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The variety of $\mathfrak{sa}(X)$

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In [3], Kada and Tomoyasu defined some cardinal characteristics concerning approximating the Stone-Čech compactification of a metrizable space by a family of its metric-dependent compactifications, and raised several questions on these characteristics. Since then, Kada, Tomoyasu and the author have been studying this subject ([4], [6] and [5]). In this article the author presents a few (rather simple) observations which were obtained in the author's recent study, which was done as a part of this continuing joint research.

1 Basic definitions and backgrounds

For topological spaces X and αX satisfying $X \subseteq \alpha X$, we say αX is a *compactification* of X if αX is compact Hausdorff and X is dense in αX. For compactifications αX, γX of X, we denote $\alpha X \geq_X \gamma X$ if there is a continuous mapping of αX onto γX which is identity on X. We also denote $\alpha X \simeq_X \gamma X$ if $\alpha X \geq_X \gamma X \geq_X \alpha X$ holds, or equivalently there is a homeomorphism between αX and γX which is identity on X. Note that $\simeq_X$ is a (class) equivalent relation on the class $\mathrm{Cpt}(X)$ of compactifications of X, and by identifying $\simeq_X$-equivalent compactifications we may consider that $\mathrm{Cpt}(X)$ is a set and that $\leq_X$ is a partial ordering of $\mathrm{Cpt}(X)$.

The following are well-known facts about $\mathrm{Cpt}(X)$.

Proposition 1. (1) $\mathrm{Cpt}(X) \neq \emptyset$ iff X is completely regular.

(2) If $\mathrm{Cpt}(X) \neq \emptyset$, $(\mathrm{Cpt}(X), \leq_X)$ forms an upper semi-lattice. In particular, $\mathrm{Cpt}(X)$ has the $\leq_X$-largest element, the *Stone-Čech compactification* of X, denoted as βX.

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An important tool to analyze the structure of $(\mathrm{Cpt}(X), \leq_X)$ is a family of Banach algebras of real-valued functions on X. Let $C^*(X)$ denote the set of bounded continuous functions from X to $\mathbb{R}$. $C^*(X)$ forms a (real) Banach algebra with respect to the uniform norm. A subalgebra (as a Banach algebra) C of $C^*(X)$ is said to be *regular* if for every closed $F \subseteq X$ and $x \in X$ there is $f \in C$ such that $f(x) = 0$ and $f(p) = 1$ for all $p \in F$. Let $\mathcal{R}(X)$ denote the class of regular subalgebras of $C^*(X)$.

For $\alpha X \in \mathrm{Cpt}(X)$ let $C_{\alpha X}$ denote the set of functions in $C^*(X)$ which can be continuously extended to a function on αX. Then $C_{\alpha X} \in \mathcal{R}(X)$ holds. The mapping which maps each $\alpha X \in \mathrm{Cpt}(X)$ to $C_{\alpha X}$ gives an isomorphism between $(\mathrm{Cpt}(X), \leq_X)$ and $(\mathcal{R}(X), \subseteq)$. See [2] for more details.

Now suppose X is a metrizable space, and d is a metric on X which is consistent with the topology of X. The *Smirnov compactification* $u_d X$ of X with respect to d is defined so that

$$C_{u_d X} = \{f \in C^*(X) \mid f \text{ is uniformly continuous with respect to } d\}.$$

Note that if X is totally bounded with respect to d, $u_d X$ is exactly the same as the completion of X with respect to d.

The following theorem shows that the class of Smirnov compactifications of a space is rich enough to "generate" its Stone-Čech compactification.

Theorem 2. (Woods [8]) For any metrizable space X,

$$\bigvee_{d \in M(X)} u_d X \simeq_X \beta X$$

holds, where $M(X)$ denotes the set of metrics on X which are consistent with the topology of X, and the join in the left-hand side is taken in the upper semilattice $(\mathrm{Cpt}(X), \leq_X)$.

Inspired with this theorem, Kada and Tomoyasu raised the following general question: For various metrizable spaces, how many metrics do we need to generate their Stone-Čech compactifications?

Definition 3. (Kada and Tomoyasu [3]; see also [4]) For a metrizable space X, define

$$\mathfrak{sa}(X) = \min\{|D| \mid D \subseteq M(X) \wedge \bigvee_{d \in D} u_d X \simeq_X \beta X\}.$$

The following are general facts about $\mathfrak{sa}(X)$:

Theorem 4. (Kada and Tomoyasu [3] for (1); Kada, Tomoyasu and Yoshinobu [6] for (2); [5] for (3))

(1) $\mathfrak{sa}(X) = 1$ holds if and only if the set of non-isolated points of X is compact.

(2) If $\mathfrak{sa}(X) \neq 1$ then $\mathfrak{sa}(X) \geq \mathfrak{d}$ (the dominating number).

(3) For an arbitrarily large cardinal θ, there exists a metrizable space X such that $\mathfrak{sa}(X) \geq \theta$.

On the other hand, if X is separable, $\mathfrak{sa}(X) \leq \mathfrak{c}(= 2^{\aleph_0})$ holds since there are at most $\mathfrak{c}$ metrics on X, and thus the problem comes within the range of set theory of reals. The authors have been working on deciding $\mathfrak{sa}(X)$ for various separable X's.

Theorem 5. (Kada, Tomoyasu and Yoshinobu [6] for (1); [5] for (2), (3))

(1) $\mathfrak{sa}(X) = \mathfrak{d}$ holds for every non-compact, locally compact separable metrizable space X.

(2) $\mathfrak{sa}(\mathbb{Q}) = \mathfrak{sa}(\mathbb{R} \setminus \mathbb{Q}) = \mathfrak{d}$.

(3) $\mathfrak{sa}(\mathbb{B}) = \mathfrak{c}$ for a Bernstein subset $\mathbb{B}$ of $\mathbb{R}$.

Having these results, the following question was raised in [5].

Question 6. Is it consistent that there exists a separable metrizable space X such that $\mathfrak{d} < \mathfrak{sa}(X) < \mathfrak{c}$?

If X is separable, X can be homeomorphically embedded into the Hilbert cube $\mathbb{H} = {}^{\omega}[0, 1]$ (with the product topology). So in such cases we regard X as a subspace of $\mathbb{H}$. Let us denote $X^* = \bar{X} \setminus X$, where $\bar{X}$ denotes the closure of X in $\mathbb{H}$.

The following theorem, observed independently by Kada and Todorčević, shows that the study of $\mathfrak{sa}(X)$ for a separable metrizable X can be reduced to combinatorics on compact subsets of a separable metrizable space.

Theorem 7. (Kada, Todorčević(see [5])) Suppose $X \subseteq \mathbb{H}$ and $\mathfrak{sa}(X) > 1$. Then the following holds:

$$\mathfrak{sa}(X) = \max\{\mathfrak{d}, \mathrm{cof}(\mathcal{K}(X^*), \subseteq)\},$$

where $\mathcal{K}(X^*)$ denotes the class of compact subsets of X^*.

Note that any separable metrizable space Y is homeomorphic to X^* for some $X \subseteq \mathbb{H}$, since Y can be regarded as a subspace of $\{f \in \mathbb{H} \mid f(0) = 0\}$, which is homeomorphic to $\mathbb{H}$ itself, and thus by letting $X = \mathbb{H} \setminus Y$ we have $Y = X^*$. Therefore Question 6 is equivalent to the following:

Question 8. Is it consistent that there exists a separable metrizable space Y such that $\mathfrak{d} < \mathrm{cof}(\mathcal{K}(Y), \subseteq) < \mathfrak{c}$?

2 $^{\omega}\infty$-bounding posets and countable compact subsets of metrizable spaces

Here we introduce a property of posets and observe the effect of forcing by posets with this property on the structure of countable compact subsets of metrizable spaces.

Definition 9. Let $\mathbb{P}$ be a poset.

(1) For an ordinal λ, $\mathbb{P}$ is *$^{\omega}\lambda$-bounding* if for any $f : \omega \longrightarrow \lambda$ in $V^{\mathbb{P}}$ there exists a function $F : \omega \longrightarrow [\lambda]^{<\omega}$ in V such that $\forall n < \omega(f(n) \in F(n))$ holds.

(2) $\mathbb{P}$ is *$^{\omega}\infty$-bounding* if $\mathbb{P}$ is $^{\omega}\lambda$-bounding for all ordinal λ.

(3) $\mathbb{P}$ is *ω-covering* if whenever X is a countable set of ordinals in $V^{\mathbb{P}}$ there exists a countable set Y in V such that $X \subseteq Y$.

Note that if $\mathbb{P}$ is $^{\omega}\infty$-bounding, it is also true that for any $f \in V^{\mathbb{P}}$ from ω to V, there exists a function $F \in V$ such that $F(n)$ is finite and $f(n) \in F(n)$ for all $n < \omega$.

Lemma 10. Suppose $\mathbb{P}$ is an $^{\omega}\infty$-bounding poset and X is a metrizable space in V. Then any $C \subseteq X$ in $V^{\mathbb{P}}$ which is countable and compact in $V^{\mathbb{P}}$ is covered by some $C_0 \subseteq X$ in V which is countable and compact in V.

Proof. Fix a metric d on X within V. We prove the lemma by induction on the Cantor-Bendixson rank α of C. The case $\alpha = 0$ is trivial, since in this case $C = \emptyset$ holds. Otherwise, argue in $V^{\mathbb{P}}$ for a while. By the compactness of C, $\alpha = \xi + 1$ for some ξ, and letting F denote the set of points of rank ξ in C, we have F is finite (non-empty) and thus is in V. Pick a positive real $d_0 \in V$ which is larger than the diameter of C (this is possible since C is compact). For each $n < \omega$ let

$$X_n = \{x \in X \mid \frac{d_0}{2^n} \geq d(x, F) \geq \frac{d_0}{2^{n+1}}\}, \text{ and } K_n = C \cap X_n.$$

Note that $\{X_n\}_{n<\omega}$ is defined within V. Note also that each K_n is a closed subset of C and thus is compact, and that $F \cup \bigcup_{n<\omega} K_n = C$ holds. Moreover, the Cantor-Bendixson rank of each K_n is strictly smaller than α, since K_n contains no points in F. Now by the induction hypothesis, for each $n < \omega$ there exists a countable compact $f(n) \in V$ such that $K_n \subseteq f(n) \subseteq X$. Then by the note after Lemma 11 there exists a function $H \in V$ such that $H(n)$ is finite and $f(n) \in H(n)$ holds for all $n < \omega$. Moreover, we may assume that each $H(n)$ consists only of countable compact subsets of X_n in V. Now let $C_0 = F \cup \bigcup_{n<\omega} \bigcup H(n)$. It is clear that C_0 is a countable set in V and $C \subseteq C_0$. To see that C_0 is compact, let $\{x_n\}$ be any sequence in C_0. Then either $\lim_{n\to\infty} d(x_n, F) = 0$ holds, or, for infinitely many n's x_n is in some fixed $\bigcup H(m)$, which is a finite union of compact sets and thus is itself compact. In any case, there exists a subsequence of $\{x_n\}$ converging to a point in C_0. □

Remark

(1) The converse of Lemma 10 is also true. For an ordinal λ, let $X = (\omega \times \lambda) \cup \{(\omega, 0)\}$ and define a metric d on X as follows: for every two distinct (m, α), $(n, \beta) \in X$ let $d((m, \alpha), (n, \beta)) = 2^{-\min\{m,n\}}$. Note that for each $f : \omega \to \lambda$, $C_f = \{(n, f(n)) \mid n < \omega\} \cup \{(\omega, 0)\}$ is a (countable) compact subset of X, and that for any compact subset C of X, $F(n) = \{\alpha < \lambda \mid (n, \alpha) \in C_0\}$ is finite for all $n < \omega$. Using these facts one can show that the conclusion of Lemma 10 for this X implies that $\mathbb{P}$ is ${}^{\omega}\lambda$-bounding.

(2) One can also show that for the conclusion of Lemma 10 only for separable metrizable spaces, the assumption that $\mathbb{P}$ is ${}^{\omega}\mathfrak{c}$-bounding is sufficient (and necessary).

The following is a useful criterion for a poset to be ${}^{\omega}\infty$-bounding.

Lemma 11. For a poset $\mathbb{P}$, the following are equivalent:

(a) $\mathbb{P}$ is ${}^{\omega}\infty$-bounding,

(b) $\mathbb{P}$ is ${}^{\omega}\omega$-bounding and ω-covering.

Proof. ((a)⇒(b)) Assume (a). It is enough to show that $\mathbb{P}$ is ω-covering. Let X be a countable set of ordinals in $V^{\mathbb{P}}$. Fix a surjection $f : \omega \to X$. By (a) there exists a function $F \in V$ on ω such that $F(n)$ is finite and $f(n) \in F(n)$ holds for all $n < \omega$. Thus $\bigcup \text{range}(F)$ is countable in V and contains X.
((b)⇒(a)) Assume (b). Let $f \in V^{\mathbb{P}}$ be any ordinal-valued function on ω. Since range(f) is a countable set of ordinals in $V^{\mathbb{P}}$ and $\mathbb{P}$ is ω-covering, there exists a countable set Y in V such that range$(f) \subseteq Y$. Now let $\{y_m\}_{m<\omega}$ be an enumeration of Y in V, and define $g : \omega \to \omega$ so that $f(n) = y_{g(n)}$ holds for each $n < \omega$. Since $g \in V^{\mathbb{P}}$ and $\mathbb{P}$ is ${}^{\omega}\omega$-covering, there exists a function $G : \omega \to \omega$ in V such that $g(n) < G(n)$ for all $n < \omega$. Define F so that $F(n) = \{y_m \mid m < G(n)\}$ for each $n < \omega$. Then $F \in V$ and for all $n < \omega$ $F(n)$ is finite and $f(n) \in F(n)$ holds. □

3 Consistency of $\mathfrak{d} < \mathfrak{sa}(X) < \mathfrak{c}$

For an infinite cardinal κ, let $\mathbb{B}(\kappa)$ denote the measure algebra on κ. It is well-known that $\mathbb{B}(\kappa)$ is ${}^{\omega}\omega$-bounding (see [1]), and is also ω-covering since it satisfies the countable chain condition. Thus by Lemma 11 $\mathbb{B}(\kappa)$ is ${}^{\omega}\infty$-bounding. The following theorem gives an affirmative answer to Question 8 and thus to Question 6 in a strong sense.

Theorem 12. Assume GCH and κ is a cardinal in V. Then in $V^{\mathbb{B}(\kappa)}$, for every cardinal θ satisfying $\mathfrak{d}(=\aleph_1) \leq \theta \leq \mathfrak{c}$ there exists a separable metrizable space X such that $\mathfrak{sa}(X) = \theta$.

Proof. We may assume $\mathfrak{d} < \theta < \mathfrak{c}$, since the cases $\theta = \mathfrak{d}$ and $\theta = \mathfrak{c}$ are already done (Theorem 5). Under this assumption we have $\theta \leq \kappa$. By Theorem 7 it is enough to show that there exists a set $Y \subseteq \mathbb{H}$ such that $\mathrm{cof}(\mathcal{K}(Y), \subseteq) = \theta$.
Case 1 $\mathrm{cf}\theta \neq \omega$.

We will use the fact that $\mathbb{B}(\kappa)$ can be factorized as $\mathbb{B}(\theta) * \dot{\mathbb{B}}(\kappa \setminus \theta)$. Argue in $V^{\mathbb{B}(\kappa)}$. Let $Y = \mathbb{H} \cap V^{\mathbb{B}(\theta)}$. Note that $|Y| = \mathfrak{c}^{V^{\mathbb{B}(\theta)}} = \theta < \mathfrak{c}$, and thus any compact subset of Y is at most countable. This shows that $\mathrm{cof}(\mathcal{K}(Y), \subseteq) \geq \theta$, since any $\subseteq$-cofinal subfamily of $\mathcal{K}(Y)$ must cover Y. On the other hand, let $\mathcal{F}$ be the family of countable compact subsets of $\mathbb{H}$ computed in $V^{\mathbb{B}(\theta)}$. Then $|\mathcal{F}| = \theta$, and since $V^{\mathbb{B}(\kappa)}$ is an ${}^{\omega}\infty$-bounding extension of $V^{\mathbb{B}(\theta)}$, by Lemma 10, $\mathcal{F}$ remains to be a $\subseteq$-cofinal subfamily of $\mathcal{K}(Y)$ in $V^{\mathbb{B}(\kappa)}$[1]. This shows that $\mathrm{cof}(\mathcal{K}(Y), \subseteq) \leq \theta$.
Case 2 $\mathrm{cf}\theta = \omega$.

Argue in $V^{\mathbb{B}(\kappa)}$ again. Let $\{\theta_n\}_{n<\omega}$ be regular uncountable cardinals such that $\sup_{n<\omega} \theta_n = \theta$. Case 1 shows that for each $n < \omega$ there exists a set $Y_n \subseteq \mathbb{H}$ such that $|Y_n| = \theta_n$ and $\mathrm{cof}(\mathcal{K}(Y_n), \subseteq) = \theta_n$. Let $\mathcal{K}_n$ be a $\subseteq$-cofinal subfamily of $\mathcal{K}(Y_n)$ such that $|\mathcal{K}_n| = \theta_n$. Now let

$$Y = \{f \in \mathbb{H} \mid \exists n < \omega (f(0) = \frac{1}{n} \wedge \bar{f} \in Y_n)\},$$

where $\bar{f} \in \mathbb{H}$ is defined by $\bar{f}(n) = f(n+1)$ $(\forall n < \omega)$ for $f \in \mathbb{H}$. Then $|Y| = \theta$ and by the same argument as in Case 1 we have $\mathrm{cof}(\mathcal{K}(Y), \subseteq) \geq \theta$. Now note that any compact subset of Y is of the form $\bigcup_{n<m} K_n$ for some $m < \omega$, where each K_n is a compact subset of $Y'_n = \{f \in Y \mid f(0) = \frac{1}{n}\}$. Therefore

$$\mathcal{K} = \{\bigcup_{n<m} K_n \mid m < \omega \wedge \forall n < m(\{\bar{f} \mid f \in K_n\} \in \mathcal{K}_n)\}$$

[1]Here we used the fact that the compactness of a countable metric space is upward-absolute. This follows from the fact that the compactness of a countable metric space is a Π^1_1-statement about its metric.

forms a $\subseteq$-cofinal subfamily of $\mathcal{K}(Y)$. It is easy to check that $|\mathcal{K}| = \theta$ holds. This shows that $\mathrm{cof}(\mathcal{K}(Y), \subseteq) \leq \theta$. □

After having the above observation, the author noticed that older studies had already suggested that if $\theta \in [\mathfrak{d}, \mathfrak{c}]$ is small enough, it is always the case that there exists a separable metrizable space X such that $\mathfrak{sa}(X) = \theta$.

Theorem 13. (van Douwen [7, Theorem 8.10(a), (b)]) For a countable metric space X, $\mathrm{cof}(\mathcal{K}(X), \subseteq) \leq \mathfrak{d}$ holds.

Corollary 14. For any cardinal θ satisfying $\mathfrak{d} \leq \theta \leq \min\{\aleph_\omega, \mathfrak{c}\}$ there exists a separable metrizable X such that $\mathfrak{sa}(X) = \theta$.

Proof. Again we may assume $\mathfrak{d} < \theta < \mathfrak{c}$, and it is enough to show that there exists a set $Y \subseteq \mathbb{H}$ such that $\mathrm{cof}(\mathcal{K}(Y), \subseteq) = \theta$.

<u>Case 1</u> $\theta = \aleph_n$ for some $n < \omega$ (thus in fact $n > 1$).

First note that $[\theta]^{\leq\aleph_0}$ has a $\subseteq$-cofinal subfamily of size θ: ω_1 is a $\subseteq$-cofinal subfamily of $[\omega_1]^{\leq\aleph_0}$ of size ω_1. If K_m is a $\subseteq$-cofinal subfamily of $[\omega_m]^{\leq\aleph_0}$ of size $\aleph_m$, then it is easy to see that $\bigcup_{\gamma<\omega_{m+1}} (f^m_\gamma)'' K_m$ is a $\subseteq$-cofinal subfamily of $[\omega_{m+1}]^{\leq\aleph_0}$ of size $\aleph_{m+1}$ (where f^m_γ denotes a surjection from ω_m to γ).

Pick any $Y \subseteq \mathbb{H}$ of size θ. Since every compact subset of Y is countable, by the same argument as in the proof of Theorem 12, $\mathrm{cof}(\mathcal{K}(Y), \subseteq) \geq \theta$ holds. On the other hand, letting $\mathcal{C}$ be a $\subseteq$-cofinal subfamily of $[Y]^{\leq\aleph_0}$ of size θ, by Theorem 13 we have

$$\mathrm{cof}(\mathcal{K}(Y), \subseteq) \leq \sum_{X \in \mathcal{C}} \mathrm{cof}(\mathcal{K}(X), \subseteq) \leq |\mathcal{C}| \cdot \mathfrak{d} = \theta.$$

<u>Case 2</u> $\theta = \aleph_\omega$.

This case can be dealt with in exactly the same way as in Case 2 in the proof of Theorem 12, using Case 1. □

Having these results, our next question becomes of the following kind, somewhat with an opposite tone to Question 6.

Question 15. Is it consistent that $\mathfrak{d} < \aleph_{\omega+1} < \mathfrak{c}$ no separable metrizable space X satisfies $\mathfrak{sa}(X) = \aleph_{\omega+1}$? More generally, is it consistent that there exists a cardinal θ such that $\mathfrak{d} < \theta < \mathfrak{c}$ with no separable metrizable space X such that $\mathfrak{sa}(X) = \theta$?

References

[1] T. Bartoszyński and H. Judah. *Set theory on the structure of the real line.* A K Peters, 1995.

[2] L. Gillman and M. Jerison. *Rings of continuous functions*. Van Nostrand, 1960.

[3] M. Kada and K. Tomoyasu. How many miles to $\beta\omega$? (General and Geometric Topology and Related Topics). *RIMS Kokyuroku*, 1370:86–101, 2004. (Japanese).

[4] M. Kada, K. Tomoyasu, and Y. Yoshinobu. How many miles to $\beta\omega$? — Approximating $\beta\omega$ by metric-dependent compactifications. *Topology Appl.*, 145:277–292, 2004.

[5] M. Kada, K. Tomoyasu, and Y. Yoshinobu. How many miles to $\beta\omega$? II (Set Theoretic and Geometric Topology and Its Applications). *RIMS Kokyuroku*, 1419:105–125, 2005. (Japanese).

[6] M. Kada, K. Tomoyasu, and Y. Yoshinobu. How many miles to βX? — ∂ miles, or just one foot. *Topology Appl.*, 153:3313–3319, 2006.

[7] E. K. van Douwen. The integers and topology. In K. Kunen and J. E. Vaughan, editors, *Handbook of Set Theoretic Topology*, pages 111–167. North-Holland, 1984.

[8] R.G. Woods. The minimum uniform compactification of a metric space. *Fund. Math.*, 147:39–59, 1995.