\alpha_n, \beta_n \in \omega_2$ so that $\beta_n \in M$. Suppose that $n \in \omega$ and that $\langle \alpha_m, \beta_m \mid m < n \rangle$ has been taken. Then let

$$\begin{aligned} \alpha_n &:= \sigma_{i^*}(\langle \beta_m \mid m < n \rangle) \\ \beta_n &:= \sup\{\sigma_i(\langle \beta_m \mid m < n \rangle) \mid i \in \omega_1\} \end{aligned}$$

Clearly $\alpha_n \leq \beta_n < \omega_2$. Moreover $\beta_n \in M$ because $\{\beta_m \mid m < n\} \subseteq M \prec \langle \mathcal{H}_\theta, \in, \vec{\sigma} \rangle$.

Now $\langle \alpha_n, \beta_n \mid n \in \omega \rangle$ is a sequence of moves in $\Game(f, i^*)$ in which BAD has played according to the winning strategy σ_{i^*}. Hence BAD wins with this moves.

On the other hand $\mathrm{cl}_f(i^* \cup \{\beta_n \mid n \in \omega\}) \subseteq M$ because M is closed under f and $i^* \cup \{\beta_n \mid n \in \omega\} \subseteq M$. Thus $\mathrm{cl}_f(i^* \cup \{\beta_n \mid n \in \omega\}) \cap \omega_1 = i^*$, that is, GOOD wins with the moves $\langle \alpha_n, \beta_n \mid n \in \omega \rangle$.

This is a contradiction. □

Now we can prove Lemma 4.1:

Proof of Lemma 4.1. Take an arbitrary function $f : [\omega_2]^{<\omega} \to \omega_2$. We find $x^* \in T$ closed under f.

By Lemma 4.2 take $i^* \in \omega$ such that GOOD has a winning strategy σ^* in $\Game(f, i^*)$. Let θ be a sufficiently large regular cardinal, and let M be an uncountable elementary submodel of $\langle \mathcal{H}_\theta, \in, f, \sigma^* \rangle$ such that $M \cap \omega_2 \in E_{i^*} \setminus \omega_1$. Note that $\omega_1 \subseteq M \cap \omega_2 \in \omega_2$.

Take an increasing sequence $\langle \alpha_n \mid n \in \omega \rangle$ converging to $M \cap \omega_2$, and let $\beta_n := \sigma^*(\langle \alpha_m \mid m \le n \rangle) \in M$ for each $n \in \omega$. Moreover let

$$x^* := \mathrm{cl}_f(i^* \cup \{\beta_n \mid n \in \omega\}) \,.$$

It suffices to show that $x^* \in T$.

First note that $\sup x^* \ge \sup_{n\in\omega} \beta_n \ge \sup_{n\in\omega} \alpha_n = M \cap \omega_2$. On the other hand, $x^* \subseteq M$ because $i^* \cup \{\beta_n \mid n \in \omega\} \subseteq M$ and M is closed under f. Hence $\sup x^* \le M \cap \omega_2$. Therefore $\sup x^* = M \cap \omega_2 \in E_{i^*}$. Moreover $\sup x^* \notin x^*$.

Note also that $\langle \alpha_n, \beta_n \mid n \in \omega \rangle$ is a sequence of moves in $\Game(f, i^*)$ in which GOOD has played according to the winning strategy σ^*. Hence $x^* \cap \omega_1 = i^*$.

Therefore $x^* \cap \omega_1 \in \omega_1$, $\sup x^* \notin x^*$ and $\sup x^* \in E_{x^* \cap \omega_1}$, that is, $x^* \in T$. □

5 Proof of Theorem 1.2 and 1.3

Here we prove Theorem 1.2 and 1.3. In fact we prove slightly more.

To state our result we introduce the following subsets of $\mathcal{P}_{\omega_1}\omega_2$ for a $\square_{\omega_1}$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in \operatorname{Lim} \omega_2 \rangle$:

$S_0^{\vec{c}}$:= the set of all $x \in \mathcal{P}_{\omega_1}\omega_2$ such that

(i) $x \cap \omega_1 \in \omega_1$ and $\sup x \notin x$,

(ii) $\text{o.t.}\, c_{\sup x} < x \cap \omega_1$,

(iii) $c_{\sup x} \subseteq x$.

$S_1^{\vec{c}}$:= the set of all $x \in \mathcal{P}_{\omega_1}\omega_2$ such that

(i) $x \cap \omega_1 \in \omega_1$ and $\sup x \notin x$,

(ii) $\text{o.t.}\, c_{\sup x} = x \cap \omega_1$,

(iii) $c_{\sup x} \subseteq x$.

The difference between $S_0^{\vec{c}}$ and $S_1^{\vec{c}}$ is the property (ii) of their elements. As we see in the following lemma, these sets have maximality properties with respect to the stationary reflection. Note that the following lemma implies that (every subsets of) $\mathcal{P}_{\omega_1}\omega_2 \setminus S_0^{\vec{c}}$ does not reflect to any ordinal in E_0^2 and that (every subset of) $\mathcal{P}_{\omega_1}\omega_2 \setminus S_1^{\vec{c}}$ does not reflect to any ordinal in E_1^2:

Lemma 5.1. *Let $\vec{c} = \langle c_\alpha \mid \alpha \in \operatorname{Lim} \omega_2 \rangle$ be a $\square_{\omega_1}$-sequence. Then the following holds:*

(1) $S_0^{\vec{c}} \cap \mathcal{P}_{\omega_1}\alpha$ contains a club in $\mathcal{P}_{\omega_1}\alpha$ for every $\alpha \in E_0^2 \setminus \omega_1$.

(2) $S_1^{\vec{c}} \cap \mathcal{P}_{\omega_1}\alpha$ contains a club in $\mathcal{P}_{\omega_1}\alpha$ for every $\alpha \in E_1^2$.

In particular both $S_0^{\vec{c}}$ and $S_1^{\vec{c}}$ are stationary in $\mathcal{P}_{\omega_1}\omega_2$.

Proof. (1) Suppose that $\alpha \in E^2_0 \setminus \omega_1$. Note that $\text{o.t.}\, c_\alpha$ is countable. Let C be the set of all $x \in \mathcal{P}_{\omega_1}\alpha$ such that $c_\alpha \subseteq x$ and $\text{o.t.}\, c_\alpha < x \cap \omega_1 \in \omega_1$. Then C is a club in $\mathcal{P}_{\omega_1}\alpha$, and $C \subseteq S^{\vec{c}}_0$.

(2) Suppose that $\alpha \in E^2_1$. Let $\langle \beta_i \mid i < \omega_1 \rangle$ be the increasing enumeration of c_α. Let C be the set of all $x \in \mathcal{P}_{\omega_1}\alpha$ such that $x \cap \omega_1$ is a countable limit ordinal, $\sup x = \beta_{x\cap\omega_1} \notin x$ and $\{\beta_i \mid i \in x \cap \omega_1\} \subseteq x$. Then it is easy to see that C is a club in $\mathcal{P}_{\omega_1}\alpha$.

We claim that $C \subseteq S^{\vec{c}}_1$. Note that if $x \in C$ then

$$c_{\sup x} = c_{\beta_{x\cap\omega_1}} = \{\beta_i \mid i \in x \cap \omega_1\}$$

by the coherency of $\vec{c}$. Hence if $x \in C$ then $c_{\sup x} \subseteq x$ and $\text{o.t.}\, c_{\sup x} = x \cap \omega_1$. Therefore $C \subseteq S^{\vec{c}}_1$. □

We prove the following:

Theorem 5.2. *Assume that* GCH *and* $\square_{\omega_1}$ *holds. Let* $\vec{c}$ *be a* $\square_{\omega_1}$*-sequence. Then there exists an* ω_2*-c.c.* ω*-distributive forcing extension in which* $\mathsf{SR}_k(S^{\vec{c}}_k)$ *holds for both* $k = 0, 1$.

In the above theorem note that both $S^{\vec{c}}_0$ and $S^{\vec{c}}_1$ are absolute between the ground model and the forcing extension because the extension preserves all cardinals and adds no new countable subsets of ordinals.

The extension of the above theorem will be obtained by making all nonreflecting stationary subsets of $S^{\vec{c}}_0$ and $S^{\vec{c}}_1$ nonstationary by a countable support iteration of club shootings.

First we describe the club shooting used in each stage:

Definition 5.3. *Let* S *be a subset of* $\mathcal{P}_{\omega_1}\omega_2$. *Then let* $\mathbb{C}(S)$ *be the forcing notion consisting of all* p *such that*

(i) p *is a function from* $d \times d$ *to* ω_1,

(ii) if $x \in S$ *and* $x \subseteq d$ *then* x *is not closed under* p.

for some $d \in \mathcal{P}_{\omega_1}\omega_2$. $p \leq q$ *if and only if* $p \supseteq q$ *for each* $p, q \in \mathbb{C}(S)$. *For each* $p \in \mathbb{C}(S)$ *we let* d_p *denote* $d \in \mathcal{P}_{\omega_1}\omega_2$ *satisfying (i) and (ii) above.*

Below we present easy facts on $\mathbb{C}(S)$:

Lemma 5.4. *Let* S *be a subset of* $\mathcal{P}_{\omega_1}\omega_2$.

(1) For every $y \in \mathcal{P}_{\omega_1}\omega_2$ *the set* $\{p \in \mathbb{C}(S) \mid y \subseteq d_p\}$ *is dense in* $\mathbb{C}(S)$.

(2) Suppose that G *is a* $\mathbb{C}(S)$*-generic filter over* V. *Then* $\bigcup G$ *is a total function from* ${\omega_2}^V \times {\omega_2}^V$ *to* ${\omega_1}^V$, *and there are no* $x \in S$ *closed under* $\bigcup G$.

(3) $\mathbb{C}(S)$ *is better.*

Proof. (1) Take an arbitrary $y \in \mathcal{P}_{\omega_1}\omega_2$ and an arbitrary $p \in \mathbb{C}(S)$. We must find $p^* \le p$ with $y \subseteq d_{p^*}$.

Let d^* be $d_p \cup y$, and take $\gamma \in \omega_1 \setminus d^*$. Then let p^* be a function from $d^* \times d^*$ to ω_1 defined as follows:

$$p^*(a) = \begin{cases} p(a) & \cdots \quad \text{if } a \in d_p \times d_p \\ \gamma & \cdots \quad \text{otherwise} \end{cases}$$

All we have to show is that if $x \in S$ and $x \subseteq d^*$ then x is not closed under p^*. This implies that p^* is a condition in $\mathbb{C}(S)$ below p and that $y \subseteq d_{p^*} = d^*$.

Suppose that $x \in S$ and $x \subseteq d^*$. First consider the case when $x \subseteq d_p$. In this case x is not closed under p because $p \in \mathbb{C}(S)$. Hence x is not closed under p^* which extends p. Next consider the case when $x \not\subseteq d_p$. In this case there exists $a \in (x \times x) \setminus (d_p \times d_p)$. Then $p^*(a) = \gamma \notin d^* \supseteq x$, and thus $p^*(a) \notin x$. Therefore x is not closed under p^*.

(2) Clear from (1).

(3) Clearly $\mathbb{C}(S)$ satisfies the properties (i) and (ii) in Definition 3.6. We check that $\mathbb{C}(S)$ satisfies (iii) and (iv).

First we check (iii). Suppose that $p, q \in \mathbb{C}(S)$ and that $p \restriction (\operatorname{dom} p \cap \operatorname{dom} q) = q \restriction (\operatorname{dom} p \cap \operatorname{dom} q)$. We must find a common extension p^* of p and q.

Let d^* be $d_p \cup d_q$, and take $\gamma \in \omega_1 \setminus d^*$. Then let p^* be a function from $d^* \times d^*$ to ω_1 defined as follows:

$$p^*(a) = \begin{cases} p(a) & \cdots \quad \text{if } a \in d_p \times d_p \\ q(a) & \cdots \quad \text{if } a \in d_q \times d_q \\ \gamma & \cdots \quad \text{otherwise} \end{cases}$$

p^* is well-defined because p and q coincide on $\operatorname{dom} p \cap \operatorname{dom} q$. All we have to show is that if $x \in S$ and $x \subseteq d^*$ then x is not closed under p^*.

Suppose that $x \in S$ and $x \subseteq d^*$. If $x \subseteq d_p$ then the same argument as in the proof of (1) shows that x is not closed under p and thus that x is not closed under p^*. Similarly, if $x \subseteq d_q$ then x is not closed under q, and hence x is not closed under p^*.

So suppose that $x \not\subseteq d_p$ and $x \not\subseteq d_q$. In this case take an $\alpha \in x \setminus d_p$ and an $\beta \in x \setminus d_q$, and let $a := \langle \alpha, \beta \rangle$. Then $a \in x \times x$ but $a \notin d_p \times d_p$ and $a \notin d_q \times d_q$. Hence $p^*(a) = \gamma \notin x$. Therefore x is not closed under p^*.

Next we check (iv). Suppose that $\langle p_n \mid n \in \omega \rangle$ is a descending sequence in $\mathbb{C}(S)$ which has a lower bound. Let p^* be a lower bound of $\langle p_n \mid n \in \omega \rangle$.

Then $\bigcup_{n \in \omega} p_n$ is a restriction of p^* to $(\bigcup_{n \in \omega} d_{p_n}) \times (\bigcup_{n \in \omega} d_{p_n})$. From this it is clear that $\bigcup_{n \in \omega} p_n \in \mathbb{C}(S)$. □

Club shootings which we iterate will be T-complete for some stationary $T \subseteq \mathcal{P}_{\omega_1}\omega_2$. Here we present a sufficient condition for $\mathbb{C}(S)$ to be T-complete:

Definition 5.5. *For $S, T \subseteq \mathcal{P}_{\omega_1}\omega_2$ let $\Phi(S,T)$ be the following principle:*

There exist a regular cardinal $\theta > 2^{\omega_2}$ and an expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}_\theta, \in \rangle$ such that if M is a countable elementary submodel of $\mathcal{M}$ with $M \cap \omega_2 \in T$ then $S \cap \mathcal{P}(M) \subseteq M$.

While we do not use, the standard argument shows that $\Phi(S,T)$ is equivalent with the following:

If θ is a sufficiently large regular cardinal, and M is a countable elementary submodel of $\langle \mathcal{H}_\theta, \in, S, T \rangle$ with $M \cap \omega_2 \in T$ then $S \cap \mathcal{P}(M) \subseteq M$.

Now we prove that $\Phi(S,T)$ is a sufficient condition for $\mathbb{C}(S)$ to be T-complete:

Lemma 5.6. *Suppose that $S, T \subseteq \mathcal{P}_{\omega_1}\omega_2$ and that $\Phi(S,T)$ holds. Then $\mathbb{C}(S)$ is T-complete.*

Proof. Let θ and $\mathcal{M}$ be witnesses of $\Phi(S,T)$. Suppose that M is a countable elementary submodel of $\mathcal{M}$ with $M \cap \omega_2 \in T$ and that $\langle p_n \mid n \in \omega \rangle$ is an $(M, \mathbb{C}(S))$-generic sequence. By Lemma 3.3 it suffices to show that $\langle p_n \mid n \in \omega \rangle$ has a lower bound. Moreover it suffices for this to show that $p^* := \bigcup_{n\in\omega} p_n$ is a condition in $\mathbb{C}(S)$.

Let d^* be $\bigcup_{n\in\omega} d_{p_n}$. Then $d^* \in \mathcal{P}_{\omega_1}\omega_2$, and p^* is a function from $d^* \times d^*$ to ω_1. We show that if $x \in S$ and $x \subseteq d^*$ then x is not closed under p^*.

Suppose that $x \in S$ and $x \subseteq d^*$. First note that $d_{p_n} \subseteq M$ for each $n \in \omega$ because d_{p_n} is a countable set which belongs to $M \prec \langle \mathcal{H}_\theta, \in \rangle$. Hence $d^* \subseteq M$, and so $x \subseteq M$. Thus $x \in M$ by $\Phi(S,T)$.

Then the set $D := \{p \in \mathbb{C}(S) \mid x \subseteq d_p\}$ belongs to M. Moreover D is dense open in $\mathbb{C}(S)$ by Lemma 5.4 (1). Hence there exists $n \in \omega$ with $p_n \in D$. Then $x \subseteq d_{p_n}$, and x is not closed under p_n because $p_n \in \mathbb{C}(S)$. Therefore x is not also closed under p^* which extends p_n. □

Next we present a stationary $T \subseteq \mathcal{P}_{\omega_1}\omega_2$ such that club shootings which we iterate will be T-complete. For a $\square_{\omega_1}$-sequence $\vec{c} = \langle c_\alpha \mid \alpha \in \operatorname{Lim}\omega_2 \rangle$ let

$$
\begin{aligned}
T^{\vec{c}} := {} & \text{the set of all } x \in \mathcal{P}_{\omega_1}\omega_2 \text{ such that} \\
& \text{(i) } x \cap \omega_1 \in \omega_1 \text{ and } \sup x \notin x, \\
& \text{(ii) } \text{o.t.}\, c_{\sup x} > x \cap \omega_1 .
\end{aligned}
$$

The main difference of $T^{\vec{c}}$ from $S^{\vec{c}}_0$ and $S^{\vec{c}}_1$ is the property (ii) of its elements. It is easy to see that $T^{\vec{c}}$ is stationary using Lemma 4.1:

Lemma 5.7. *$T^{\vec{c}}$ is stationary in $\mathcal{P}_{\omega_1}\omega_2$ for every $\square_{\omega_1}$-sequence $\vec{c}$.*

Proof. Suppose that $\vec{c} = \langle c_\alpha \mid \alpha \in \operatorname{Lim}\omega_2 \rangle$ is a $\square_{\omega_1}$-sequence.

For each $i \in \omega_1$ let $E_i := \{\alpha \in E^2_0 \mid \text{o.t.}\, c_\alpha > i\}$. Note that $E_i \cap \beta$ contains a club in β for every $\beta \in E^2_1$. Hence E_i is a stationary subset of E^2_0.

Here note also that

$$T^{\vec{c}} = \{x \in \mathcal{P}_{\omega_1}\omega_2 \mid x \cap \omega_1 \in \omega_1 \ \wedge\ \sup x \notin x \ \wedge\ \sup x \in E_{x\cap\omega_1}\} .$$

Therefore $T^{\vec{c}}$ is stationary by Lemma 4.1. □

We want to show something like that if S is a nonreflecting subset of $S_0^{\vec{c}}$ or $S_1^{\vec{c}}$ then $\mathbb{C}(S)$ is $T^{\vec{c}}$-complete. For this we slightly reduce $S_0^{\vec{c}}$ and $S_1^{\vec{c}}$ as follows:

We call a sequence $\vec{\pi} = \langle \pi_\alpha \mid \alpha \in \omega_2 \rangle$ a *surjection system* if π_α is a surjection from ω_1 to α for each $\alpha \in \omega_2$. For a $\square_{\omega_1}$-sequence $\vec{c}$, a surjection system $\vec{\pi} = \langle \pi_\alpha \mid \alpha \in \omega_2 \setminus \omega_1 \rangle$ and $k = 0, 1$ let

$$S_k^{\vec{c},\vec{\pi}} := \{x \in S_k^{\vec{c}} \mid \forall \alpha \in x,\ x \cap \alpha = \pi_\alpha``(x \cap \omega_1)\} .$$

Note that $S_k^{\vec{c}} \setminus S_k^{\vec{c},\vec{\pi}}$ is nonstationary.

We claim the following.

Lemma 5.8. *Suppose that $\vec{c} = \langle c_\alpha \mid \alpha \in \operatorname{Lim} \omega_2 \rangle$ is a $\square_{\omega_1}$-sequence and that $\vec{\pi} = \langle \pi_\alpha \mid \alpha \in \omega_2 \setminus \omega_1 \rangle$ is a surjection system.*

(1) Let S be a subset of $S_0^{\vec{c},\vec{\pi}}$ which does not reflect to any ordinal in $E_0^2 \setminus \omega_1$. Then $\mathbb{C}(S)$ is $T^{\vec{c}}$-complete.

(2) Let S be a subset of $S_1^{\vec{c},\vec{\pi}}$ which does not reflect to any ordinal in E_1^2. Then $\mathbb{C}(S)$ is $T^{\vec{c}}$-complete.

To prove Lemma 5.8 we need the following easy lemma:

Lemma 5.9. *Suppose that $\vec{c} = \langle c_\alpha \mid \alpha \in \operatorname{Lim} \omega_2 \rangle$ is a $\square_{\omega_1}$-sequence. Let θ be a sufficiently large regular cardinal and M be a countable elementary submodel of $\langle \mathcal{H}_\theta, \in, \vec{c} \rangle$. Moreover let α^* be an ordinal in E_0^2 such that $\alpha^* < \sup(M \cap \omega_2)$, $\alpha^* \notin M$ and $\sup(M \cap \alpha^*) = \alpha^*$. Then $\text{o.t.}\, c_{\alpha^*} = M \cap \omega_1$.*

Proof. Let $\beta^* := \min(M \setminus \alpha^*)$. Then $\beta^* \in M \cap \omega_2$, and $\sup(M \cap \beta^*) = \alpha^* < \beta^*$. Moreover it easily follows from the elementarity of M that $\beta^* \in E_1^2$. Let $\langle \beta_i \mid i \in \omega_1 \rangle$ be the increasing enumeration of c_{β^*}. We claim that $\sup(M \cap \beta^*) = \beta_{M \cap \omega_1}$.

First note that $c_{\beta^*} \in M$ by the elementarity of M. Hence $\{\beta_i \mid i \in M \cap \omega_1\} \subseteq M$. Thus

$$\sup(M \cap \beta^*) \geq \sup\{\beta_i \mid i \in M \cap \omega_1\} = \beta_{M \cap \omega_1} .$$

On the other hand assume that $\sup(M \cap \beta^*) > \beta_{M \cap \omega_1}$. Then we can take $\beta \in M \cap \beta^*$ with $\beta \geq \beta_{M \cap \omega_1}$. Let j be the least ordinal $< \omega_1$ such that $\beta_j \geq \beta$. Then $j \geq M \cap \omega_1$ because $\beta \geq \beta_{M \cap \omega_1}$. But $j \in M \cap \omega_1$ by the elementarity of M. This is a contradiction. Therefore $\sup(M \cap \beta^*) \leq \beta_{M \cap \omega_1}$.

Now we have shown that $\sup(M \cap \beta^*) = \beta_{M \cap \omega_1}$. Recall that $\alpha^* = \sup(M \cap \beta^*)$. Hence $\alpha^* = \beta_{M \cap \omega_1}$. Then $c_{\alpha^*} = \{\beta_i \mid i \in M \cap \omega_1\}$ by the coherency of $\vec{c}$. Therefore $\text{o.t.}\, c_{\alpha^*} = M \cap \omega_1$. □

Now we prove Lemma 5.8:

Proof of Lemma 5.8. For simplicity of our notation let S_0, S_1 and T denote $S_0^{\vec{c},\vec{\pi}}$, $S_1^{\vec{c},\vec{\pi}}$ and $T^{\vec{c}}$ respectively.

(1) By Lemma 5.6 it suffices to show that $\Phi(S, T)$ holds. Let θ be a sufficiently large regular cardinal, and let M be a countable elementary submodel of $\langle \mathcal{H}_\theta, \in, S, \vec{c}, \vec{\pi} \rangle$ with $M \cap \omega_2 \in T$. Moreover suppose that $x \in S$ and $x \subseteq M$. We show that $x \in M$. Before starting note that $x \cap \omega_1 \leq M \cap \omega_1 \in \omega_1$.

First we claim the following:

Claim 1. $\sup x \in M$.

Proof of Claim. On the contrary assume that $\sup x \notin M$. Then note that $M \cap \omega_1 \leq \text{o.t.}\, c_{\sup x}$: If $\sup x = \sup(M \cap \omega_2)$ then $M \cap \omega_1 < \text{o.t.}\, c_{\sup x}$ because $M \cap \omega_2 \in T$. On the other hand, if $\sup x < \sup(M \cap \omega_2)$ then $M \cap \omega_1 = \text{o.t.}\, c_{\sup x}$ by Lemma 5.9.

Note also that $x \cap \omega_1 > \text{o.t.}\, c_{\sup x}$ because $x \in S_0$. Hence $M \cap \omega_1 \leq \text{o.t.}\, c_{\sup x} < x \cap \omega_1$. This contradicts that $x \subseteq M$. $\square$(*Claim*)

Next we claim the following:

Claim 2. $x \cap \omega_1 < M \cap \omega_1$.

Proof of Claim. Assume not. Then $x \cap \omega_1 = M \cap \omega_1$. First note that $M \cap \alpha = \pi_\alpha``(M \cap \omega_1)$ for each $\alpha \in M \cap \omega_2$ by the elementarity of M. Hence

$$M \cap \sup x \;=\; \bigcup_{\alpha \in x} \pi_\alpha``(M \cap \omega_1) \;=\; \bigcup_{\alpha \in x} \pi_\alpha``(x \cap \omega_1) \;=\; x\,.$$

The last equality follows from $x \in S_0$.

Here note that $S \cap \mathcal{P}_{\omega_1}(\sup x)$ is nonstationary by the assumption on S. Moreover $\sup x \in M \prec \langle \mathcal{H}_\theta, \in, S\rangle$ by Claim 1. Hence there exists a function $f \in M$ from $[\sup x]^{<\omega}$ to $\sup x$ such that every element of $S \cap \mathcal{P}_{\omega_1}(\sup x)$ is not closed under f. But $x = M \cap \sup x$, and so x is closed under f by the elementarity of M. Because $x \in S \cap \mathcal{P}_{\omega_1}(\sup x)$ this is a contradiction. $\square$(*Claim*)

Now $x = \bigcup\{\pi_\alpha``(x \cap \omega_1) \mid \alpha \in c_{\sup x}\}$ because $x \in S_0$. Hence x is definable in $\langle \mathcal{H}_\theta, \in, \vec{c}, \vec{\pi}\rangle$ from the parameters $x \cap \omega_1$ and $\sup x$. But both $x \cap \omega_1$ and $\sup x$ belong to M by Claim 1 and 2, and $M \prec \langle \mathcal{H}_\theta, \in, \vec{c}, \vec{\pi}\rangle$. Therefore $x \in M$.

(2) We show that $\Phi(S, T)$ holds. Let θ, M and x be as in the proof of (1). We show that $x \in S$.

First we claim the following:

Claim 3. $\sup x \in M$.

Proof of Claim. First note that $\sup x < \sup(M \cap \omega_2)$: Otherwise $\sup x = \sup(M \cap \omega_2)$, and

$$M \cap \omega_1 < \text{o.t.}\, c_{\sup x} = x \cap \omega_1$$

because $M \cap \omega_2 \in T$ and $x \in S_1$. This contradicts that $x \subseteq M$.

Now assume that $\sup x \notin M$. Then $M \cap \omega_1 = \text{o.t.}\, c_{\sup x}$ by Lemma 5.9. Hence $M \cap \omega_1 = x \cap \omega_1$ because $x \in S_1$. Then the same argument as in the proof of Claim 2 shows that $M \cap \sup x = x$.

Let β^* be $\min(M \setminus \sup x)$. Then $\beta^* \in E^2_1$, and thus $S \cap \mathcal{P}_{\omega_1}\beta^*$ is nonstationary by the assumption on S. Because $\beta^* \in M \prec \langle \mathcal{H}_\theta, \in, S\rangle$ there exists a function $f \in M$ from $[\beta^*]^{<\omega}$ to β^* such that every element of $S \cap \mathcal{P}_{\omega_1}\beta^*$ is not closed under f. But $x = M \cap \sup x = M \cap \beta^*$, and so x is closed under f by the elementarity of M. This contradicts that $x \in S$. $\square$(*Claim*)

Note that $x \cap \omega_1 = \text{o.t.}\, c_{\sup x} \in M \cap \omega_1$ by Claim 3 and the elementarity of M. The rest of the proof is similar as (1).

First $x = \bigcup\{\pi_\alpha``(x \cap \omega_1) \mid \alpha \in c_{\sup x}\}$, and thus x is definable in $\langle \mathcal{H}_\theta, \in, \vec{c}, \vec{\pi}\rangle$ from the parameters $\sup x$ and $x \cap \omega_1$. Moreover both $\sup x$ and $x \cap \omega_1$ belongs to M, and $M \prec \langle \mathcal{H}_\theta, \in, \vec{c}, \vec{\pi}\rangle$. Therefore $x \in M$. □

Now we can prove Theorem 5.2 by combining lemmata above:

Proof of Theorem 5.2. Take a surjection system $\vec{\pi}$ in V. We make all nonreflecting subsets of $S_0^{\vec{c},\vec{\pi}}$ and $S_1^{\vec{c},\vec{\pi}}$ nonstationary by a countable support iteration of club shootings.

First note that $S_k^{\vec{c},\vec{\pi}}$ and $T^{\vec{c}}$ are absolute in all ω_2-c.c. ω-distributive forcing extensions of V. Let S_0, S_1 and T denote $S_0^{\vec{c},\vec{\pi}}$, $S_1^{\vec{c},\vec{\pi}}$ and $T^{\vec{c}}$ respectively. Note also that $|\mathbb{C}(S)| = \omega_2$ for every $S \subseteq \mathcal{P}_{\omega_1}\omega_2$ in all such extensions.

Then, by Lemmata 3.4, 3.5, 3.8, 5.4, 5.8, by GCH and by the standard book keeping method, we can construct a countable support iteration $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta \mid \xi \leq \omega_3, \eta < \omega_3\rangle$ with the following properties:

(i) $\mathbb{P}_\xi$ has the ω_2-c.c. and is ω-distributive for each $\xi \leq \omega_3$.

(ii) If $\eta < \omega_3$ then $\Vdash_\eta$ "$\dot{\mathbb{Q}}_\eta = \mathbb{C}(\dot{S})$" for some $\mathbb{P}_\eta$-name $\dot{S}$ such that either

$$\Vdash_\eta \text{“}\dot{S} \subseteq S_0 \ \wedge\ \dot{S} \text{ does not reflect to any ordinal in } E_0^2\text{”}\ ,$$

or

$$\Vdash_\eta \text{“}\dot{S} \subseteq S_1 \ \wedge\ \dot{S} \text{ does not reflect to any ordinal in } E_1^2\text{”}\ .$$

Hence $\Vdash_\eta$ "$\dot{\mathbb{Q}}_\eta$ is T-complete and better $\wedge$ $|\dot{\mathbb{Q}}_\eta| \leq \omega_2$".

(iii) If $\xi < \omega_3$ and $\dot{S}$ is a $\mathbb{P}_\xi$-name such that either

$$\Vdash_\xi \text{“}\dot{S} \subseteq S_0 \ \wedge\ \dot{S} \text{ does not reflect to any ordinal in } E_0^2\text{”}$$

or

$$\Vdash_\xi \text{“}\dot{S} \subseteq S_1 \ \wedge\ \dot{S} \text{ does not reflect to any ordinal in } E_1^2\text{”}$$

then there exists $\eta \in \omega_3 \setminus \xi$ such that $\Vdash_\eta$ "$\dot{\mathbb{Q}}_\eta = \mathbb{C}(\dot{S})$".

Then $\mathbb{P}_{\omega_3}$ has the ω_2-c.c. and is ω-distributive. Let G be a $\mathbb{P}_{\omega_3}$-generic filter over V. Then the standard argument shows that the following both hold in $V[G]$:

- If $S \subseteq S_0$ and S does not reflect to any ordinal in $E_0^2 \setminus \omega_1$ then S is nonstationary.
- If $S \subseteq S_1$ and S does not reflect to any ordinal in E_1^2 then S is nonstationary.

That is, $\mathsf{SR}_k(S_k)$ holds for both $k = 0, 1$ in $V[G]$. But note that $S_k^{\vec{c}} \setminus S_k$ is nonstationary. Therefore $\mathsf{SR}_k(S_k^{\vec{c}})$ holds for both $k = 0, 1$ in $V[G]$.

This completes the proof. □

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