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Global asymptotic stability for a class of difference equations

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1 Introduction

Consider the following nonlinear difference equation with variable coefficients:

$$x_{n+1}=qx_{n}-\sum_{j=0}^{m}a_{j}f_{j}(x_{n-j}),\quad n=0,1,2,\cdots, \tag{1.1}$$

where $0<q\leq 1,\ \ a_{j}\geq 0,\ 0\leq j\leq m$ and $\sum_{j=0}^{m}a_{j}>0$. We now assume that

$$\begin{cases} f(x)\in C(-\infty,+\infty) \text{ is a strictly monotone increasing function,} \\ f(0)=0,\quad 0<\frac{f_{j}(x)}{f(x)}\leq 1,\ \ x\neq 0,\ \ 1\leq j\leq m,\quad \text{and} \\ \text{if } f(x)\neq x, \text{ then } \lim_{x\rightarrow-\infty}f(x) \text{ is finite, otherwise } f(x)=x. \end{cases} \tag{1.2}$$

The above difference equation has been studied by many literatures (see for example, [1]-[9] and references therein).

Definition 1.1 *The solution y^{*} of (1.1) is called uniformly stable, if for any $\epsilon>0$ and non-negative integer n_{0}, there is a constant $\delta=\delta(\epsilon)>0$ such that* $\sup\{|y_{n_{0}-i}-y^{*}|\mid 0\leq i\leq m\}<\delta$, *implies that the solution $\{y_{n}\}_{n=0}^{\infty}$ of (1.1) satisfies $|y_{n}-y^{*}|<\epsilon,\ n=n_{0},n_{0}+1,\cdots$.*

Definition 1.2 *The solution y^{*} of (1.1) is called globally attractive, if every solution of (1.1) tends to y^{*} as $n\rightarrow\infty$.*

Definition 1.3 *The solution y^{*} of (1.1) is called globally asymptotically stable, if it is uniformly stable and globally attractive.*

In this paper, we study "semi-contractive" functions and global asymptotic stability of difference equations. In Section 2, we first define semi-contractivity of functions and show the related results on the global asymptotic stability of difference equations.

2 Semi-contractive function

Assume that

$$g(z_{0},z_{1},\cdots,z_{m})\in C(R^{m+1})\quad \text{and}\quad g(y,y,\cdots,y)=y \text{ has a unique solution } y=y^{*}. \tag{2.1}$$

Definition 2.1 *The function $g(z_0, z_1, \cdots, z_m)$ is said to be semi-contractive at y^*, if*
(i) for any constants $\underline{z} < y^*$ and $z_i \geq \underline{z}$, $0 \leq i \leq m$, there exists a constant $y^* < \bar{z} < +\infty$ such that $g(z_0, z_1, \cdots, z_m) \leq \bar{z}$, and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $\underline{\underline{z}} > \underline{z}$ such that $\underline{\underline{z}} \leq g(z_0, z_1, \cdots, z_m)$, or
(ii) for any constants $\bar{z} > y^*$ and $z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $y^* > \underline{z} > -\infty$ such that $g(z_0, z_1, \cdots, z_m) \geq \underline{z}$, and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $\bar{\bar{z}} < \bar{z}$ such that $\bar{\bar{z}} \geq g(z_0, z_1, \cdots, z_m)$.

Lemma 2.1 *If $g(y) \in C(R)$ is a strictly monotone decreasing function such that $g(g(y)) > y$ for any $y < y^*$, then $g(z)$ is semi-contractive for y^*.*

Lemma 2.2 *Assume* (2.1) *and that each $g_i(z_0, z_1, \cdots, z_m)$, $0 \leq i \leq m$ is semi-contractive for y^*. Then for any $b_{n,i} \geq 0$, $n \geq 0$, $0 \leq i \leq m$ such that $\sum_{i=0}^{m} b_{n,i} = 1$ and $\lim_{n\to\infty} b_{n,i} = b_i$, $0 \leq i \leq m$, it holds that $\sum_{i=0}^{m} b_{n,i} g_i(z_0, z_1, \cdots, z_m)$ is semi-contractive for y^*.*

Collorary 2.1 *Assume* (2.1) *and that $g(z_0, z_1, \cdots, z_m)$ is semi-contractive for y^*. Then for any $0 \leq q_n < 1$, $g_n(z_0, z_1, \cdots, z_m)$ and k such that*

$$\begin{cases} \lim\limits_{n\to\infty} q_n = q < 1, \quad \text{and} \quad 0 \leq k \leq m, \\ \lim\limits_{n\to\infty} g_n(z_0, z_1, \cdots, z_m) = g(z_0, z_1, \cdots, z_m) \quad \text{for any} \quad z_0, z_1, \cdots, z_m \in (-\infty, +\infty), \end{cases} \tag{2.2}$$

it holds that $q_n z_k + (1 - q_n) g_n(z_0, z_1, \cdots, z_m)$ is semi-contractive for y^*.

Collorary 2.2 *Assume that each $g_i(z) \in C(R)$ and $g_i(y) = y$ has a unique solution $y = y^*$, $0 \leq i \leq m$, and each $g_i(z_i)$, $0 \leq i \leq m$ is semi-contractive for y^*, then for any $b_{n,i} \geq 0$, $n \geq 0$, $0 \leq i \leq m$ such that $\sum\limits_{i=0}^{m} b_{n,i} = 1$ and $\lim_{n\to\infty} b_{n,i} = b_i$, $0 \leq i \leq m$, it holds that $\sum_{i=0}^{m} b_{n,i} g_i(z_i)$ is semi-contractive for y^*. In particular, for any $0 \leq q_n < 1$ and k such that $\lim_{n\to\infty} q_n = q < 1$ and $0 \leq k \leq m$, it holds that $q_n z_k + (1 - q_n) \sum\limits_{i=0}^{m} b_{n,i} g_i(z_i)$ is semi-contractive for y^*.*

Remark 2.1 If $g(z_0, z_1, \cdots, z_m) > 0$ for any $z_i > 0$, $0 \leq i \leq m$, then there are cases that we may restrict our attention only to $z_i > 0$, $0 \leq i \leq m$ and the unique positive solution $y^* > 0$ of $g(y^*, y^*, \cdots, y^*) = y^*$, whether or not $g(y, y, \cdots, y) = y$ has other solutions $y \leq 0$.

Example 2.1 Examples of semi-contractive function $g(z_0, z_1, \cdots, z_m)$ for y^*.
(i) $g(z_0, z_1, \cdots, z_m) = z_m e^{c(1-z_m)}$, $y^* = 1$ and $c \leq 2$ (see [1]).
(ii) $g(z_0, z_1, \cdots, z_m) = z_0 \exp\{c(1 - \sum_{i=0}^{m} a_i z_i)\}$, $y^* = 1/(\sum_{i=0}^{m} a_i)$ and $c \leq 2$, where $a_0 > 0$, $a_i \geq 0$, $1 \leq i \leq m$ and $(\sum_{i=1}^{m} a_i)/a_0 \leq 2/e$.
This is equivalent that $h(u_0, u_1, \cdots, u_m) = u_0 - c \sum_{i=0}^{m} b_i (e^{u_i} - 1)$ is semi-contractive for $u^* = 0$ and $c \leq 2$, where $z_i = y^* e^{u_i}$, $b_0 = y^* a_0 > 0$, $b_i = y^* a_i \geq 0$, $1 \leq i \leq m$, $\sum_{i=0}^{m} b_i = 1$, and $(\sum_{i=1}^{m} b_i)/b_0 \leq 2/e$ (see [8]).
(iii) $g(z_0, z_1, \cdots, z_m) = c(1 - e^{z_m})$, $y^* = 0$ and $c \leq 1$ (see [3]).
(iv) $g(z_0, z_1, \cdots, z_m) = \frac{cz_m}{1+bz_m^p}$, $x^* = ((c-1)/b)^{1/p}$ and $c \leq \frac{p}{p-2}$, where $p > 2$ and $b > 0$ (see [1]).

We consider the following difference equation

$$y_{n+1} = q_n y_{n-k} + (1 - q_n) g_n(y_n, y_{n-1}, \cdots, y_{n-m}), \qquad n = 0, 1, \cdots, \tag{2.3}$$

where we assume (2.1) and

$$\begin{cases} 0 \le q_n < 1, \quad \lim_{n\to\infty} q_n = q < 1, \quad k \in \{0, 1, \cdots, m\}, \quad \text{and} \\ \lim_{n\to\infty} g_n(z_0, z_1, \cdots, z_m) = g(z_0, z_1, \cdots, z_m) \quad \text{for any} \quad z_0, z_1, \cdots, z_m \in (-\infty, +\infty). \end{cases} \tag{2.4}$$

Theorem 2.1 *If $g(z_0, z_1, \cdots, z_m)$ is semi-contractive for y^*, then y^* of (2.3) is globally asymptotically stable for any $0 \le q < 1$.*

Collorary 2.3 *Assume that there exists a constant $0 \le q_0 < 1$ and some $0 \le k \le m$ such that $q_0 z_k + (1 - q_0)g(z_0, z_1, \cdots, z_m)$, is semi-contractive for y^*. Then, for any $q_0 \le q_n < 1$ and $g_n(z_0, z_1, \cdots, z_m)$ which satisfy (2.4), the solution y^* of (2.3) is globally asymptotically stable.*

Remark 2.2 (i) The corresponding continuous case (2.3) is the following differential equation

$$\begin{cases} y'(t) = -p(t)\{y(t) - \frac{1}{1-q_n} g_n(y(n), y(n-1), \cdots, y(n-m))\}, \ n \le t < n+1, \quad n = 0, 1, 2, \cdots, \\ p(t) > 0, \quad q_n = e^{-\int_n^{n+1} p(t)dt} < 1. \end{cases}$$

(ii) In Theorem 2.1, a *semi-contractivity condition* is a *delays and q_n-independent condition* for the solution y^* of (2.3) to be globally asymptotically stable.

By Theorem 2.1 and Example 2.1, we obtain the following result:

Example 2.2 Examples of delays and q-independent stability conditions.
(i) Ricker model $y_{n+1} = qy_n + (1-q)y_{n-m}e^{c(1-y_{n-m})}, \quad n = 0, 1, 2, \cdots$. The positive equilibrium $y^* = 1$ is globally asymptotically stable, if $c \le 2$ (see [1]).
(ii) Ricker model with delaled-density dependence $y_{n+1} = qy_n + (1-q)y_n \exp\{c(1 - \sum_{i=0}^m a_i y_{n-i})\}$. The positive equilibrium $y^* = 1/(\sum_{i=0}^m a_i)$ is globally asymptotically stable, if $c \le 2$, where $a_0 > 0, \ a_i \ge 0, \ 1 \le i \le m$ and $(\sum_{i=1}^m a_i)/a_0 \le 2/e$ (see [8]).
(iii) Wazewska-Czyzewska and Lasota model $y_{n+1} = qy_n + (1-q)c\sum_{i=0}^{m} b_i e^{-\gamma y_{n-i}}, \quad n = 0, 1, 2, \cdots,$
where $\gamma > 0, \ b_i \ge 0, \ 0 \le i \le m,$ and $\sum_{i=0}^m b_i = 1$.
The positive equilibrium y^* is the positive solution of the equation $y^* = ce^{-\gamma y^*}$. Put $x_n = \gamma(y^* - y_n)$. Then, this equation is equivalent to

$$x_{n+1} = qx_n - (1-q)\gamma y^* \sum_{i=0}^{m} b_i(e^{x_{n-i}} - 1), \quad \text{where } b_i \ge 0, \ 0 \le i \le m, \quad \sum_{i=0}^{m} b_i = 1. \tag{2.5}$$

Thus, the positive equilibrium y^* is globally asmptotically stable, if $c \le e/\gamma$ which is equivalent that the zero solution of (2.5) is globally asymptotically stable if $\gamma y^* \le 1$ (see [3]).
(iv) Bobwhite quail population model $y_{n+1} = qy_n + (1-q)\frac{cy_{n-m}}{1+by_{n-m}^p}, \ n = 0, 1, 2, \cdots,$ where $c > 1, \ b > 0$. The positive equilibrium $y^* = ((c-1)/b)^{1/p}$ is globally asymptotically stable, if $c \le \frac{p}{p-2}$ for $p > 2$ (see [1]).

We have the following counter example:

Example 2.3 Examples of q-dependent and delay-dependent stability conditions.
(i) A model in hematopoiesis $y_{n+1} = qy_n + (1-q)e^{2(1-y_n)}, \quad n = 0, 1, 2, \cdots$.
The equilibrium $y^* = 1$ is *globally asymptotically stable* if $q \in [1/3, 1)$, and 2-cycle if $q \in [0, 1/3)$ (see [2]).
(ii) A delayed model in hematopoiesis $y_{n+1} = qy_n + (1-q)e^{2(1-y_{n-2})}, \quad n = 0, 1, 2, \cdots$.
The characteristic equation takes the form $\lambda^3 - q\lambda^2 = -2(1-q)$. Then for $q = q_2 = \frac{3-\sqrt{3}}{2} =$

$0.633975\cdots > 1/3$, the roots are $-1 < \lambda_1 < 0$, $|\lambda_2| = |\lambda_3| = 1$. For $q_2 < q < 1$, the equilibrium $y^* = 1$ is locally attractive but it becomes unstable for $q = q_2$, and *Hopf bifurcation occurs* (see [2]).
(iii) Ricker's equation with delayed-density dependence $y_{n+1} = y_n \exp\{c_n(1-\sum_{i=0}^m b_{n,i}y_{n-i})\}$, $n = 0, 1, \cdots$, which is equivalent to $x_{n+1} = x_n - c_n\sum_{i=0}^m b_{n,i}(e^{x_{n-i}} - 1)$, $n = 0, 1, \cdots$, where c_n, $b_{n,i} > 0$, $\sum_{i=0}^m b_{n,i} = 1$ and $y_n = e^{x_n}$.
The positive equiliblium $y^* = 1$ is *globally asymptotically stable* if $\limsup_{n\to\infty}\sum_{i=n}^{n+m} r_i < \frac{3}{2} + \frac{1}{2(m+1)}$ (see [7]).
(iv) A model of the growth of bobwhite quail populations $y_{n+1} = qy_n + (1-q)\frac{cy_n}{1+y_{n-m}^p}$, $n = 0, 1, \cdots$,
where $c, p > 0$. If $c \le 1$, then for any $0 < q < 1$, $\lim_{n\to\infty} y_n = 0$. If $c > 1$, then the positive equilibrium $y^* = (c-1)^{1/p}$ of the model exists. Moreover, if $p \le \frac{2c}{(c-1)(1-q)}$ for $m = 0$, or $p < \frac{c}{(c-1)(1-q)}\frac{3m+4}{2(m+1)^2}$ for $m \ge 1$, then the positive equiliblium y^* is globally asymptotically stable (see [4]).

3 Delays-independent stability conditions for (1.1)

After setting

$$r_1 = a_0,\ r_2 = \sum_{i=1}^m a_i,\ r = r_1 + r_2,\ \varphi(x) = qx - r_1 f(x),\ \hat{z}(q) = (-1+\sqrt{1+4q})/(2q), \quad (3.1)$$

we have the following result.

Theorem 3.1 *Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$, and suppose that*

$$r_1 < q, \quad r \le q + (1-q)\ln(q/r_1) \quad \textit{and} \quad (q/r_1)^q e^{r-q}(r_1 - r_2) + (1-q) \ge 0, \quad (3.2)$$

or

$$\begin{cases} r_1 \le q, \quad r > q + (1-q)\ln(q/r_1), \quad qr_2 \le r_1, \\ r - r_2(q/r_1)^q e^{r-q} - (1-q)(\bar{L} - 1) \ge 0 \ \textit{and} \ \bar{L} = \ln\frac{r-q-(1-q)\ln(q/r_1)}{r_2} \le 0, \end{cases} \quad (3.3)$$

or

$$\begin{cases} r_1 > q, \quad r \le 1 + q, \quad r - r_2(q/r_1)^q e^{r-q} - (1-q)(\ln(q/r_1) - 1) \ge 0, \\ \textit{and} \ \frac{r}{q}(q/r_1)^q e^{r-q} \le \frac{e^{\hat{z}(q)}}{1-\hat{z}(q)}. \end{cases} \quad (3.4)$$

Then, the zero solution of (1.1) *is globally asymptotically stable.*

Numerical result 3.1 Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$.
(i) The last inequality in (3.4) can be eliminated from (3.4).
(ii) Under the condition $\frac{r_2}{r_1} \le \frac{2}{e}$ and $r \le 1 + q$, the third inequality of (3.4) is satisfied, and hence the zero solution of (1.1) is globally asymptotically stable.

Example 3.1 Wazewska-Czyzewska and Lasota model (see [9]).

$$y_{n+1} = qy_n + (1-q)c\sum_{i=0}^m b_i e^{-\gamma y_{n-i}}, \quad \text{where } c,\ \gamma > 0,\ b_i \ge 0 \text{ and } \sum_{i=0}^m b_i = 1. \quad (3.5)$$

(3.5) is equivalent to (2.5). For equation (3.5), the positive equilibrium of (3.5), say y^*, is globally asymptotically stable, if $\gamma y^* \le 1$ (see [3] and Example 2.2 iii)). For the case $\gamma y^* > 1$, by using the generalized Yorke condition, [6, Theorem 8] extended these to $\gamma y^* \leq (1+q^{m+1})/(1-q^{m+1})$ with some restricted conditions "$V_k(q) < 0,\ W_k(q) < 0$". Note that the last condition contains the restriction $(q+q^2+\cdots+q^m)q^m \leq 1$ for $0 < q < 1$. On the other hand, by applying Theorem 3.1 and Numerical result 3.1 to (2.5) for $a_i = (1-q)\gamma y^* b_i,\ 0 \leq i \leq m$, we obtain another sufficient condition, for example, $\sum_{i=1}^m b_i \leq \frac{2}{e} b_0$ and $\gamma y^* \leq (1+q)/(1-q)$ for the solution y^* of (3.5) to be globally asymptotically stable. Note that $e^x - 1 < x/(1-x)$ for $0 < x < 1$ and $\frac{1+q^{m+1}}{1-q^{m+1}} < \frac{1+q}{1-q}$ for $0 < q < 1$. Thus, compared with [6, Proof of Theorem 2] (and [1]-[9] and references therein), one can see that our results offer new stability conditions to (3.5).

4 Semi-contractivity with a sign condition

For $0 \leq q < 1$, consider the following nonautonomous equation

$$x_{n+1} = qx_n - \sum_{j=0}^{m} a_{n,j} f_j(x_{n-j}), \qquad n = 0, 1, \cdots, \tag{4.1}$$

where $0 < q \leq 1$, $a_{n,j} \geq 0$, $0 \leq j \leq m$, $n = 0, 1, \cdots$, and $\sum_{j=0}^{m} a_{n,j} > 0$, and we assume that there is a function $f(x)$ such that (1.2) holds.

For (4.1) and any $0 \leq l_n \leq m$, we can derive the following equation.

$$\begin{cases} x_{n+1} = \{q^{l_n+1} x_{n-l_n} + (1-q) \sum_{k=0}^{l_n} q^k \sum_{j=0}^{m-k} a_{n-k,j} f_j(x_{n-k-j})\} \\ \qquad - \sum_{k=1}^{l_n} q^k \sum_{j=m-k+1}^{m} a_{n-k,j} f_j(x_{n-k-j}), \quad n = 2m, 2m+1, \cdots . \end{cases} \tag{4.2}$$

Similar to the proofs of [5, Lemmas 2.3 and 2.4], we have the following two lemmas for (4.1).

Lemma 4.1 *Let $\{x_n\}_{n=0}^{\infty}$ be the solution of* (4.1). *If there exists an integer $n \geq m$ such that $x_{n+1} \geq 0$ and $x_{n+1} > x_n$, then there exists an integer $\underline{g}_n \in [n-m, n]$ such that*

$$x_{\underline{g}_n} = \min_{0 \leq j \leq m} x_{n-j} < 0. \tag{4.3}$$

If there exists an integer $n \geq m$ such that $x_{n+1} \leq 0$ and $x_{n+1} < x_n$, then there exists an integer $\bar{g}_n \in [n-m, n]$ such that

$$x_{\bar{g}_n} = \max_{0 \leq j \leq m} x_{n-j} > 0. \tag{4.4}$$

After setting

$$\begin{cases} \bar{r}_1 = \sup_{n \geq m} \sum_{k=0}^{m} q^k \sum_{j=0}^{m-k} a_{n-k,j}, \quad \bar{r}_2 = \sup_{n \geq m} \sum_{k=1}^{m} q^k \sum_{j=m-k+1}^{m} a_{n-k,j}, \\ \bar{r} = \bar{r}_1 + \bar{r}_2, \quad \bar{\varphi}(x) = \bar{q}x - \bar{r}_1 f(x), \quad \bar{q} = q^{m+1}, \quad \bar{\bar{z}} = (-1 + \sqrt{1+4\bar{q}})/(2\bar{q}), \end{cases} \tag{4.5}$$

and

$$\bar{g}(z_0, z_1, \cdots, z_m; \bar{q}) = \bar{\varphi}(z_0) + \sum_{k=1}^{m} q^k \sum_{j=m-k+1}^{m} a_{n-k,j} g(z_j), \tag{4.6}$$

we are able to prove the following results.

If there exists an integer $n \ge m$ such that $x_{n+1} \ge 0$ and $x_{n+1} > x_n$, then by (4.3) and (4.2) with $l_n = n - \underline{g}_n$, we have that

$$x_{n+1} \le \bar{\varphi}(x_{\underline{g}_n}) - \bar{r}_2 f(L_n), \quad L_n = \min_{0 \le j \le 2m} x_{n-j}. \tag{4.7}$$

If there exists an integer $n \ge m$ such that $x_{n+1} \le 0$ and $x_{n+1} < x_n$, then by (4.4) and (4.2) with $l_n = n - \bar{g}_n$, we have that

$$x_{n+1} \ge \bar{\varphi}(x_{\bar{g}_n}) - \bar{r}_2 f(R_n), \quad R_n = \max_{0 \le j \le 2m} x_{n-j}. \tag{4.8}$$

Lemma 4.2 *Suppose that the solution x_n of (4.1) is oscillatory about 0. If for some real number $L < 0$, there exists a positive integer $n_L \ge 2m$ such that $x_n \ge L$ for $n \ge n_L$, then for any integer $n \ge n_L + 2m$,*

$$x_{n+1} \le R_L \text{ for } n \ge n_L + 2m, \quad \text{and} \quad x_{n+1} \ge S_L \text{ for } n \ge n_L + 4m, \tag{4.9}$$

where $R_L = \max_{L \le x \le 0} \varphi(x) - r_2 f(L) > 0$ and $S_L = \min_{0 \le x \le R_L} \varphi(x) - r_2 f(R_L) < 0$. Moreover, if $S_L > L$ for any $L < 0$, then $\lim_{n \to \infty} x_n = 0$.

Assume that $g(z_0, z_1, \cdots, z_m)$ is continuous for $(z_0, z_1, \cdots, z_m) \in R^{m+1}$ and $g(y^*, y^*, \cdots, y^*) = y^*$ has a unique solution y^*.

Definition 4.1 *The function $g(z_0, z_1, \cdots, z_m)$ is said to be semi-contractive with a sign condition z_0 for y^*, if*
(i) for any constants $\underline{z} < y^*$ and $z_i \ge \underline{z}$, $0 \le i \le m$ with $z_0 \le y^*$, there exists a constant $y^* < \bar{z} < +\infty$ such that $g(z_0, z_1, \cdots, z_m) \le \bar{z}$ and for any $\underline{z} \le z_i \le \bar{z}$, $0 \le i \le m$ with $z_0 \ge y^*$, there exists a constant $\bar{\underline{z}} > \underline{z}$ such that $\bar{\underline{z}} \le g(z_0, z_1, \cdots, z_m)$,
or
(ii) for any constants $\bar{z} > y^*$ and $z_i \le \bar{z}$, $0 \le i \le m$ with $z_0 \ge y^*$, there exists a constant $y^* > \underline{z} > -\infty$ such that $g(z_0, z_1, \cdots, z_m) \ge \underline{z}$ and for any $\underline{z} \le z_i \le \bar{z}$, $0 \le i \le m$ with $z_0 \le y^*$, there exists a constant $\bar{\bar{z}} < \bar{z}$ such that $\bar{\bar{z}} \ge g(z_0, z_1, \cdots, z_m)$.

Then by (4.7), (4.8) and Lemma 4.2, we can obtain the following result.

Theorem 4.1 *If $\bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - \bar{r}_2 f(z_1)$ is semi-contractive with a sign condition z_0 for $x^* = 0$, then the zero solution of (4.1) is globally asymptotically stable.*

Note that if $\bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - \bar{r}_2 f(z_1)$ is semi-contractive with a sign condition z_0 for $x^* = 0$, then the zero solution $x^* = 0$ of (4.1) is uniformly stable and hence $x^* = 0$ is globally asymptotically stable.

For the special case $f(x) = e^x - 1$, we establish the following sufficient conditions for $0 < q < 1$ which are some extentions of the result in [5] for $q = 1$.

Theorem 4.2 *Suppose that $f(x) = e^x - 1$ and that one of the following condition is fulfilled:*

$$\begin{cases} \bar{r}_2 \le 1 \quad \text{and} \quad \frac{\bar{r}}{\bar{q}} e^{\bar{r}_2} \le \frac{e^{\bar{\hat{z}}}}{1-\hat{z}} & \text{if} \quad \bar{r}_1 \le \bar{q}, \\ \bar{r} \le 1 + \bar{q} \quad \text{and} \quad \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} \le \frac{e^{\bar{\hat{z}}}}{1-\hat{z}} & \text{if} \quad \bar{r}_1 > \bar{q}, \end{cases} \tag{4.10}$$

or

$$\begin{cases} \bar{r}_2 \le 1, \quad \frac{\bar{r}}{\bar{q}} e^{\bar{r}_2} > \frac{e^{\bar{\hat{z}}}}{1-\hat{z}} \quad \text{and} \quad G_3(\delta) > 0 & \text{if} \quad \bar{r}_1 \le \bar{q}, \\ \bar{r} \le 1 + \bar{q}, \quad \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} > \frac{e^{\bar{\hat{z}}}}{1-\hat{z}} \quad \text{and} \quad G_1(\alpha) > 0 & \text{if} \quad \bar{r}_1 > \bar{q}, \end{cases} \tag{4.11}$$

$$\text{with} \quad \begin{cases} G_1(x) = \bar{q}\Big(\bar{q}\ln(\bar{q}/\bar{r}_1) + \bar{r} - \bar{q} - \bar{r}_2 e^x\Big) + \bar{r} - \bar{r}(\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}-\bar{r}_2 e^x} - x, \\ G_3(x) = (\bar{r}_1 + (1+\bar{q})\bar{r}_2) - \bar{q}\bar{r}_2 e^x - \bar{r}e^{\bar{r}_2 - \bar{r}_2 e^x} - x, \end{cases} \tag{4.12}$$

where α and δ are the lowest solutions of $G_1(x) = 0$ and $G_3(x) = 0$, respectively, and and $\bar{\hat{z}}$ is a positive solution of $\bar{q}z^2 + z - 1 = 0$. Then, the solution $x^* = 0$ of (4.1) is globally asymptotically stable..

As an immediate consequence we have the following corollary.

Collorary 4.1 *Assume that $f(x) = e^x - 1$ and that*

$$\bar{r} \le 1 + \bar{q} \quad \text{and} \quad \bar{r}_1 \ge \bar{q}\bar{r}_2. \tag{4.13}$$

If

$$(i)\ \frac{\bar{r}}{\bar{q}}(\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} \le \frac{e^{\bar{\hat{z}}}}{1-\bar{\hat{z}}}, \quad \text{or} \quad (ii)\ \frac{\bar{r}}{\bar{q}}(\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} > \frac{e^{\bar{\hat{z}}}}{1-\bar{\hat{z}}} \quad \text{and} \quad G_1(\alpha) > 0, \tag{4.14}$$

then, the zero solution of (4.1) is globally asymptotically stable.

Example 4.1 Consider a model $x_{n+1} = qx_n - \sum_{i=0}^{m} a_i(e^{-x_{n-i}} - 1)$, $n = 0, 1, 2, \cdots$, where $a_i \ge 0$, $0 \le i \le m$, and $\sum_{i=0}^{m} a_i > 0$. This equation is equivalent to (2.5), if $\sum_{i=0}^{m} a_i = (1-q)\gamma y^*$ and $0 < q < 1$. By Corollary 4.1, the zero solution $x^* = 0$ is globally asymptotically stable for $\bar{r} \le 1 + \bar{q}$, if for the setting (4.5) and $\hat{r}_1 = \bar{q}(\frac{1+\bar{q}}{\bar{q}}(1-\bar{\hat{z}})e^{1-\bar{\hat{z}}})^{1/\bar{q}}$, it holds that $\frac{\bar{r}_2}{\bar{r}_1} \le \frac{1+\bar{q}}{\bar{r}_1} - 1$. Since $e^x - 1 < x/(1-x)$ for $0 < x < 1$ and we do not need the restriction $(q + q^2 + \cdots + q^m)q^m \le 1$ for $0 < q < 1$ in [6, Theorem 2], our results improve some of [6, Theorem 8] (see [5]).

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