数理解析研究所講究録
第 1596 巻 2008 年 127-134

RANK-ONE PERTURBATION OF WEIGHTED SHIFTS SEPARATING GAPS OF OPERATORS

Eun Young Lee[*]

Department of Mathematics, Kyungpook National University,
Daegu 702-701, Korea
E-mail: eee-222@hanmail.net

Abstract

The weak hyponormalities of Hilbert space operators make important roles to study the gaps of operators. In particular, p-hyponormality, p-paranormality, and absolute p-paranormality has been considered to detect gaps of operators. But examples of those operators with weak hyponormality are not developed well still. In this note we consider rank-one peturbation of weighted shifts to detect examples for those operators and characterize weak hyponormalities of those operators. In addition, we discuss some related examples being distinct those weak hyponormalities.

1. Introduction. This is based on the joint work with G. Exner, I. Jung, and M. Lee([EJLL]) and was talked at the 2008 RIMS conference: Inequalities on linear operators and its applications, which was held at Kyoto University on January 30-February 1 in 2008.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The study of operators with weak hyponormality has been discussed for recent 30 years (see [Fur]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* $(0 < p < \infty)$ if $(T^*T)^p \geq (TT^*)^p$. In particular, if $p = \frac{1}{2}$, then T is *semi-hyponormal* ([Xi]). And T is said to be ∞*-hyponormal* if T is p-hyponormal for all $p \in (0, \infty)$ ([MS]). Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. For each $p > 0$, an operator T is *absolute-p-paranormal* if $\||T|^pTx\| \geq \|Tx\|^{p+1}$ for all unit vector $x \in \mathcal{H}$. Every absolute-q-paranormal operator is absolute-p-paranormal for $q \leq p$ ([Fur]). We call simply absolute-1-paranormal as paranormal. And T is *p-paranormal* if $\| \, |T|^p \, U \, |T|^p \, x\| \geq \| \, |T|^p \, x\|^2$ for all unit vectors $x \in \mathcal{H}$. In particular, the 1-paranormality is referred to as the paranormality. Every q-paranormal operator is p-paranormal for $q \leq p$ ([Fuj]). The implications among classes of operators mentioned above are as follows:

- p-hyponormal $\Rightarrow$ p-paranormal $\Rightarrow$ absolute-p-paranormal $(0 < p < 1)$;

[*] *2000 Mathematics Subject Classification.* Primary 47A50, 47B20; Secondary 47A55.

[†] *Key words and phrases:* rank-one perturbation, p-hyponormal operators, absolute p-paranormal operators.

- p-hyponormal $\Rightarrow$ absolute-p-paranormal $\Rightarrow$ p-paranormal ($p > 1$).

Seeing that examples for those operators are not abundant, it is worthwhile to develop examples to distinguish those classes. In [JLP] and [JLL] block matrix operators were considered to classify the above operators, but it was proved in their models that p-paranormality is equivalent to absolute-p-paranormality. Also, models of composition operators were discussed in [JLP] and [BJ] to classify those operators with weak hyponormality, but it also was shown that two such weak hyponormalities are equivalent ([BJ]). However, our rank-one perturbation models classify completely such two weak hyponormalities. In this paper we discuss rank-one perturbations of weighted shifts.

The paper consists of three sections. In Section 2 we characterize quasinormality and p-hyponormality for rank-one perturbation of a weighted shift, and obtain examples being distinct the classes of p-hyponormal operators. In Section 3, we also characterize absolute p-paranormal and p-paranormal operators, which provides examples being distinct the classes of such operators. Especially, we discuss via numerical table that the absolute p-paranormality is different from the p-paranormality as we said above.

Some of the calculations in this paper were obtained through computer experiments using the software tool *Mathematica* [Wol].

2. p-hyponormality Let W_α be a weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ of nonnegative real numbers. Let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for $\mathcal{H} = \ell^2(\mathbb{Z}_+)$. Obviously, W_α is hyponormal if and only if W_α is p-hyponormal for any[some] $p \in (0, \infty)$. In particular, W_α is normal if and only if $\alpha_n = 0$ for all $(n \geq 0)$, which is equivalent to that W_α is quasinormal. Hence the weighted shifts can not separate classes of p-hyponormal operators. But rank-one perturbations of weighted shifts with a positive real parameter separate the classes of p-hyponormal operators positively.

2.1. Characterizations of quasinormality. We consider a rank-one perturbation of weighted shift

$$T(k,t) := W_\alpha + t(e_k \otimes e_k), \quad k \in \mathbb{N} \tag{2.1}$$

with parameter $t \in [0, \infty)$.

Proposition 2.1. *Let $T := T(k,t)$ be as in* (2.1). *Then $T(k,t)$ is quasinormal if and only if it holds that*

i) *if $\alpha_k \neq 0$, then $\alpha_i = 0$ $(0 \leq i \leq k-1)$ and $\alpha_i = \sqrt{\alpha_k^2 + t^2}$ $(i \geq k+1)$;*

ii) *if $\alpha_k = 0$, then $\alpha_i = 0$ $(0 \leq i \leq k)$ and $\alpha_i = t$ $(i \geq k+1)$.*

2.2. Characterizations for p-hyponormality.

Theorem 2.2. *Let $T(k,t)$ be as in* (2.1). *Suppose that $p \in (0, \infty)$. Then*

(i) *$T(0,t)$ is p-hyponormal if and only if $\alpha_1^2 \geq \alpha_0^2 + t^2$ and $\alpha_{i+1} \geq \alpha_i$ for $i \in \mathbb{N}$;*

(ii) *$T(k,t)$ is p-hyponormal if and only if $\alpha_i \leq \alpha_{i+1}$ $(0 \leq i \leq k-3)$, $\alpha_{i+k+1} \geq \alpha_{i+k}$ $(i \in \mathbb{N})$ and it holds that:*

$$\delta_{11} > 0, \delta_{11}\delta_{22} - \delta_{12}^2 > 0, \text{ and } \delta_{33}\left(\delta_{11}\delta_{22} - \delta_{12}^2\right) - \delta_{11}\delta_{23}^2 \geq 0. \tag{2.2}$$

where

$$\begin{aligned}
\delta_{11} &= -\alpha_{k-2}^{2p} + \left\{(t^2+\alpha_k^2-\alpha_{k-1}^2+\gamma_k)\lambda_k^p + (\alpha_{k-1}^2 - t^2 - \alpha_k^2 + \gamma_k)\mu_k^p\right\}/(2\gamma_k);\\
\delta_{12} &= \delta_{21} = t\alpha_{k-1}(\mu_k^p - \lambda_k^p)/\gamma_k; \delta_{22} = (\alpha_k^2 - \alpha_{k-1}^2)(\mu_k^p - \lambda_k^p)/\gamma_k;\\
\delta_{23} &= \delta_{32} = t\alpha_k(\lambda_k^p - \mu_k^p)/\gamma_k;\\
\delta_{33} &= \alpha_{k+1}^{2p} - \left\{(t^2+\alpha_{k-1}^2-\alpha_k^2+\gamma_k)\lambda_k^p + (\alpha_k^2 - t^2 - \alpha_{k-1}^2 + \gamma_k)\mu_k^p\right\}/(2\gamma_k);\\
\lambda_k &= (t^2+\alpha_{k-1}^2+\alpha_k^2-\gamma_k)/2;\ \mu_k = (t^2+\alpha_{k-1}^2+\alpha_k^2+\gamma_k)/2;\\
\gamma_k &= [(t^2+\alpha_{k-1}^2+\alpha_k^2)^2 - 4(\alpha_{k-1}\alpha_k)^2]^{1/2} \quad (\textit{with } \alpha_{-1} := 0).
\end{aligned}$$

2.3. Examples for distinction of p-hyponormalities. Let W_α be a weighted shift with weight sequence α satisfying

$$\alpha_n = 0\ (0 \le n \le k-2),\ \alpha_{k-1} = \sqrt{x},\ \alpha_k = 1,\ \alpha_n = 2\ (n \ge k+1).$$

Let $T := T(k,t) = W_\alpha + te_k \otimes e_k$ for $0 \le x \le 1$, $t \in [0,\infty)$, and $\gamma = \sqrt{(1+x+t^2)^2 - 4x}$. Applying Theorem 2.2 with x, t, and γ, we obtain that $(T^*T)^p \ge (TT^*)^p$ if and only if $A^p \ge B^p$ for $0 < p < \infty$, where

$$A = \begin{pmatrix} x & t\sqrt{x} & 0 \\ t\sqrt{x} & t^2+1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & t^2+x & t \\ 0 & t & 1 \end{pmatrix}.$$

To compute A^p and B^p, first we find eigenvalues and eigenvectors of A and B so that we may have $D = P^{-1}AP$ and $E = Q^{-1}BQ$ in usual fashion; in fact, $D = \mathrm{Diag}\{\lambda, \mu, 4\}$, $E = \mathrm{Diag}\{0, \lambda, \mu\}$, $\lambda := \frac{1}{2}(1+x+t^2-\gamma)$, $\mu := \frac{1}{2}(1+x+t^2+\gamma)$, and

$$P = \begin{pmatrix} \frac{x-t^2-1-\gamma}{2t\sqrt{x}} & \frac{x-t^2-1+\gamma}{2t\sqrt{x}} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{x+t^2-1-\gamma}{2t} & \frac{x+t^2-1+\gamma}{2t} \\ 0 & 1 & 1 \end{pmatrix}.$$

By a direct computation, $\Delta = A^p - B^p = (\delta_{ij})_{3\times 3}$ with
$\delta_{11} = \frac{1}{2\gamma}\{\lambda^p(-x+t^2+1+\gamma) + \mu^p(x-t^2-1+\gamma)\}$,
$\delta_{12} = \delta_{21} = \frac{1}{\gamma}(\mu^p - \lambda^p)t\sqrt{x}$, $\delta_{22} = \frac{1}{\gamma}(1-x)(\mu^p - \lambda^p)$,
$\delta_{23} = \delta_{32} = \frac{1}{\gamma}(\lambda^p - \mu^p)t$,
$\delta_{33} = 4^p - \frac{1}{2\gamma}\{\mu^p(1-x-t^2+\gamma) + \lambda^p(x+t^2-1+\gamma)\}$,
$\delta_{ij} = 0$ otherwise.
And, we write $d^{(i)}$ $(i = 1,2,3)$ for the determinant of the $i \times i$ upper left corner of the matrix Δ. Since $x - t^2 - 1 + \gamma > 0$ and $0 < \lambda < \mu$, $d^{(1)} = \delta_{11} > 0$. By simple calculation, we obtain

$$\begin{aligned}
f_1(x,t,p) := \frac{2\gamma^2}{\mu^p - \lambda^p}\cdot d^{(2)} &= \lambda^p\left[1 - \gamma(x-1) - 2x + x^2 + t^2 + xt^2\right]\\
&\quad - \mu^p\left[1 + \gamma(x-1) + x^2 + t^2 + x(t^2-2)\right].
\end{aligned}$$

And by more computation, we obtain that $d^{(3)} = \frac{(\mu^p-\lambda^p)}{4\gamma^3} f_2(x,t,p)$, where

$$
\begin{aligned}
f_2(x,t,p) \ : \ &= 2(\lambda^p-\mu^p)t^2\left\{\mu^p(-1+\gamma+x-t^2)+\lambda^p(1+\gamma-x+t^2)\right\} \\
&+\left\{2\cdot 4^p\gamma+\mu^p(-1-\gamma+x+t^2)-\lambda^p(-1+\gamma+x+t^2)\right\} \\
&\cdot\left[2(\lambda^p-\mu^p)xt^2+(1-x)\left\{\mu^p(-1+\gamma+x-t^2)+\lambda^p(1+\gamma-x+t^2)\right\}\right].
\end{aligned}
$$

Since $\mu > \lambda$, $d^{(3)} \geq 0$ if and only if $f_2(x,t,p) \geq 0$ for $0 \leq x \leq 1$, $t \in [0,\infty)$ and $p > 0$. Hence T is p-hyponormal if and only if $f_1 > 0$ and $f_2 \geq 0$. And we obtain the regions for p-hyponormalities in Figure 2.1.

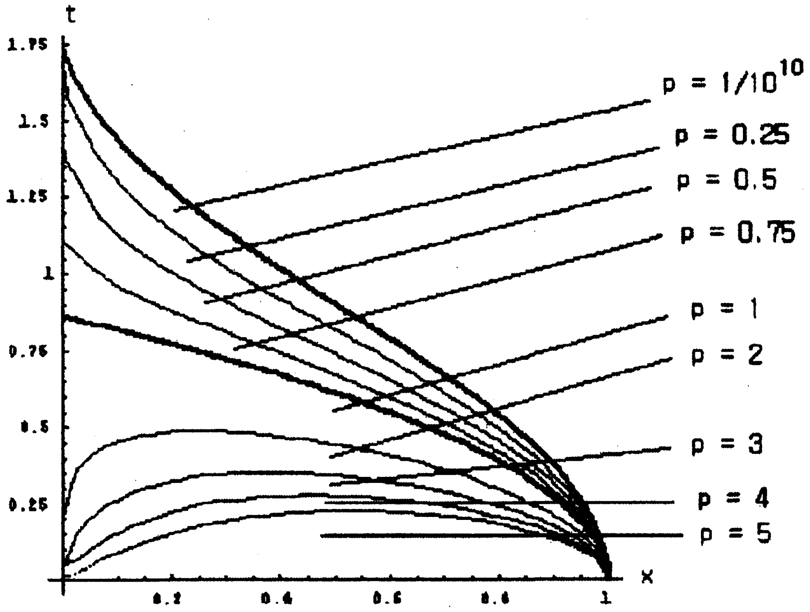

Figure 2.1

3. Weak hyponormalities There are several kinds of weak hyponormalities that are weaker than p -hyponormality, for examples, p-paranormality, absolute p-paranormality p-paranormality, absolute p-paranormality, $A(p)$-class, normaloid, and spectraloid. It is not known whether the p-paranormality is different from the absolute p-paranormality for each $p \in (0,\infty)\backslash\{1\}$. In this section we discuss p-paranormal and absolute p-paranormal operators and continue Example 2.3 to discuss distinction between p -paranormality and absolute p-paranormality$p$-paranormality.

3.1. Absolute p-paranormality. Let $T \in \mathcal{L}(\mathcal{H})$. Then it follows from [Fur, p.174] that T is absolute p-paranormal if and only if $T^*(T^*T)^pT-(p+1)T^*Ts^p+ps^{p+1} \geq 0$ for all $s \in \mathbb{R}_+$.

Theorem 3.1. *Let $T := T(k,t)$ be as in* (2.1). *Suppose $k \geq 2$. Then the following assertions are equivalent:*

(i) *T is absolute p-paranormal;*

(ii) $\alpha_{n+1} \geq \alpha_n$, $n \in \mathbb{N}_0\backslash\{k-i : i = 0,1,2\}$; *and for all $s \in \mathbb{R}_+$,*

$$
\Omega_k(s) := \begin{pmatrix} \omega_{11} & \phi_2\alpha_{k-1}\alpha_{k-2} & t\phi_2\alpha_{k-2} \\ \phi_2\alpha_{k-1}\alpha_{k-2} & \omega_{22} & t\alpha_{k-1}(\phi_3-(p+1)s^p) \\ t\phi_2\alpha_{k-2} & t\alpha_{k-1}(\phi_3-(p+1)s^p) & \omega_{33} \end{pmatrix} \geq 0,
$$

where

$$\begin{aligned}
\omega_{11} &: = \omega_{11}(p,t) = \phi_1\alpha_{k-1}^2 - (p+1)\alpha_{k-2}^2 s^p + ps^{p+1};\\
\omega_{22} &: = \omega_{22}(p,t) = \phi_3\alpha_{k-1}^2 - (p+1)\alpha_{k-1}^2 s^p + ps^{p+1};\\
\omega_{33} &: = \omega_{33}(p,t) = t^2\phi_3 + \alpha_k^2\alpha_{k+1}^{2p} - (p+1)s^p(t^2+\alpha_k^2) + ps^{p+1};\\
\phi_1 &: = \phi_1(k,p) = (\lambda_k^p + \mu_k^p)/2 + (\lambda_k^p - \mu_k^p)(t^2 - \alpha_{k-1}^2 + \alpha_k^2)/(2\gamma_k);\\
\phi_2 &: = \phi_2(k,p) = t\alpha_{k-1}(\mu_k^p - \lambda_k^p)/\gamma_k;\\
\phi_3 &: = \phi_3(k,p) = (\lambda_k^p + \mu_k^p)/2 - (\lambda_k^p - \mu_k^p)(t^2 - \alpha_{k-1}^2 + \alpha_k^2)/(2\gamma_k).
\end{aligned}$$

Proposition 3.2. *Under the same notation with Theorem 3.1, it holds that*
i) $T(0,t)$ *is absolute p-paranormal if and only if* $\alpha_1^2 \geq t^2 + \alpha_0^2$ *and* $\alpha_{n+1} \geq \alpha_n$ $(n \geq 1)$;
ii) $T(1,t)$ *is absolute p-paranormal if and only if* $\alpha_{n+1} \geq \alpha_n$ $(n \geq 2)$ *and for all* $s \in \mathbb{R}_+$,

$$\begin{pmatrix} \alpha_0^2\delta - (p+1)s^p\alpha_0^2 + ps^{p+1} & t\alpha_0(\delta - (p+1)s^p) \\ t\alpha_0(\delta - (p+1)s^p) & t^2\delta + \alpha_1^2\alpha_2^{2p} - (p+1)s^p(t^2+\alpha_1^2) + ps^{p+1} \end{pmatrix} \geq 0,$$

where $\delta = \phi_3(1,p)$.

The following remark comes immediately from Proposition 3.2 above.

Remark 3.3. $T(0,t)$ *is absolute paranormal if and only if* $T(0,t)$ *is absolute p-paranormal for all[some]* $p \in (0,\infty)$.

3.2. p-paranormality. Let $T = U|T| \in \mathcal{L}(\mathcal{H})$. Then is follows from [YY, Proposition 3] that T is p-paranormal if and only if $|T|^p U^* |T|^{2p} U |T|^p - 2s|T|^{2p} + s^2 \geq 0$ for all $s \in \mathbb{R}_+$. Let $T(k,t)$ be as in (2.1) and let $T(k,t) = U(k,t)\,|T(k,t)|$ be a polar decomposition. Then $U(k,t)$ has the form such that the $(i+1,i)$-terms are $1,\cdots,1,F_k,1,\cdots$ $(k \geq 1)$, where F_k is $(k+1,k)$ term of $U(k,t)$ and

$$F_k = \frac{1}{\alpha_{k-1}\alpha_k}\begin{pmatrix} (\alpha_{k-1}\phi_3(k,\frac{1}{2}) - t\phi_2(k,\frac{1}{2})) & (t\phi_1(k,\frac{1}{2}) - \alpha_{k-1}\phi_2(k,\frac{1}{2})) \\ -\alpha_k\phi_2(k,\frac{1}{2}) & \alpha_k\phi_1(k,\frac{1}{2}) \end{pmatrix};$$

and others are 0. In particular, $U(0,t) = W_\beta + \frac{t}{\sqrt{t^2+\alpha_0^2}} e_0 \otimes e_0$, where $\beta : \beta_0 = \frac{\alpha_0}{\sqrt{t^2+\alpha_0^2}}, \beta_k = 1$ $(k \geq 1)$. For brevity we write $u_{ij}(k)$ is the (i,j) term of F_k. By the similar method of Theorem 3.1 and the above characterization for p-paranormality, we obtain the following results, but we omit the detail proof here.

Proposition 3.4. *Let* $T(k,t)$ *be as in* (2.1) *and let* u_{ij} *be as above. Then*
(i) $T(0,t)$ *is p-paranormal if and only if* $\alpha_{n+1} \geq \alpha_n$ $(n \geq 1)$ *and* $\alpha_1^2 \geq \alpha_0^2 + t^2$;
(ii) $T(1,t)$ *is p-paranormal if and only if* $\alpha_{n+1} \geq \alpha_n$ $(n \geq 1)$ *and, for all* $s \in \mathbb{R}_+$

$$\begin{pmatrix} \psi_1 - 2\phi_1(1,p)s + s^2 & \psi_2 - 2\phi_2(1,p)s \\ \psi_2 - 2\phi_2(1,p)s & \psi_3 - 2\phi_3(1,p)s + s^2 \end{pmatrix} \geq 0$$

with

$$\begin{aligned}
\psi_1 &= \phi_3(1,p)[\phi_1(1,\tfrac{p}{2})u_{11}(1) + \phi_2(1,\tfrac{p}{2})u_{12}(1)]^2 + \alpha_2^{2p}[\phi_1(1,\tfrac{p}{2})u_{21}(1) + \phi_2(1,\tfrac{p}{2})u_{22}(1)]^2,\\
\psi_2 &= \phi_3(1,p)[\phi_1(1,\tfrac{p}{2})u_{11}(1) + \phi_2(1,\tfrac{p}{2})u_{12}(1)][\phi_2(1,\tfrac{p}{2})u_{11}(1) + \phi_3(1,\tfrac{p}{2})u_{12}(1)]\\
&\quad + \alpha_2^{2p}[\phi_1(1,\tfrac{p}{2})u_{21}(1) + \phi_2(1,\tfrac{p}{2})u_{22}(1)][\phi_2(1,\tfrac{p}{2})u_{21}(1) + \phi_3(1,\tfrac{p}{2})u_{22}(1)]\\
\psi_3 &= \phi_3(1,p)[\phi_2(1,\tfrac{p}{2})u_{11}(1) + \phi_3(1,\tfrac{p}{2})u_{12}(1)]^2 + \alpha_2^{2p}[\phi_2(1,\tfrac{p}{2})u_{21}(1) + \phi_3(1,\tfrac{p}{2})u_{22}(1)]^2.
\end{aligned}$$

The following remark follows immediately from Proposition 3.4 (i).

Remark 3.5. *$T(0,t)$ is paranormal if and only if $T(0,t)$ is p-paranormal for all [some] $p \in (0,\infty)$.*

Theorem 3.6. *Let $T(k,t)$ be as in (2.1) and let u_{ij} and ϕ_j be as above. Suppose $k \geq 2$. Then $T(k,t)$ is p-paranormal if and only if $\alpha_{n+1} \geq \alpha_n$ $(0 \leq n \leq k-3;\ n \geq k+1)$ and, for all $s \in \mathbb{R}_+$*

$$\Psi_k := \begin{pmatrix} \varphi_{11} - 2\alpha_{k-2}^{2p}s + s^2 & \varphi_{12} & \varphi_{13} \\ \varphi_{12} & \varphi_{22} - 2s\phi_1(p) + s^2 & \varphi_{23} - 2\phi_2(p)s \\ \varphi_{13} & \varphi_{23} - 2\phi_2(p)s & \varphi_{33} - 2\phi_3(p)s + s^2 \end{pmatrix} \geq 0$$

where

$$\begin{aligned}
\varphi_{11} &: = \alpha_{k-2}^{2p}\phi_1(p); \\
\varphi_{12} &: = \alpha_{k-2}^{2p}\phi_2(p)[\phi_1(\tfrac{p}{2})u_{11}(k) + \phi_2(\tfrac{p}{2})u_{12}(k)]; \\
\varphi_{13} &: = \alpha_{k-2}^{2p}\phi_2(p)[\phi_2(\tfrac{p}{2})u_{11}(k) + \phi_3(\tfrac{p}{2})u_{12}(k)]; \\
\varphi_{22} &: = \phi_3(p)[\phi_1(\tfrac{p}{2})u_{11}(k) + \phi_2(\tfrac{p}{2})u_{12}(k)]^2 + \alpha_{k+1}^{2p}[\phi_1(\tfrac{p}{2})u_{21}(k) + \phi_2(\tfrac{p}{2})u_{22}(k)]^2; \\
\varphi_{23} &: = \phi_3(p)[\phi_1(\tfrac{p}{2})u_{11}(k) + \phi_2(\tfrac{p}{2})u_{12}(k)][\phi_2(\tfrac{p}{2})u_{11}(k) + \phi_3(\tfrac{p}{2})u_{12}(k)] \\
&\quad +\alpha_{k+1}^{2p}[\phi_1(\tfrac{p}{2})u_{21}(k) + \phi_2(\tfrac{p}{2})u_{22}(k)][\phi_2(\tfrac{p}{2})u_{21}(k) + \phi_3(\tfrac{p}{2})u_{22}(k)]; \\
\varphi_{33} &: = \phi_3(p)[\phi_2(\tfrac{p}{2})u_{11}(k) + \phi_3(\tfrac{p}{2})u_{12}(k)]^2 + \alpha_{k+1}^{2p}[\phi_2(\tfrac{p}{2})u_{21}(k) + \phi_3(\tfrac{p}{2})u_{22}(k)]^2;
\end{aligned}$$

(we write $\phi_i(p)$ for $\phi_i(k,p)$ for brevity).

Remark 3.7. Recall that $T \in \mathcal{L}(\mathcal{H})$ is an *$A(p)$-class operator* if $(T^* |T|^{2p} T)^{\frac{1}{p+1}} \geq |T|^2$ $(0 < p < \infty)$. We can apply our method to this $A(p)$ class operators. We leave these computations to interesting readers.

3.3. Examples for weak hyponormalities (continued from Example 2.3). Let $T := T(k,t) = W_\alpha + te_k \otimes e_k$ $(0 \leq x \leq 1)$ be as in Example 2.3. In this example, we discuss operators $T(x,t)$ with absolute-p-paranormality but not p-paranormality for $p \in (0,1)$, and operators with p-paranormality but not absolute-p-paranormality for $p \in (1,\infty)$. In Table 3.1, $\Omega_k^{(2)}$ is the determinant of lower right 2×2 submatrix of Ω_k in Theorem 3.1, and $\Psi_k^{(1)}$ is the (2, 2)-term of Ψ_k and $\Psi_k^{(2)}$ is the determinant of lower right 2×2 submatrix of Ψ_k in Theorem 3.6 (Note that (1,2), (1,3), (2,1), and (3,1) terms of Ω_k and Ψ_k are zero and (1,1) term of Ψ_k is positive in this example.)

Algorithm 3.8. Under the same notation with Theorems 2.2, 3.1, and 3.4, we give steps to obtain examples being distinct p-hyponormal, absolute-p-paranormality, and p-paranormality.

I. Take p, x, t such that $T(x,t)$ does not satisfy p-hyponormality in Figure 2.1, i.e., $f := f(x,t,p) < 0$;

II. For p, x, t taken in Step I, check the positivity of ω_{11}, ω_{22}, and $\Omega_k^{(2)}$, for all $s \in \mathbb{R}_+$;

III. For p, x, t taken in Step I, check the positivity $\Psi_k^{(1)}$, and $\Psi_k^{(2)}$, for all $s \in \mathbb{R}_+$.

We give examples being distinct absolute-p-paranormality, and p-paranormality for $0 < p < 1$ and $p > 1$, respectively, as following.

Example 3.9 (Absolute p-paranormal but not p-paranormal for $p < 1$). If we take $p = .25$; $x = .4$; $t = 1.166$, then we have the following:

I. $f(x,t,p) \approx -2.24293$;

II. for all $s \in \mathbb{R}_+$, $\omega_{11} \approx .276285 + .25s^{5/4} > 0$; $\omega_{22} \approx .482308 - .5s^{1/4} + .25s^{5/4} > 0$; $\Omega_k^{(2)} \approx .682086 - 1.30999s^{1/4} + .625s^{1/2} + .883958s^{5/4} - .862361s^{3/2} + .0625s^{5/2} > 0$;

III. for all $s \in \mathbb{R}_+$, $\Psi_k^{(1)} \approx .865735 - 1.38143s + s^2 > 0$;

$\Psi_k^{(2)} \approx (-1.26429 + s)(-1.26321 + s)(.849122 - 1.26546s + s^2) \not\geq 0$.

Hence T is absolute p-paranormal but not p-paranormal.

Example 3.10 (p-paranormal but not absolute p-paranormal for $p > 1$). If we take $p = 2$; $x = .7$; $t = 1.347$, then we have the following:

I. $f(x,t,p) \approx -2411.31$;

II. for all $s \in \mathbb{R}_+$, $\omega_{11} \approx 1.23206 + 2s^3 > 0$; $\omega_{22} \approx 6.43369 - 2.1s^2 + 2s^3 > 0$

$\Omega_k^{(2)} \approx 4(-3.34444+s)(-3.25175+s)(1.01729+s)(1.66817+s)(1.39443-1.36088s+s^2) \not\geq 0$;

III. for all $s \in \mathbb{R}_+$, $\Psi_k^{(1)} \approx 17.8439 - 3.52017s + s^2 > 0$;

$\Psi_k^{(2)} \approx 72.0573 - 24.6872s + 121.774s^2 + 21.9021s^3 + s^4 \geq 0$.

Hence T is p-paranormal but not absolute p-paranormal.

Repeating these processes in Examples 3.9 and 3.10 with Algoritm 3.8 and some scales, we have the following table 3.1, which shows that the absolute-p-paranormality is different from p-paranormality in some numerical computations.

Table 3.1

p	x	t	f	ω_{11}	ω_{22}	$\Omega_k^{(2)}$	$\Psi_k^{(1)}$	$\Psi_k^{(2)}$	p-H	A-p-P	p-P
.25	.4	1	−	+	+	+	+	+	NO	YES	YES
.25	.4	1.166	−	+	+	+	+	−	NO	YES	NO
.50	.8	.5	−	+	+	+	+	+	NO	YES	YES
.50	.8	.85	−	+	+	+	+	−	NO	YES	NO
.75	.6	.6	−	+	+	+	+	+	NO	YES	YES
.75	.6	1.189	−	+	+	+	+	−	NO	YES	NO
1	.5	1.33523	−	+	+	+	+	+	NO	YES	YES
1	.5	1.33524	−	+	+	−	+	−	NO	NO	NO
2	.7	1	−	+	+	+	+	+	NO	YES	YES
2	.7	1.347	−	+	+	−	+	+	NO	NO	YES
3	.2	1.6	−	+	+	+	+	+	NO	YES	YES
3	.2	1.6452	−	+	+	−	+	+	NO	NO	YES
4	.9	1.367	−	+	+	+	+	+	NO	YES	YES
4	.9	1.368	−	+	+	−	+	+	NO	NO	YES
5	.5	1.5	−	+	+	+	+	+	NO	YES	YES
5	.5	1.5537	−	+	+	−	+	+	NO	NO	YES

p-H = p-hyponormal; A-p-H = absolute p-hyponormal; p-P = p-paranormal

References

[BJ] C. Burnap and I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl. **337**(2008), 686-694.

[BJL] C. Burnap, I. Jung, and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53**(2005), 381-397.

[CF] R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equation and Operator Theory, 17(1993), 202-246.

[FN] M. Fujii and Y. Nakatsu, *On subclasses of hyponormal operators*, Proc. Japan Acad., **51**(1975), 243-246.

[Fuj] M. Fujii, *Some classes of operators related to paranormal and log-hyponormal operators*, Math. Japonica, 51(2000), 395-402.

[Fur] T. Furuta, *Invitation to linear operators*, Taylor & Francis Inc. London and New York(2001).

[JLL] I. Jung, M. Lee, and P. Lim, *Gaps of operators, II*, Glasgow Math. J. **47**(2005), 461-469.

[JLP] I. Jung, P. Lim, and S. Park, *Gaps of operators*, J. Math. Anal. Appl. **304**(2005), 87-95.

[MS] S. Miyajima and I. Saito, ∞-*hyponormal operators and their spectral properties*, Acta Sci. Math. (Szeged), **67**(2001), 357-371.

[Xi] D. Xia, *Spectral Theory of Hyponormal Operators*, Birkhäuser Verlag, Boston, 1983.

[Wol] Wolfram Research, Inc. Mathematica, Version 3.0, Wolfram Research Inc. Champaign, IL,(1996).

[YY] T. Yamazaki and M. Yanagida, *A further generalized of paranormal operators*, Sci. Math. **3**(2000), 23-31.

[EJLL] G. Exner, I. Jung, E. Lee and M. Lee, *Rank-one perturbations separating weak hyponormalities*, preprint