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Reverse of Bebiano-Lemos-Providência inequality and Complementary Furuta inequality

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We give a simultaneous extension of Bebiano-Lemos-Providência inequality and Araki-Cordes one: Let A and B be positive operators. Then for each $r \geq 0$

$$\| A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \| \leq \| A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \| \qquad \text{for} \quad s \geq t \geq 0.$$

In succession, we prove a reverse inequality: Let A and B be positive operators. Then for each $r \geq 0$

$$\| A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \| \geq \| A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \| \qquad \text{for} \quad t \geq s \geq r \geq 0.$$

Furthermore, we discuss reverse of generalized BLP inequality in a general setting, in which we point out that the restriction $t \geq s \geq r$ in the above is quite reasonable.

1 Introduction

Let A be a bounded linear operator acting on a Hilbert spase H. Then A is positive, denoted by $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is invertible and positive.

Recently, Bebiano-Lemos-Providência showed an interesting norm inequality (**BLP**): Let A and B be positive operators. Then

(1) $$\| A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}} \| \leq \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}} \| \qquad \text{for} \quad s \geq t \geq 0.$$

If we delete $A^{\frac{1}{2}}$ on both sides in (1), we have the Araki-Cordes inequality (**AC**)

(2) $$\| A^p B^p A^p \| \leq \| ABA \|^p \qquad \text{for} \quad 0 \leq p \leq 1.$$

In this sense, BLP inequality (1) is regarded as an extension of AC inequality (2).

In this note, we first give a simultaneous extension of (1) and (2) as follows generalized BLP inequalty (**GBLP**): Let A and B be positive operators. Then for each $r \geq 0$

(3) $$\| A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \| \leq \| A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \| \qquad \text{for} \quad s \geq t \geq 0.$$

Next we discuss a reverse inequaltiy of (3)(**R-GBLP**): Let A and B be positive operators. Then for each $r \geq 0$,

(4) $$\| A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \| \geq \| A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \| \qquad \text{for} \quad t \geq s \geq r \quad \text{and} \quad s > 0.$$

As a corollary, the case $r = 1$ in (4) coresspponds to the reverse of BLP(**R-BLP**) inequality and the case $r = 0$ in (4) corresponds to the reverse of AC inequality (2)(**R-AC**).

2 A generalization of BLP inequality

In this section, we generalize BLP inequality (1). For this, we cite Furuta inequality: Let $A \geq B$ for A and B be positive operators. Then for each $r \geq 0$.

$$(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}} \leq (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \qquad \text{for} \quad p \geq 0, \quad q \geq 1 \quad \text{with} \quad (1+r)q \geq p+r.$$

We prove the following theorem including BLP inequality and AC one.

Theorem. 1. *Let A and B be positive operators. Then for each $r \geq 0$*

$$\| A^{\frac{r+t}{2}}B^tA^{\frac{r+t}{2}} \| \leq \| A^{\frac{r}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{r}{2}} \| \qquad \textit{for} \quad s \geq t \geq 0.$$

Proof. Since this inequality is AC inequality when $r = 0$, we may assume $r > 0$. It suffices to prove that

$$A^{\frac{r}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{r}{2}} \leq 1 \Longrightarrow A^{\frac{r+t}{2}}B^tA^{\frac{r+t}{2}} \leq 1 \qquad \text{for} \quad s \geq t \geq 0.$$

If $A^{\frac{r}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{r}{2}} \leq 1$, then $(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}} \leq A^{-r}$. We put

$$A_1 = A^{-r}, \quad B_1 = (A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}} \quad r_1 = \frac{s}{r} \quad \text{and} \quad p = q = \frac{s}{t}.$$

Then

$$A_1 \geq B_1 \geq 0, \quad r_1 \geq 0, \quad p = q \geq 1 \qquad \text{and} \qquad (1+r_1)q \geq p + r_1,$$

so that Furuta inequality implies

$$(A_1^{\frac{r_1}{2}}B_1^pA_1^{\frac{r_1}{2}})^{\frac{1}{q}} \leq (A_1^{\frac{r_1}{2}}A_1^pA_1^{\frac{r_1}{2}})^{\frac{1}{q}},$$

that is,

$$(A_1^{\frac{s}{2r}}B_1^{\frac{s}{t}}A_1^{\frac{s}{2r}})^{\frac{t}{s}} \leq (A_1^{\frac{s}{2r}}A_1^{\frac{s}{t}}A_1^{\frac{s}{2r}})^{\frac{t}{s}}.$$

Since we have

$$A = A_1^{-\frac{1}{r}}, \; B = (A^{-\frac{s}{2}}B_1^{\frac{s}{t}}A^{-\frac{s}{2}})^{\frac{1}{s}} = (A_1^{\frac{s}{2r}}B_1^{\frac{s}{t}}A_1^{\frac{s}{2r}})^{\frac{t}{s}},$$

it follows that

$$\begin{aligned} A^{\frac{r+t}{2}}B^tA^{\frac{r+t}{2}} &= (A_1^{-\frac{1}{r}})^{\frac{r+t}{2}}(A_1^{\frac{s}{2r}}B_1^{\frac{s}{t}}A_1^{\frac{s}{2r}})^{\frac{t}{s}}(A_1^{-\frac{1}{r}})^{\frac{r+t}{2}} \\ &\leq A_1^{-\frac{r+t}{2r}}(A_1^{-\frac{s}{2}}A_1^{\frac{s}{t}}A_1^{-\frac{s}{2}})^{\frac{t}{s}}A_1^{-\frac{r+t}{2r}} = A_1^{-\frac{r+t}{2r}}A_1^{\frac{r+t}{r}}A_1^{-\frac{r+t}{2r}} = I. \end{aligned}$$

□

Remark. It is obvious that the case $r = 1$ in Theorem 1 is just the BLP inequality and the case $r = 0$ is AC one.

3 Reverse inequalities

In this section, we show a reverse inequality of Theorem 1, in which we use the well-known formula(Lemma of Furuta),

(5) $$(A^{\frac{1}{2}}BA^{\frac{1}{2}})^p = A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{p-1}B^{\frac{1}{2}}A^{\frac{1}{2}} \quad \text{for} \quad p \geq 1$$

and Löwner-Heinz inequality (**LH**)

(6) $$A \geq B \geq 0 \quad \text{implies} \quad A^p \geq B^p \qquad \text{for all } 0 \leq p \leq 1.$$

Theorem. 2. *Let A and B be positive operators. Then for each $r \geq 0$*

(7) $$\| A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \| \geq \| A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \| \qquad \textit{for} \quad t \geq s \geq r \quad \textit{and} \quad s > 0.$$

Proof. It suffices to prove that

$$A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \leq 1 \Longrightarrow A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \leq 1 \qquad \text{for} \quad t \geq s \geq r \quad \text{and} \quad s > 0.$$

Suppose that $[\frac{t}{s}] = 2l-1$ or $2l$ for some natural number l. Then following equations are obtained by (5).

$$\begin{aligned}
&A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \\
&= A^{\frac{r+s}{2}} B^{\frac{s}{2}} (B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\frac{t}{s}-1} B^{\frac{s}{2}} A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}} B^s A^{\frac{s}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}-2} A^{\frac{s}{2}} B^s A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}} (B^s A^s) B^{\frac{s}{2}} (B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\frac{t}{s}-3} B^{\frac{s}{2}} (A^s B^s) A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}} (B^s A^s) B^s A^{\frac{s}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}-4} A^{\frac{s}{2}} B^s (A^s B^s) A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}} (B^s A^s)(B^s A^s) B^{\frac{s}{2}} (B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\frac{t}{s}-5} B^{\frac{s}{2}} (A^s B^s)(A^s B^s) A^{\frac{r+s}{2}} \\
&\vdots \\
&= A^{\frac{r+s}{2}} \overbrace{(B^s A^s) \cdots (B^s A^s)}^{(l-1)\text{-times}} B^{\frac{s}{2}} (B^{\frac{s}{2}} A^s B^{\frac{s}{2}})^{\frac{t}{s}-(2l-1)} B^{\frac{s}{2}} \overbrace{(A^s B^s) \cdots (A^s B^s)}^{(l-1)\text{-times}} A^{\frac{r+s}{2}}
\end{aligned}$$

Since $t \geq s \geq r \geq 0$, we have

$$0 \leq \frac{s}{r+t} \leq 1 \ , \ 0 \leq \frac{s}{t} \leq 1 \ , \ 0 \leq \frac{t(r+s) - 2(l-h)rs}{t(r+t)} \leq 1 \quad \text{for} \quad h = 1, 2, \cdots , l$$

and

$$0 \leq \frac{t(s-r) + 2(l-k)rs}{t(r+t)} \leq 1 \quad \text{for} \quad k = 0, 1, \cdots , l-1 \ .$$

We suppose that $A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \leq 1$, that is,

$$B^t \leq A^{-(r+t)} \quad \text{and} \quad A^{(r+t)} \leq B^{-t}.$$

So the Löwner-Heinz inequality (6) implies that

$$A^s \le B^{-\frac{st}{r+t}}, \quad B^s \le A^{-\frac{s(r+t)}{t}}, \quad B^{\frac{t(r+s)-2(l-h)rs}{(r+t)}} \le A^{-\frac{t(r+s)-2(l-h)rs}{t}} \quad \text{for} \quad h = 1, 2, \cdots, l$$

and

$$A^{\frac{t(s-r)+2(l-k)rs}{t}} \le B^{-\frac{t(s-r)+2(l-k)rs}{(r+t)}} \quad \text{for} \quad k = 0, 1, \cdots, l-1.$$

Now we assume that $[\frac{t}{s}] = 2l - 1$ for some natural number l. Since $0 \le \frac{t}{s} - (2l-1) \le 1$, we have

$$\begin{aligned}
& A^{\frac{r}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{r}{2}} \\
&= A^{\frac{r+s}{2}} \overbrace{(B^sA^s)\cdots(B^sA^s)(B^sA^s)}^{(l-1)\text{-times}} B^{\frac{s}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}-(2l-1)}B^{\frac{s}{2}} \overbrace{(A^sB^s)(A^sB^s)\cdots(A^sB^s)}^{(l-1)\text{-times}} A^{\frac{r+s}{2}} \\
&\le A^{\frac{r+s}{2}}(B^sA^s)\cdots(B^sA^s)(B^sA^s)B^{\frac{s}{2}}(B^{\frac{s}{2}}B^{-\frac{st}{r+t}}B^{\frac{s}{2}})^{\frac{t}{s}-(2l-1)}B^{\frac{s}{2}}(A^sB^s)(A^sB^s)\cdots(A^sB^s)A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}}(B^sA^s)\cdots(B^sA^s)(B^sA^s)B^{\frac{t(s+r)-2(l-1)rs}{r+t}}(A^sB^s)(A^sB^s)\cdots(A^sB^s)A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}} \overbrace{(B^sA^s)\cdots(B^sA^s)}^{(l-2)\text{-times}} B^sA^sB^{\frac{t(s+r)-2(l-1)rs}{r+t}}A^sB^s \overbrace{(A^sB^s)\cdots(A^sB^s)}^{(l-2)\text{-times}} A^{\frac{r+s}{2}} \\
&\le A^{\frac{r+s}{2}}(B^sA^s)\cdots(B^sA^s)B^sA^{\frac{t(s-r)+2(l-1)rs}{t}}B^s(A^sB^s)\cdots(A^sB^s)A^{\frac{r+s}{2}} \\
&\le A^{\frac{r+s}{2}}(B^sA^s)\cdots(B^sA^s)B^{\frac{t(s+r)-2(l-2)rs}{r+t}}(A^sB^s)\cdots(A^sB^s)A^{\frac{r+s}{2}} \\
&\vdots \\
&\le A^{\frac{r+s}{2}}B^sA^{\frac{t(s-r)+2(l-(l-1))rs}{t}}B^sA^{\frac{r+s}{2}} \\
&\le A^{\frac{r+s}{2}}B^{\frac{ts+rt}{r+t}}A^{\frac{r+s}{2}} \\
&\le A^{\frac{r+s}{2}}A^{-(r+s)}A^{\frac{r+s}{2}} \\
&= I.
\end{aligned}$$

On the other hand, we assume that $[\frac{t}{s}] = 2l$ for some natural number l. Since $0 \le \frac{t}{s} - 2l \le 1$, similarly we have the follwing, in which the first equality is ensured in the first paragraph.

$$\begin{aligned}
& A^{\frac{r}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{r}{2}} \\
&= A^{\frac{r+s}{2}} \overbrace{(B^sA^s)\cdots(B^sA^s)(B^sA^s)}^{(l-1)\text{-times}} B^{\frac{s}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}-(2l-1)}B^{\frac{s}{2}} \overbrace{(A^sB^s)(A^sB^s)\cdots(A^sB^s)}^{(l-1)\text{-times}} A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}} \overbrace{(B^sA^s)\cdots(B^sA^s)(B^sA^s)}^{(l-1)\text{-times}} B^sA^{\frac{s}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}-2l}A^{\frac{s}{2}}B^s \overbrace{(A^sB^s)(A^sB^s)\cdots(A^sB^s)}^{(l-1)\text{-times}} A^{\frac{r+s}{2}} \\
&\le A^{\frac{r+s}{2}}(B^sA^s)\cdots(B^sA^s)(B^sA^s)B^sA^{\frac{s}{2}}(A^{\frac{s}{2}}A^{-\frac{s(r+t)}{t}}A^{\frac{s}{2}})^{\frac{t}{s}-2l}A^{\frac{s}{2}}B^s(A^sB^s)(A^sB^s)\cdots(A^sB^s)A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}}(B^sA^s)\cdots(B^sA^s)(B^sA^s)B^sA^{\frac{t(s-r)+2ls}{t}}B^s(A^sB^s)(A^sB^s)\cdots(A^sB^s)A^{\frac{r+s}{2}} \\
&\le A^{\frac{r+s}{2}}(B^sA^s)\cdots(B^sA^s)(B^sA^s)B^sB^{-\frac{t(s-r)+2lrs}{r+s}}B^s(A^sB^s)(A^sB^s)\cdots(A^sB^s)A^{\frac{r+s}{2}} \\
&= A^{\frac{r+s}{2}} \overbrace{(B^sA^s)\cdots(B^sA^s)}^{(l-2)\text{-times}} B^sA^sB^{\frac{t(r+s)-2(l-1)rs}{r+t}}A^sB^s \overbrace{(A^sB^s)\cdots(A^sB^s)}^{(l-2)\text{-times}} A^{\frac{r+s}{2}} \\
&\vdots \\
&\le I.
\end{aligned}$$

Hence the proof is complete. □

4 Complementary Furuta inequality

In this section, we consider **R-GBLP**, in which Kamei's theorem(**Theorem K**) on complements of Furuta inequality corresponds to our result. So now recall it due to Kamei.

Theorem K. *If $A \geq B > 0$, then for $0 < p \leq \frac{1}{2}$*

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \leq A^{2p} \quad \textit{for} \quad 0 \leq t \leq p \tag{8}$$

and for $\frac{1}{2} \leq p \leq 1$

$$A^t \natural_{\frac{1-t}{p-t}} B^p \leq A \quad \textit{for} \quad 0 \leq t \leq p. \tag{9}$$

Here $\natural_q$ for $q \notin [0,1]$ has been used as

$$A \natural_q B := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q A^{\frac{1}{2}} \qquad \textit{for } A, B > 0.$$

First we prove the following Theorem.

Theorem. 3. *Let $A, B \geq 0$ and $0 < p \leq 1$. Then*

$$\| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}} \|^{\frac{p+s}{p(1+s)}} \quad \geq \quad \| A^{\frac{1}{2}}(A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}} \| \tag{10}$$

for all $s \geq 0$ with $s \geq 1 - 2p$.

Proof. It suffices to show that

$$B^{1+s} \leq A^{-(1+s)} \;\Rightarrow\; A^{\frac{1}{2}}(A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}} \leq 1 \tag{11}$$

for $0 < p \leq 1$ and $s \geq 0$ with $s \geq 1 - 2p$. So we put

$$A_1 = A^{-(1+s)}, \; B_1 = B^{1+s}.$$

Then (11) is rephrased as

$$A_1 \geq B_1 > 0 \;\Rightarrow\; A_1^{\frac{s}{1+s}} \natural_{\frac{1}{p}} B_1^{\frac{p+s}{1+s}} \leq A_1.$$

for $0 < p \leq 1$ and $s \geq 0$ with $s \geq 1 - 2p$. Moreover if we replace

$$t_1 = \frac{s}{1+s}, \; p_1 = \frac{p+s}{1+s},$$

then we have $\frac{1-t_1}{p_1-t_1} = \frac{1}{p}$, and $\frac{1}{2} \leq p_1 (\leq 1)$ if and only if $1 - 2p \leq s$, so that (11) has the following equivalent expression:

$$A_1 \geq B_1 > 0 \;\Rightarrow\; A_1^{t_1} \natural_{\frac{1-t_1}{p_1-t_1}} B_1^{p} \leq A_1 \quad \text{for} \quad 0 \leq t_1 < p_1.$$

Fortunately, since $\frac{1}{2} \leq p_1 \leq 1$, it has been ensured by Theorem A due to Kamei. □

Next we show a reserve inequality of BLP inequality (**R-BLP**) is obtained as colloary of Teorem 3.

Proof of **R-BLP** We put $p = \frac{s}{t}$ for $t \geq s \geq 0$. Then we have $1 - 2p \leq s$ if and only if $\frac{t}{t+2} \leq s$. Since $s \geq 1$ is assumed, $\frac{t}{t+2} \leq s$ holds for arbitrary $t > 0$, so that Theorem 3 is applicable.

Now we take $B_1 = B^{\frac{1+t}{t}}$, i.e., $B = B_1^{\frac{t}{1+t}}$. Then Araki-Cordes inequality and Theorem 3 imply that

$$\| A^{\frac{1+t}{2}} B_1^t A^{\frac{1+t}{2}} \| \geq \| A^{\frac{1+s}{2}} B_1^{\frac{t(1+s)}{1+t}} A^{\frac{1+s}{2}} \|^{\frac{1+t}{1+s}} = \| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}} \|^{\frac{p+s}{p(1+s)}}$$

$$\geq \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{1}{p}} A^{\frac{1}{2}} \| = \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B_1^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}} \|,$$

as desired.

R-BLP is generalized a bit as follows:

Corollary. *For $A,\ B > 0$ and $r \geq 0$*

(12) $$\| A^{\frac{r+t}{2}} B^t A^{\frac{r+t}{2}} \| \geq \| A^{\frac{r}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{r}{2}} \|$$

holds for all $t \geq s \geq r$.

Proof. It is proved by applying **R-BLP** to $A_1 = A^r$, $B_1 = B^r$ and $t_1 = \frac{t}{r}$, $s_1 = \frac{s}{r}$. □

Finally we consider a reverse inequality of generalized BLP inequality which corresponds to another Kamei's complement (7): If $A \geq B > 0$, then for $0 < p \leq \frac{1}{2}$

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \leq A^{2p} \quad \text{for} \quad 0 \leq t < p.$$

Theorem. 4. *Let $A, B \geq 0$ and $0 < p \leq \frac{1}{2}$. Then*

(13) $$\| A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}} \|^{\frac{(2p+s)(p+s)}{p(1+s)}} \quad \geq \quad \| A^{p+\frac{s}{2}} (A^{\frac{s}{2}} B^{p+s} A^{\frac{s}{2}})^{\frac{2p+s}{p}} A^{p+\frac{s}{2}} \|$$

for all $0 \leq s \leq 1 - 2p$.

Proof. A proof is quite similar to that of Theorem 3. We put

$$A_1 = A^{-(1+s)},\ B_1 = B^{1+s};\ t_1 = \frac{s}{1+s},\ p_1 = \frac{p+s}{1+s}.$$

Then (7) gives us that

$$A_1 \geq B_1 > 0 \ \Rightarrow\ A_1^{t_1} \natural_{\frac{2p_1-t_1}{p_1-t_1}} B_1^{p_1} \leq A_1^{2p_1},$$

for $0 \leq t_1 < p_1 \leq \frac{1}{2}$, so that

$$A^{-(1+s)} \geq B^{1+s} \ \Rightarrow\ A^{-s} \natural_{\frac{2p+s}{p}} B^{p+s} \leq A^{-2(p+s)}$$

for $0 \leq s \leq 1 - 2p$. Obviously it implies the desired norm inequality (12). □

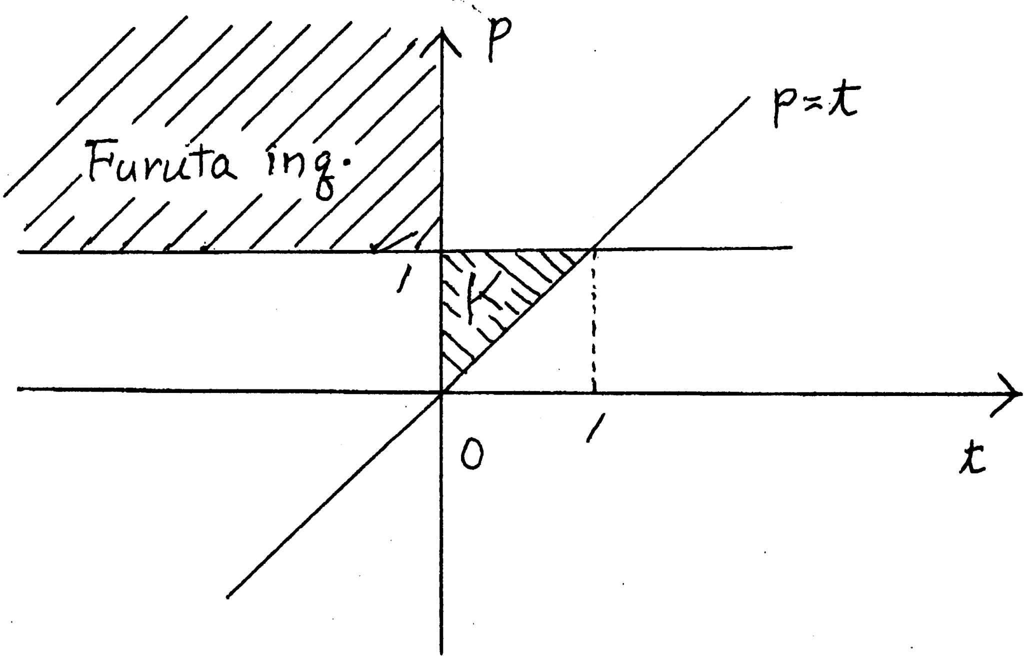
5 Remarks

We first consider a relation between Furuta inequality and Theorem K. Furuta inequality has the following representation by α-geometric mean $\sharp_\alpha$: For $A \geq B \geq 0$

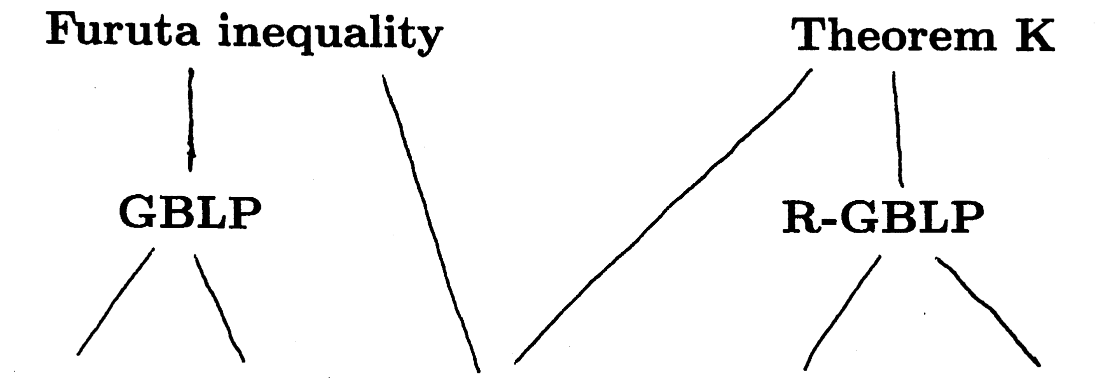
$$A^t \sharp_{\frac{1-t}{p-t}} B^p \; \leq \; A \quad \text{for } p \geq 1, t \leq 0$$

where $A \sharp_\alpha B$ for $0 \leq \alpha \leq 1$ is defined by

$$A \sharp_\alpha B := A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \qquad \text{for } A, B > 0.$$



As a consequence we have relations among the inequalitys discussed abave.



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