数理解析研究所講究録
第 1600 巻 2008 年 21-47

Why Renormalizable NonCommutative Quantum Field Theories?

Vincent Rivasseau
Laboratoire de Physique Théorique
Bât. 210
Université Paris XI
F–91405 Orsay Cedex, France

November 12, 2007

Abstract

A new version of scale analysis and renormalization theory has been found on the non-commutative Moyal space. Hopefully it may connect nicely to Alain Connes's interpretation of the standard model in terms of non-commutative geometry. However it is also a non-trivial extension of this circle of ideas. It could be useful both for physics beyond the standard model or for standard physics in strong external field. The good news is that quantum field theory is better behaved on noncommutative than on ordinary space-time. Some models at least should be completely finite, even beyond perturbation theory. We discuss why the $\phi_4^{\star 4}$ theory can be built with some extensions of the traditional methods of constructive theory. In this way noncommutative field theory might become a possible alternative to supersymmetry or to string theory, whose key properties are also to tame ultraviolet divergencies. We suggest that by gluing together many Grosse-Wulkenhaar theories at high energy one can obtain an effective commutative field theory at lower energy.

1 Introduction

I was recently proposed at the occasion of a set of lectures at the Research Institute of Mathematical Sciences in Kyoto to answer the question which is the title of this paper. In fact I have neither a unique nor a precise answer to give, but rather many different partial answers. Renormalizable noncommutative quantum field theories, hereafter called RNCQFT seem to me important to study because they combine many nice features which hint to the potential solution of many different physical riddles: the Higgs mechanism, a cure for the Landau ghost, the strange spectrum of plateaux in the quantum Hall effect, possibly the emergence of effective strings in confinement, even perhaps a simplified road towards quantization of gravity. I realize this is a good occasion to try to clarify my own ideas and to put in shape some written defense of this subject, based on a condensed version of [1, 2], together with some updating of what happened in the last few months. But I must warn the reader that since RNCQFT is only three years old and fast evolving, nothing is fixed yet. This short review might be soon outdated, some proposals might fade away, and some new discoveries may throw us into unexpected directions.

I'll try to answer briefly at a completely general level why the three separate elements, renormalizability, quantum field theory and non commutative geometry (in that perhaps slightly surprising order) appear to me as key components of theoretical physics. Then it should be clear why the formalism which combines these three elements together, namely RNCQFT, should also be studied.

I'll continue with a brief review of what has been accomplished so far in the subject and a possible preliminary classification of RNCQFT's. I'll develop a little longer the constructive aspects developed recently. Finally I'll end up with a list of possible applications and conclude with some speculations about how this formalism may connect to our ordinary world.

Acknowledgments

I would like to thank Prof. Keiichi Ito, from Setsunan University in Osaka, and the Research Institute of Mathematical Physics in Kyoto for their warm hospitality and for giving me this fresh opportunity to present again in a slightly different form this set of ideas. My very warm thanks also extend to all the people who contributed to the elaboration of this material, in particular M. Disertori, R. Gurau, J. Magnen, A. Tanasa, F. Vignes-

Tourneret, J.C. Wallet and R. Wulkenhaar. I would like also to sincerely apologize to the many people whose work in this area would be worth of citation but who have not been cited here.

2 Why R, NC, QFT?

2.1 Why renormalization?

The history of renormalization is quite an extraordinary one. In less than about half a century it metamorphosed from an obscure technique to remove infinities in quantum field theory into a truly ubiquitous concept, so central to physics that I do not hesitate to put it first in this list.

The most interesting physical phenomena, whether they pertain to quantum field theory, phase transitions, turbulence, condensed matter behavior and so on in general occur over many scales with at least approximate scaling laws. The exponents of these laws usually show some beautiful universal character. The main step to understand this universality was made by K. Wilson when he connected two previously different areas, renormalization in quantum field theory and the classical evolution of dynamical systems [3]. This uncovered the analogy between time evolution and the evolution of effective actions under change of the observation scale.

I consider not excessive to compare the importance of renormalization in physics to that of DNA for biology. Indeed renormalization theory gives us the key to understand self-replication over scales. An other comparison with biology may come from Darwinism. In physics fixed points of the renormalization group do appear because they are the only ones to survive renormalization group flows. In practice Gaussian fixed points, corresponding to perturbatively renormalizable theories can be fully analyzed, keeping their fundamental structure unchanged, but with a few rescaled parameters. One should not believe that such just renormalizable theories are rare: for instance the Fermi surface singularities make the usual Fermionic non relativistic theories of condensed matter just renormalizable in *any* space dimension [4, 5].

Let us make an additional remark which points to another fundamental similarity between renormalization group flow and time evolution. Both seem naturally *oriented* flows. Microscopic laws are expected to determine macroscopic laws, not the converse. Time runs from past to future and entropy increases rather than decreases. This is philosophically at the heart of standard determinism. A key feature of Wilson's RG is to have defined in a mathematically precise way *which* short scale information should be forgotten through coarse graining: it is the part corresponding to the *irrelevant operators* in the action. But coarse graining is also fundamental for the second law in statistical mechanics, which is the only law in classical physics which is "oriented in time" and also the one which can be only understood in terms of change of scales.

Whether this arrow common to RG and to time evolution is of a cosmological origin remains to be better understood. We remark simply here that in the distant past the big bang has to be explored and understood on a logarithmic time scale. At the beginning of our universe important physics occurs at very short distance. As time passes and the universe evolves, physics at longer distances, lower temperatures and lower momenta becomes literally visible. Hence the history of the universe itself can be summarized as a giant unfolding of the renormalization group.

This unfolding can then be specialized into many different technical versions depending on the particular physical context, and the particular problem at hand.

However there is one domain, namely quantum gravity, which seems difficult to reconcile with renormalization. Ordinary quantum gravity is not renormalizable in the ordinary sense. We learn from string theory that one should expect surprises in the new notions of scale and renormalization group that govern physics at Planckian or transPlanckian energies. In particular the string dualities mix ordinary notions of short and long lengths: winding modes around short circles cannot be distinguished from translation modes around long circles and vice versa. This feature is also present in RNCQFT (but in a much more accessible mathematical setting) because instead of being based on the heat kernel, RNCQFT's are based on the Mehler kernel which exhibit duality between small and large lengths.

2.2 Why Quantum Field Theory?

Here probably nobody should argue, so this is the easy part. In the strictest sense quantum field theory or QFT provides a quantum description of particles and interactions which is compatible with special relativity. But the mathematical essence of quantum field theory is to treat by functional integration systems of infinitely many degrees of freedom. Therefore its formalism applies beyond particle physics to many non relativistic problems in statistical mechanics and condensed matter. Even classical mechanics can benefit from a field theoretic reformulation. Hence QFT is a truly ubiquitous formalism. At the mathematical level it is much more advanced than for instance string theory or loop quantum gravity.

Over the years relativistic QFT has evolved into the *standard model* which explains in great detail most experiments in particle physics and is contradicted by none. But it suffers from at least two flaws. First it lives on a rigid and flat space time background and is not yet compatible with general relativity. Second, the standard model incorporates so many different Fermionic matter fields coupled by Bosonic gauge fields that it seems more some kind of new Mendeleyev table than a fundamental theory. For these two reasons QFT and the standard model are not supposed to remain valid without any changes until the Planck length where gravitation should be quantized. They could in fact become inaccurate much before that scale.

2.3 Why noncommutative geometry?

In general noncommutative geometry corresponds to an extension of the commutative algebra of functions on an ordinary manifold into a noncommutative algebra.

As soon as ordinary coordinates functions no longer commute a fundamental dimensioned area appears proportional to their commutator. But there exists certainly a fundamental length in our world, namely $\ell_P = \sqrt{\hbar G/c^3}$, obtained by combining the three fundamental constants of physics, Newton's gravitation constant G, Planck's constant $\hbar$ and the speed of light c. Its value is about 1.6 10^{-35} meters. Below this length gravity should be quantized, and the energy required for a particle to probe physics at such small distances seems to create a black hole horizon which prevents the very observation of this physics. Most experts agree that this means that ordinary commutative flat space-time should be modified. Noncommmutative geometry seems a very natural possibility in this respect. The fact that black hole entropy involves the area of an horizon seems also to point to the Planck *area* as being more fundamental than the Planck *length*, just as should be the case if there is a non-trivial commutator between coordinates.

Following initial ideas of Schrödinger and Heisenberg [6, 7] who tried to extend the noncommutativity of phase space to ordinary space, noncommutative quantum field theory was first formalized by Snyder [8] in the hope that it should behave better than ordinary field theory in the ultraviolet regime.

After initial work by Michel Dubois-Violette, Richard Kerner and John Madore [9], Alain Connes, Ali Chamseddine and others have forcefully advocated that the *classical* Lagrangian of the current standard model arises much more naturally on simple non-commutative geometries than on ordinary commutative Minkowski space [10], and leads naturally to the classical Einstein Hilbert action for gravity. The noncommutative reformulation initially threw light on the Higgs mechanism and later on more and more detailed aspects of the standard model. We have now a fairly compelling picture: the detailed Lagrangian of the standard model can be reproduced very simply from the principle of a spectral action corresponding to a Dirac operator on a manifold which is ordinary space-time R^4 twisted by a simple non commutative finite dimensional "internal" algebra [11, 12].

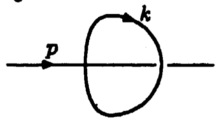
The interest for non commutative geometry came out also from string theory. Open string field theory may be recast as a problem of noncommutative multiplication of string states [13]. String theorists realized in the late 90's that NCQFT is an effective theory of strings [14, 15]. Roughly this is because in addition to the symmetric tensor $g_{\mu\nu}$ the spectrum of the closed string also contains an antisymmetric tensor $B_{\mu\nu}$. There is no reason for this antisymmetric tensor not to freeze at some lower scale into a classical field, just as $g_{\mu\nu}$ is supposed to freeze into the classical metric of Einstein's general relativity. But such a freeze of $B_{\mu\nu}$ precisely induces an effective non commutative geometry. In the simplest case of flat Riemannian metric and trivial constant antisymmetric tensor, the geometry is simply of the Moyal type; it reduces to a constant anticommutator between (Euclidean) space-time coordinates. This made NCQFT popular among string theorists. A good review of these ideas can be found in [16].

Let us remark also that NCQFT is also the right setting for down to earth applications such as quantum physics in strong external field (e.g. in condensed matter the Quantum Hall effect [17]-[18]-[19]).

3 RNCQFT: the present state

NCQFT combines nicely the last two of the three elements above. The Connes-Chamseddine version of the standard model remains in the line of Einstein's classical unification of Maxwell's electrodynamics equations through the introduction of a new four dimensional space-time. Climbing in energy, the next logical step seems to find the quantum version of these ideas, which ought to be quantum field theory on non-commutative geometry, or NCQFT. We may indeed see noncommutativity, initially confined to some internal space, invade more fully spacetime itself as we approach the Planck scale. Going down in energy from the Planck scale (at which at least part of string theory may be correct), we may also meet NCQFT's as effective theories. Therefore it is tempting to think that there ought to be some intermediate regime between QFT and possibly string theory (or another theory of quantum gravity) where NCQFT is the right formalism. The ribbon graphs of NCQFT may be interpreted either as "thicker particle world-lines" or as "simplified open strings world-sheets" in which only the ends of strings appear but not yet their internal oscillations.

These two lines of arguments both point to develop NCQFT. However remember we really want in fact RNCQFT because we argued that renormalizable theories are the building blocks of physics, the ones who survive RG flows.

Until recently a big stumbling block remained on this road. The simplest NCQFT on Moyal space, such as $\phi^{\star 4}_4$, were found not to be renormalizable because of a surprising phenomenon called *uv/ir mixing*. The $\phi^{\star 4}_4$ theory still has infinitely many ultraviolet divergent graphs but fewer than the ordinary ϕ^4_4 theory. However ultraviolet convergent graphs, such as the non-planar tadpole 

generate unexpected infrared divergences which are not of the renormalizable type [20].

However three years ago the solution out of this riddle was found. H. Grosse and R. Wulkenhaar in a brilliant series of papers discovered how to renormalize $\phi^{\star 4}_4$ [21, 22] on four dimensional flat non-commutative Moyal space.

The first renormalization proof [22] was based on the matrix representation of the Moyal product. It relies on adding to the usual propagator a marginal harmonic potential, as required by Langmann-Szabo duality [23]. We now call such an addition which allowed renormalization the *vulcanization*[1] of the model. Vulcanization means that NCQFT on Moyal spaces has

[1] Vulcanization is a technological operation which adds sulphur to natural rubber to improve its mechanical properties and its resistance to temperature change, and temperature is a scale in imaginary time...

to be based on the Mehler kernel, which governs propagation in a harmonic potential, rather than on the heat kernel, which governs ordinary propagation in commutative space. Grosse and Wulkenhaar were able to compute for the first time the Mehler kernel in the *matrix base* which transforms the Moyal product into a matrix product. They combined this computation with an extensive analysis of all possible contractions of ribbon graphs in the RG equations for the corresponding class of so-called quasi-matrix models. In this way they proved perturbative renormalizability of the theory, up to some estimates which were finally proven in [24].

These founding papers opened up the subject of renormalizable non commutative field theories, hereafter called RNCQFT. By matching correctly propagator and interaction to respect symmetries, Grosse and Wulkenhaar had followed one of the main successful thread of quantum field theory.

The initial renormalization proof was improved by introducing multiscale analysis, first in the matrix base [24], then in position space [25]. The β-function was computed at one loop in [26], then shown to vanish (first up to three loop order and then to all orders) in [27, 28] at the self-duality point $\Omega = 1$ (where Ω is the frequency of the harmonic term). The exciting conclusion is that the $\phi^{\star 4}_4$-theory is asymptotically safe, hence free of any Landau ghost. Wave function renormalization exactly compensates the renormalization of the four-point function, so that the flow between the bare and the renormalized coupling is bounded.

Essentially most of the standard tools of field theory such as parametric [29, 30] and Mellin representations [31], dimensional regularization and renormalization [32] and the Connes-Kreimer Hopf algebra formulation of renormalization [33] have now been generalized to RNCQFT. Other models have been also developed In [34, 35] renormalization to all orders of the duality-covariant orientable Gross-Neveu model was shown; the one-loop beta function of the model was computed in [36]. The Dirac operator in [34, 35] is *not* the square root of the harmonic oscillator Hamiltonian appearing in the ϕ^4-model of [22] but is of the *covariant* type studied (for scalar fields) in [37, 38], hence describes the influence of a constant magnetic background field. Its spectrum has infinite degeneracy. This fact can also be seen from a different structure of the propagator in position space [39]. It makes the renormalization of the Gross-Neveu model technically more difficult, but to understand such *covariant* models is important for the future application of RNCQFT to condensed matter problems such as the quantum Hall effect.

Concerning other just renormalizable scalar models, in [40, 41] the noncommutative $\phi_6^{\star 3}$-model at the self-duality point was built and shown just renormalizable and exactly solvable. Self-duality relates this model to the Kontsevich-model. For $\phi_4^{\star 3}$, see [42]. The $\phi_3^{\star 6}$-model has been shown renormalizable with x-space techniques in [43]. That model however is not expected to have a non-perturbative definition because it should be unstable at large ϕ.

A remaining very important and difficult goal is to properly "vulcanize" gauge theories such as Yang-Mills in four dimensional Moyal space or Chern-Simons on the two dimensional Moyal plane plus one additional ordinary commutative time direction. We do not need to look at complicated gauge groups since the $U(1)$ pure gauge theory is non trivial and interacting on non commutative geometry even without matter fields. What is not obvious is to find a proper compromise between gauge and Langmann-Szabo symmetries which still has a well-defined perturbation theory around a computable vacuum after gauge invariance has been fixed through appropriate Faddeev-Popov or BRS procedures. We should judge success in my opinion by one main criterion, namely renormalizability. Non commutative action for gauge fields which can be induced through integration of a scalar renormalizable matter field minimally coupled to the gauge field have been computed independently by de Goursac, Wallet and Wulkenhaar [44], and by Grosse and Wohlgenannt. [45]. The result exhibits both gauge symmetry and LS covariance, hence vulcanization, but the vacuum looked non trivial so that to check whether the associated perturbative expansion is really renormalizable seems difficult. Recently a new progress was accomplished by Grosse and Wulkenhaar [46]. They showed firstly that this vulcanized gauge action is the gauge part of a more general action including Higgs fields that can be deduced from the Connes-Lott spectral action, and secondly they found the equation obeyed by a radial non-trivial vacuum. In this work, it appears much more clearly that the harmonic potential of the initial Grosse Wulkenhaar model is intimately related to the symmetry breaking of Higgs model which produces also this non-trivial gauge vacuum.

4 A short classification of RNCQFT

We can now propose a preliminary classification of these models into different categories, according to the behavior of their propagators:

- ordinary models at $0 < \Omega < 1$ such as $\Phi_4^{\star 4}$ (which has non-orientable graphs) or $(\bar{\phi}\phi)^2$ models (which has none). Their propagator, roughly $(p^2 + \Omega^2\tilde{x}^2 + A)^{-1}$ is LS covariant and has good decay both in matrix space and direct space. They have non-logarithmic mass divergencies and definitely require "vulcanization" i.e. the Ω term.

- self-dual models at $\Omega = 1$ in which the propagator is LS invariant. Their propagator is even better. In the matrix base it is diagonal, e.g. of the form $G_{m,n} = (m + n + A)^{-1}$, where A is a constant. The supermodels seem generically ultraviolet fixed points of the ordinary models, at which non-trivial Ward identities force the vanishing of the beta function. The flow of Ω to the $\Omega = 1$ fixed point is very fast (exponentially fast in RG steps).

- covariant models such as orientable versions of LSZ or Gross-Neveu (and presumably orientable gauge theories of various kind: Yang-Mills, Chern-Simons...). They may have only logarithmic divergencies and apparently no perturbative UV/IR mixing. However the vulcanized version still appears the most generic framework for their treatment. The propagator is then roughly $(p^2 + \Omega^2\tilde{x}^2 + 2\Omega\tilde{x} \wedge p)^{-1}$. In matrix space this propagator shows definitely a weaker decay than for the ordinary models, because of the presence of a non-trivial saddle point. In direct space the propagator no longer decays with respect to the long variables, but only oscillates. Nevertheless the main lesson is that in matrix space the weaker decay can still be used; and in x space the oscillations can never be completely killed by the vertices oscillations. Hence these models retain therefore essentially the power counting of the ordinary models, up to some nasty details concerning the four-point subgraphs with two external faces. Ultimately, thanks to a little conspiration in which the four-point subgraphs with two external faces are renormalized by the mass renormalization, the covariant models remain renormalizable. This is the main message of [35].

- self-dual covariant models which are of the previous type but at $\Omega = 1$. Their propagator in the matrix base is diagonal and depends only on one index m (e.g. always the left side of the ribbon). It is of the form $G_{m,n} = (m + A)^{-1}$. In x space the propagator oscillates in a way that often exactly compensates the vertices oscillations. These models have definitely worse power counting than in the ordinary case, with e.g. quadratically divergent four point-graphs (if sharp cut-offs are used).

Nevertheless Ward identities can presumably still be used to show that they can still be renormalized. This probably requires a much larger conspiration to generalize the Ward identities of the supermodels.

Notice that the status of non-orientable covariant theories is not yet clarified.

5 Constructive NCQFT

Constructive field theory [47, 48] builds rigorously the correlation functions for particular field theories whose Taylor expansions in the coupling are those of ordinary perturbative field theory. Since any formal power series is asymptotic to infinitely many smooth functions, perturbative field theory alone does not provide any well defined recipe to compute to arbitrary accuracy a given physical quantity, so in a deep sense it is no theory at all. Therefore for uncompromising minds, the only meaningful quantum field models are those of constructive field theory.

In field theory infinite volume quantities are expressed by connected functions. One main advantage of perturbative field theory is that connected functions are simply the sum of the connected Feynman graphs. But the expansion diverges because there are too many such graphs.

In fact connectedness does not require the full knowledge of a Feynman graph, with all its loop structure, but only the knowledge of a spanning tree. To summarize constructive theory, it is about working with the trees and hiding the quantum loops. Hence the constructive golden rule:

"Thou shall not know the loops, or thou shall diverge!"

Therefore tree formulas are among the main technical tools in constructive field theory. These formulas lie at the root of all the constructive expansions such as the cluster expansion of Glimm, Jaffe and Spencer [49]. They have been improved over the years by Brydges, Battle, Federbush, Kennedy and others. The final form presented below is due to [50]; it is a canonical formula which distributes graphs according to underlying trees in a completely symmetric and positivity-preserving way.

Theorem 5.1. *Let F be a smooth function of $n(n-1)/2$ line variables x_ℓ, $\ell = (i,j)$, $1 \le i < j \le n$. We have*

$$F(1) = \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} [\int_0^1 dw_\ell] \Big\} \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial x_\ell} F \Big\} [x^{\mathcal{F}}(\{w\})] \tag{5.1}$$

where $F(1)$ means $F(1,1,...,1)$ and

- *the sum over $\mathcal{F}$ is over all forests over n vertices,*
- *the interpolated value $x_\ell^{\mathcal{F}}(\{w\})$ is 0 if $\ell = (i,j)$, with i and j in different connected components with respect to $\mathcal{F}$, and is the infimum of the $w_{\ell'}$ for ℓ' running over the unique path from i to j' in $\mathcal{F}$.*
- *Furthermore the real symmetric matrix $x_{i,j}^{\mathcal{F}}(\{w\})$ (completed by 1 on the diagonal $i = j$) is* positive.

The constructive program launched by A. Wightman and pursued by J. Glimm, A. Jaffe and followers in the 70's partial failed because no natural four dimensional field theory could be identified and fully built. This is because the only theories asymptotically free in the ultraviolet limit, namely non-Abelian gauge theories, are very complicated (gauge fixing is marred by the so-called Gribov problem). Moreover in these theories ultraviolet asymptotic freedom comes at the price of infrared slavery: non-perturbative long range effects such as quark confinement are not fully understood until now, even at a non-rigorous level. The constructive program went on, but mostly as a set of rigorous techniques applied to many different areas of mathematical physics [51, 52].

For mathematical physicists who like me came from the constructive field theory program, the Landau ghost has always been a big frustration.Since non Abelian gauge theories are so complicated and lead to confinement in the infrared regime, there is no four dimensional rigorous field theory without unnatural cutoffs up to now[2]. It is therefore very exciting to build non perturbatively the $\phi_4^{\star 4}$ theory, even if it lives on the unexpected Moyal space and does not obey the usual axioms of ordinary QFT.

However in order to build $\phi_4^{\star 4}$ we need to extend first in a proper way constructive methods This because the standard constructive cluster and Mayer expansion of Glimm-Jaffe-Spencer [49] does not apply here because the interaction is non-local. This problem can be overcome by a new expansion called the loop-vertex tree expansion [58]. This expansion also solves an old problem of ordinary constructive field theory as well, namely how to compute the thermodynamic limit of a ϕ^4 theory with fixed cutoffs without using any intermediate non-canonical lattice discretization [59]. Hence it

[2]We have only built renormalizable theories for two dimensional Fermions [53]-[54] and for the infrared side of ϕ_4^4[55]-[56].

provides a first example where the stimulation of NCQFT leads to a new tool for *ordinary* field theory.

As a first step we have proved Borel summability [57] in the coupling constant for the connected functions in a way which has to be *uniform* in the slice index. The full construction of $\phi_4^{\star 4}$-theory now requires to extend these tools to a multiscale analysis. The full control of the bounded RG trajectory should then presumably follow as in [60].

We now summarize the loop-vertex tree expansion of [58] for a matrix model which mimics a single slice of the $\phi_4^{\star 4}$ model. This matrix ϕ^4 model is made of a Gaussian independent identically distributed measure on N by N real or complex matrices perturbed by a positive $\frac{\lambda}{N}\,\mathrm{Tr}\,\phi^\star\phi\phi^\star\phi$ interaction. The $N \to \infty$ limit is given by planar graphs. It can be studied through various methods such as orthogonal polynomials, supersymmetric saddle point analysis and so on. However none of these methods seems exactly suited to constructive results such as uniform Borel summability in N (Theorem 5.3 below).

Consider the complex case (the real case being similar). The normalized interacting measure is

$$d\nu(\Phi) = \frac{1}{Z(\lambda, N)} e^{-\frac{\lambda}{N}\,\mathrm{Tr}\,\Phi^\star\Phi\Phi^\star\Phi} d\mu(\Phi) \tag{5.2}$$

where

$$d\mu = \pi^{-N^2} e^{-\frac{1}{2}\,\mathrm{Tr}\,\Phi^\star\Phi} \prod_{i,j} d\Re\Phi_{ij} d\Im\Phi_{ij} \tag{5.3}$$

is the normalized Gaussian measure with covariance

$$< \Phi_{ij}\Phi_{kl} >=< \bar{\Phi}_{ij}\bar{\Phi}_{kl} >= 0, \quad < \bar{\Phi}_{ij}\Phi_{kl} >= \delta_{ik}\delta_{jl}. \tag{5.4}$$

For the moment assume the coupling λ to be real positive and small. We want to prove a uniform Borel summability theorem as $N \to \infty$ of the pressure $N^{-2}\log Z(\lambda, N)$, which is the suitably normalized sum of connected vacuum graphs. This should at first sight be easy because the limit as $N \to \infty$ of this quantity is given by the sum of all connected *planar* vacuum graphs, hence is *analytic* in λ. But there is a subtlety: in the matrix base it seems that one needs to know all the loops to find the correct scaling (a factor N per vertex compensates the $1/N$ in the coupling). Indeed contrary to vector models, each propagator carries *two* delta functions for matrix indices, one for the right and one for the left. At each vertex *four* indices meet. A spanning tree provides only $n-1$ lines, hence $2(n-1)$ identifications. Therefore about $2n$ indices remain to be summed, hence two per vertex if we do not know the loop structure. Of course in a vector model only *two* indices meet at each vertex, and each propagator carries one delta function, so that knowing a tree there remains about one index to sum per vertex. This is why vector models can be treated through cluster expansions.

The solution to this riddle is to sort of exchange the role of vertices and propagators. We decompose the Φ functional integral according to an intermediate Hermitian field σ acting either on the right or on the left index. For instance the normalization $Z(\lambda, N)$ can be written as:

$$Z(\lambda, N) = \int d\mu_{GUE}(\sigma^R) e^{-\operatorname{Tr}\log(1\otimes 1 + i\sqrt{\frac{\lambda}{N}} 1\otimes\sigma^R)} \tag{5.5}$$

where $d\mu_{GUE}$ is the standard Gaussian measure on an Hermitian field σ^R, that is the measure with covariance $< \sigma^R_{ij}\sigma^R_{kl} >= \delta_{il}\delta_{jk}$. It is convenient to view $\mathbb{R}^{N^2}$ as $\mathbb{R}^N \otimes \mathbb{R}^N$. For instance the operator $H = \sqrt{\frac{\lambda}{N}}[1 \otimes \sigma^R]$ transforms the vector $e_m \otimes e_n$ into $\sqrt{\frac{\lambda}{N}} e_m \otimes \sum_k \sigma^R_{kn} e_k$. Remark that this is a Hermitian operator because σ^R is Hermitian. The $e^{-\operatorname{Tr}\log}$ represents the Gaussian integration over Φ, hence a big N^2 by N^2 determinant.

By duality of the matrix vertex, there is an exactly similar formula but with a left Hermitian field σ^L acting on the left index, and with $[\sigma^L \otimes 1]$ replacing $[1 \otimes \sigma^R]$. From now on we work only with the right field and drop the R superscript for simplicity.

We define the *loop vertex* V to be

$$V = -\operatorname{Tr}\log(1 \otimes 1 + 1 \otimes iH), \tag{5.6}$$

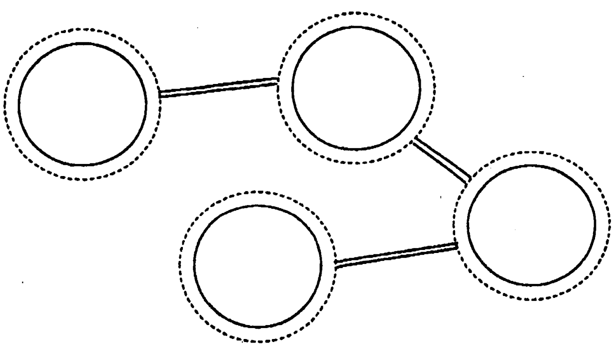
and expand the exponential in (5.5) as $\sum_n \frac{V^n}{n!}$. To compute the connected graphs we give a (fictitious) index $v = 1, ..., n$ to all the σ fields of a given loop vertex V_v. At any order n the functional integral over $d\nu(\sigma)$ is obviously also equal to the same integral but with a Gaussian measure $d\nu(\{\sigma^v\})$ with degenerate covariance $< \sigma^v_{ij}\sigma^{v'}_{kl} >= \delta_{il}\delta_{jk}$.

Then we apply the tree formula and we get

Theorem 5.2.

$$logZ(\lambda,N)=\sum_{n=1}^{\infty}\sum_{T}\Big\{\prod_{\ell\in T}\big[\int_0^1 dw_\ell \sum_{i_\ell,j_\ell,k_\ell,l_\ell}\big]\Big\}\int d\nu_T(\{\sigma^v\},\{w\})$$
$$\Big\{\prod_{\ell\in T}\big[\delta_{i_\ell l_\ell}\delta_{j_\ell k_\ell}\frac{\delta}{\delta\sigma^{v(\ell)}_{i_\ell,j_\ell}}\frac{\delta}{\delta\sigma^{v'(\ell)}_{k_\ell,l_\ell}}\big]\Big\}\prod_v V_v \qquad (5.7)$$

In this way we have an expansion whose tree lines are intermediate field propagators. No wonder they were not seen in the standard cluster expansion, because these lines come from the former vertices of the ordinary theory! See fro instance as an example of a particular tree on loop vertices:



In this way we can prove:

Theorem 5.3. *The series (5.7) is absolutely convergent for λ small enough and Borel summable* uniformly *in N.*

Proof: Left indices provide a particular cyclic order at each loop vertex. The σ field acts only on right indices, hence left indices are conserved, and there is a single global N factor per loop vertex coming from the trace over the left index. But there is a single trace over right indices corresponding to turning around the tree with of a product of resolvents bounded by 1!

We have started already to work on the multiscale version of this result, which is a bit more complicated. The naive idea is that one should now optimize the expansion by building the tree in (5.7) in priority between "high momentum" loop vertices, for instance whose with highest values of the left index. But because in the matrix base for $\phi^{\star 4}$ the same σ field with low indices (for instance σ_{00}) in present both in "high momentum" and "low momentum" loop vertices, it seems at first sight that this "optimization of the tree over scales" cannot be done in a positivity preserving way. Fortunately it is not necessary to preserve the imaginary character of all the fields in loop vertices, when they are smeared against a positive Gaussian measure. The corresponding formulas will be definitely more complicated than in a single slice, but preliminary results [61] indicate that the whole construction can be made without any cluster or Mayer expansion at any stage.

Let us add some words about the way in which ordinary constructive field theory can be renewed by these ideas. Using the "loop vertex" tree expansion in the commutative context it is presumably possible to repeat classical constructions such as those of the infrared limit of ordinary critical ϕ^4_4 theory *without* using any discretization, lattice of cubes, cluster and Mayer expansions at any stage.

This was done for a single scale model in [59], where we have shown that integration by parts combined wit the loop vertex expansion can prove the scaled decrease of the correlation functions, through a Fredholm type inequality.

It is an interesting non-trivial problem to generalize this intermediate field method to higher order interactions than ϕ^4, for instance ϕ^{2n}. More intermediate fields are obviously required.

Let us remark that we also expect many applications of this new method to constructive gluing of different expansions in ordinary QFT.

6 RNCQFT: possible applications

We would like now to comment further on possible areas of physical applications of RNCQFT:

- **Quantum Hall Effect**

  NCQFT and in particular the non commutative Chern Simons theory has been recognized as effective theory of the quantum Hall effect already for some time [17]-[18]-[19]. But the discovery of the vulcanized RG holds promises for a better explanation of how these effective actions are generated from the microscopic level.

  In this case there is an interesting reversal of the initial GW (Grosse-Wulkenhaar) problematic. In the $\phi^{\star 4}_4$ theory the vertex is given a priori by the Moyal structure, and it is LS invariant. The challenge was to find the right propagator which makes the theory renormalizable, and it turned out to have LS duality.

Now to explain the (fractional) quantum Hall effect, which is a bulk effect whose understanding requires electron interactions, we can almost invert this logic. The propagator is known since it corresponds to non-relativistic electrons in two dimensions in a constant magnetic field. It has LS duality. But the effective theory should be anionic hence not local. Here again we can argue that among all possible non-local interactions, a few renormalization group steps should select the only ones which form a renormalizable theory with the corresponding propagator. In the commutative case (i.e. zero magnetic field) local interactions such as those of the Hubbard model are just renormalizable in any dimension because of the extended nature of the Fermi-surface singularity. Since the non-commutative electron propagator (i.e. in non zero magnetic field) looks very similar to the GW propagator (it is in fact a generalization of the Langmann-Szabo-Zarembo propagator) we can conjecture that the renormalizable interaction corresponding to this propagator should be given by a Moyal product. That's why we hope that non-commutative field theory and a suitable generalization of the Grosse-Wulkenhaar renormalization group might be the correct framework for a microscopic *ab initio* understanding of the fractional quantum Hall effect which is currently lacking.

- **Charged Polymers in Magnetic Field**

  Just like the heat kernel governs random motion, the covariant Mahler kernel should govern random motion of charged particles in presence of a magnetic field. Ordinary polymers can be studied as random walk with a local self repelling or self avoiding interaction. They can be treated by QFT techniques using the $N = 0$ component limit or the supersymmetry trick to erase the unwanted vacuum graphs. Many results, such as various exact critical exponents in two dimensions, approximate ones in three dimensions, and infrared asymptotic freedom in four dimensions have been computed for self-avoiding polymers through renormalization group techniques. In the same way we expect that charged polymers under magnetic field might be studied through the new non commutative vulcanized RG. The relevant interactions again should be of the Moyal rather than of the local type, and there is no reason that the replica trick could not be extended in this context. Ordinary observables such as N point functions would be only translation *covariant*, but translation invariant physical observables such as density-density correlations should be recovered out of gauge invariant observables. In this way it might be possible to deduce new scaling properties of these systems and their exact critical exponents through the generalizations of the techniques used in the ordinary commutative case [62].

More generally we hope that the conformal invariant two dimensional theories, the RG flows between them and the c theorem of Zamolodchikov [63] might have appropriate *magnetic* generalizations involving vulcanized flows and Moyal interactions.

- **Quark Confinement**

  Quark confinement corresponds to a strong coupling non-perturbative regime of non-Abelian gauge theory on ordinary commutative space. In [15] a mapping is proposed between ordinary and non-commutative gauge fields which do not preserve the gauge groups but preserve the gauge equivalent classes. The effective physics of confinement should be governed by a non-local interaction, as is the case in effective strings or bags models. In the initial matrix model approach of 'tHooft [64] to this problem, the planar graphs dominate because a gauge group $SU(N)$ with N large. But the planar limit emerges more naturally out of NCQFT since it is then a renormalization group effect. The large N *matrix* limit in NCQFT's parallels the large N *vector* limit which allows to understand the formation of Cooper pairs in supraconductivity [65]. In that case N is not arbitrary but is roughly the number of effective quasi particles or sectors around the extended Fermi surface singularity at the superconducting transition temperature. We called this phenomenon a *dynamical* large N *vector* limit. RNCQFT's provides us with the first clear example of a *dynamical* large N *matrix* limit. We hope therefore that it should be ultimately useful to understand bound states in ordinary commutative non-Abelian gauge theories, hence quark confinement.

- **Quantum Gravity**

  Although ordinary renormalizable QFT's now seem to have renormalizable analogs on the Moyal space, there is no renormalizable commutative field theory for spin 2 particles, so that the RNCQFT alone should not allow quantization of gravity. However quantum gravity might enter the picture at a later and more advanced stage. The two current main tentatives to quantize gravity are string theory and loop gravity[3]. We remarked already that NCQFT appears as some effective version of string theory. But ribbon graphs have borders hence corresponds to open strings world sheets, whenever gravity occurs in the closed strings sector. Therefore it may have to do with doubling the ribbons of some NCQFT in an appropriate way. Because there is no reason not to quantize the antisymmetric tensor B which defines the non commutative geometry as well as the symmetric one g which defines the metric, we should clearly no longer limit ourselves to Moyal spaces. A first step towards a non-commutative approach to quantum gravity might be to search for the proper analog of vulcanization in more general non-commutative geometries such as solvable symmetric spaces [67].

The loop gravity approach is based on a background invariant formulation in which a huge symmetry group, those of diffeomorphisms is quotiented out. It seems at first sight farther from NCQFT. But some contact may appear when we better understand the role of new symmetries in RNCQFT, such as the LS duality.

However we have to admit that any theory of quantum gravity will probably remain highly conjectural for many decades or even centuries.

7 How to recover the ordinary world?

If at some energy scale in the *terra incognita* that lies between the Tev and the Planck scale noncommutativity escapes some internal space of the Connes-Chamseddine type and invades ordinary space-time itself, it might manifest itself first in the form of a tiny non-zero commutator between pairs of space time variables. From that scale up, we should use the non-commutative scale decomposition and the non-commutative renormalization group rather than the ordinary one. Although we don't know yet how to build renormalizable non-commutative gauge theories, we may hope that the flow corresponding to QED (which like ϕ^4_4 suffered from the Landau ghost in the ordinary

[3]There is also the intriguing possibility that the ordinary Einstein-Hilbert theory might have a non trivial renormalization group fixed point, so might be renormalizable in a non-perturbative sense [66].

commutative world) should become milder and may grind to a halt in the non-commutative world.

Noncommutative models with harmonic potential and non zero θ break both Lorentz and translation invariance. If renormalizable noncommutative gauge theories are built out of some Dirac analog of the Grosse-Wulkenhaar propagator with harmonic potential, as is envisioned in [46], they will also break translation invariance. Hence if such models have anything to do with physics beyond the standard model, one should explain how they can connect to our ordinary Lorentz and translation invariant commutative world.

My initial impression was that perhaps only models of the covariant type can make a connection to such ordinary physics, because such covariant models do not really break translation invariance for gauge invariant physical quantities [1].

But covariant models are much more singular, and in particular they do not seem to have $\Omega = 1$ fixed points of the asymptotically safe type. Therefore I would like to propose an other possible scenario, which will be developed in a future joint publication with R. Gurau and A. Tanasa [68].

In this scenario ordinary fields at lower energies do not emerge from a single confined model of the GW type but from the zero modes of a whole bunch of such models, which should be glued together in a coherent way. We know that although a lattice breaks rotation invariance, the long distance effective theory can be rotation-invariant. Furthermore the Laplacian can emerge naturally for instance from standard nearest-neighbor ferromagnetic coupling, but also from other generic types of short range couplings.

We therefore consider a regular (or eventually random) four dimensional lattice Λ. To each lattice site or cell would be associated a different copy of the GW model and these copies would be independent of each other except for their zero modes or perhaps for a few low values of the modes in the matrix base. Each GW model would have its own confining harmonic potential roughly centered around the center of the cell. Each would exhibit a fixed point in the ultraviolet regime. In the infrared regime the zero modes of these GW models would form the degrees of freedom of the ordinary commutative field theories of our world and govern long range physics.

Such a model at first sight resembles a field theory with a naive lattice cutoff. It has a particular scale Λ_θ (which may or may not be the Planck scale) essentially given by the θ parameter which would give the area or 4d volume of the elementary lattice cells. But it has several advantages over a naive cutoff.

- There would be no true cutoff in energy. Physics would not stop at scale Λ_θ. As one climbs in energy in our commutative world, for instance using more and more powerful colliders, I do not see how to avoid focusing on tinier and tinier regions of space-time. From scale Λ_θ on, one would enter into a particular "GW worldlet" corresponding to the inside of a given cell. This "worldlet" can have no ultraviolet cutoff and remain mathematically consistent up to infinite energy.

- The bare coupling for the commutative world, hence at scale Λ_θ is also the renormalized coupling for the GW worldlets. It would be the interaction corresponding to the GW zero mode, hence form our world lower scales it would appear local. It corresponds to the renormalizable interactions of ordinary field theory, since they appear to correspond also to the renormalizable ones for the GW worldlets.

- We would like to investigate whether the noncommutativity of space time which killed the Landau ghost could be a substitute for supersymmetry to tame ultraviolet flows, but without introducing new particles. Supersymmetry tames ultraviolet flows by adding loops of superpartners to the ordinary loops. One of the main arguments for supersymmetry is that it makes the three flows of the standard model $U(1)$, $SU(2)$ and $SU(3)$ couplings better converge at a single unification scale (see [69] and references therein for a discussion of this subtle question). Replacing commutative flows by noncommutative flows at some scale before that unification scale might also do the job.

However this scenario should be much elaborated if it is ever to become a credible alternative to supersymmetry. In particular discovering some natural way to glue the "GW wordlets" seems necessary in order to develop the model further. A proposal will be given in [68] but at the moment it is neither canonical nor unique.

References

[1] V. Rivasseau, Noncommutative renormalization, in *Quantum Spaces*, Progress in Mathematical Physics **53**, Birkhäuser, (2007), arXiv:0705.0705,

[2] V. Rivasseau and F. Vignes-Tourneret, Noncommutative Renormalization, in *Rigorous Quantum Field Theory*, a Festschrift for Jacques Bros, Birkhäuser Progress in Mathematics **251**, (2007), hep-th/0409312.

[3] K. Wilson, Renormalization group and critical phenomena, II Phase space cell analysis of critical behavior, *Phys. Rev. B* **4**, (1974), 3184.

[4] J. Feldman and E. Trubowitz, Perturbation theory for Many Fermion Systems, Helv. Phys. Acta **63** 156, (1991) .

[5] J. Feldman and E. Trubowitz, The flow of an Electron-Phonon System to the Superconducting State, Helv. Phys. Acta **64** 213, (1991).

[6] E. Schrödinger, Über die Unanwendbarkeit des Geometrie im Kleinen Naturwiis. **31**, (1934), 342.

[7] W. Heisenberg, Die Grenzen des Anwendbarkeit des bisherigen Quantentheorie Z. Phys. **110** (1938), 251.

[8] H. S. Snyder, Deformation quantization for actions of the affine group Phys Rev **71** (1947), 38.

[9] M. Dubois-Violette, R. Kerner and J. Madore, Noncommutative differential geometry and new models of gauge theory J. Math. Phys. **31**, (1990), 323-330.

[10] A. Connes, *Noncommutative geometry*, Academic Press Inc., San Diego (1994)
available at http://www.alainconnes.org/downloads.html.

[11] J. W. Barrett, A Lorentzian version of the non-commutative geometry of the standard model of particle physics, arXiv:hep-th/0608221

[12] A. Connes, Non-commutative geometry and the standard model with neutrino mixing, arXiv:hep-th/0608226.

[13] E. Witten, Non-commutative geometry and string field theory Nuclear Physics B, **268**, (1986), 253-294.

[14] A. Connes, M. R. Douglas, and A. Schwarz, Noncommutative geometry and matrix theory: Compactification on tori, JHEP **02** (1998) 003, hep-th/9711162.

[15] N. Seiberg and E. Witten, String theory and noncommutative geometry, JHEP **09** (1999) 032, `hep-th/9908142`.

[16] M. R. Douglas and N. A. Nekrasov, Noncommutative field theory, Rev. Mod. Phys. **73**, (2001), 977–1029, `hep-th/0106048`.

[17] L. Susskind, The Quantum Hall Fluid and Non-Commutative Chern Simons Theory, hep-th/0101029

[18] A. Polychronakos, Quantum Hall states as matrix Chern-Simons theory, JHEP 0104 (2001), 011, `hep-th/0103013`

[19] S. Hellerman and M. Van Raamsdonk, Quantum Hall Physics = Noncommutative Field Theory, JHEP 0110 (2001) 039, `hep-th/ 0103179`

[20] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, Noncommutative perturbative dynamics, *JHEP* **02** (2000) 020, `hep-th/9912072`.

[21] H. Grosse and R. Wulkenhaar, Power-counting theorem for non-local matrix models and renormalization, *Commun. Math. Phys.* **254** (2005), no. 1, 91–127, `hep-th/0305066`.

[22] H. Grosse and R. Wulkenhaar, Renormalization of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base, *Commun. Math. Phys.* **256** (2005), no. 2, 305–374, `hep-th/0401128`.

[23] E. Langmann and R. J. Szabo, Duality in scalar field theory on noncommutative phase spaces, *Phys. Lett.* **B533** (2002) 168–177, `hep-th/0202039`.

[24] V. Rivasseau, F. Vignes-Tourneret and R. Wulkenhaar, Renormalization of noncommutative ϕ^4-theory by multi-scale analysis, Commun. Math. Phys. **262** (2006) 565, [arXiv:hep-th/0501036].

[25] R. Gurau, J. Magnen, V. Rivasseau and F. Vignes-Tourneret, Renormalization of non-commutative ϕ^4_4 field theory in x-space, Commun. Math. Phys. **267** (2006) 515, [arXiv:hep-th/0512271].

[26] H. Grosse and R. Wulkenhaar, The beta-function in duality-covariant noncommutative ϕ^4-theory, Eur. Phys. J. C **35** (2004) 277, [arXiv:hep-th/0402093].

[27] M. Disertori and V. Rivasseau, Two and three loops beta function of non commutative ϕ_4^4 theory, Eur. Phys. J. C **50** (2007) 661 [arXiv:hep-th/0610224].

[28] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, Vanishing of beta function of non commutative ϕ_4^4 theory to all orders, Phys. Lett. B **649** (2007) 95 [arXiv:hep-th/0612251].

[29] R. Gurau and V. Rivasseau, "Parametric representation of noncommutative field theory," Commun. Math. Phys. **272** (2007) 811 [arXiv:math-ph/0606030].

[30] V. Rivasseau and A. Tanasa, "Parametric representation of 'critical' noncommutative QFT models," arXiv:math-ph/0701034.

[31] R. Gurau, A. P. C. Malbouisson, V. Rivasseau and A. Tanasa, "Non-Commutative Complete Mellin Representation for Feynman Amplitudes," Lett. Math. Phys. **81**, 161 (2007) [arXiv:0705.3437 [math-ph]].

[32] R. Gurau and A. Tanasa, "Dimensional regularization and renormalization of non-commutative QFT," arXiv:0706.1147 [math-ph].

[33] A. Tanasa and F. Vignes-Tourneret, "Hopf algebra of non-commutative field theory," arXiv:0707.4143 [math-ph].

[34] F. Vignes-Tourneret, "Renormalization of the orientable non-commutative Gross-Neveu model," Annales Henri Poincaré **8** (2007) 427 [arXiv:math-ph/0606069].

[35] F. Vignes-Tourneret, " Renormalisation des théories de champs non commutatives," Ph.D. thesis, arXiv:math-ph/0612014.

[36] A. Lakhoua, F. Vignes-Tourneret and J. C. Wallet, "One-loop beta functions for the orientable non-commutative Gross-Neveu model," arXiv:hep-th/0701170.

[37] E. Langmann, R. J. Szabo and K. Zarembo, "Exact solution of noncommutative field theory in background magnetic fields," Phys. Lett. B **569** (2003) 95 [arXiv:hep-th/0303082].

[38] E. Langmann, R. J. Szabo and K. Zarembo, "Exact solution of quantum field theory on noncommutative phase spaces," JHEP **0401** (2004) 017 [arXiv:hep-th/0308043].

[39] R. Gurau, V. Rivasseau and F. Vignes-Tourneret, "Propagators for noncommutative field theories," Annales Henri Poincaré **7** (2006) 1601 [arXiv:hep-th/0512071].

[40] H. Grosse and H. Steinacker, "Renormalization of the noncommutative ϕ^3-model through the Kontsevich model," Nucl. Phys. B **746** (2006) 202 [arXiv:hep-th/0512203].

[41] H. Grosse and H. Steinacker, "Exact renormalization of a noncommutative ϕ^3 model in 6 dimensions," arXiv:hep-th/0607235.

[42] H. Grosse and H. Steinacker, "A nontrivial solvable noncommutative ϕ^3 model in 4 dimensions," JHEP **0608** (2006) 008 [arXiv:hep-th/0603052].

[43] Z. Wang and S. Wan, Renormalization of Orientable Non-Commutative Complex Φ^6_3 Field Theory in x Space, hep-th.wzht.23443, to appear in Ann. Henri Poincaré.

[44] Axel de Goursac, J.C. Wallet and R. Wulkenhaar, Non Commutative Induced Gauge Theory `hep-th/0703075`.

[45] H. Grosse and M. Wohlgenannt, "Renormalization and Induced Gauge Action on a Noncommutative Space," arXiv:0706.2167 [hep-th].

[46] H. Grosse and R. Wulkenhaar, "8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory," arXiv:0709.0095 [hep-th].

[47] J. Glimm and A. Jaffe, Quantum physics. A functional integral point of view, Springer, 2nd edition (1987).

[48] V. Rivasseau, From perturbative to constructive renormalization, Princeton University Press (1991).

[49] J. Glimm A. Jaffe and T. Spencer, Convergent expansion about mean field theory II. Convergence of the expansion Ann. Phys. **101** 631-669 (1976)

[50] A. Abdesselam and V. Rivasseau, Trees, forests and jungles: a botanical garden for cluster expansions, in Constructive Physics, ed by V. Rivasseau, Lecture Notes in Physics 446, Springer Verlag, 1995.

[51] Constructive Physics, Proceedings of the International Workshop at Ecole Polytechnique, Palaiseau, July 1994, ed by V. Rivasseau, Lecture Notes in Physics 446, Springer Verlag (1995).

[52] Constructive Field Theory and Applications: Perspectives and Open Problems, Journ. Math. Phys. 41, 3764 (2000).

[53] K. Gawedzki and A. Kupiainen, Gross-Neveu model through convergent perturbation expansions, *Comm. Math. Phys.* **102**, 1 (1985).

[54] J. Feldman, J. Magnen, V. Rivasseau and R. Sénéor, A renormalizable field theory: the massive Gross-Neveu model in two dimensions, *Comm. Math. Phys.* **103**, 67 (1986).

[55] K. Gawedzki and A. Kupiainen, Massless ϕ^4_4 theory: Rigorous control of a renormalizable asymptotically free model, *Comm. Math. Phys.* **99**, 197 (1985).

[56] J. Feldman, J. Magnen, V. Rivasseau and R. Sénéor, Construction of infrared ϕ^4_4 by a phase space expansion, *Comm. Math. Phys.* **109**, 437 (1987).

[57] A. Sokal, An improvement of Watson's theorem on Borel summability, *Journ. Math. Phys.* **21**, 261 (1980).

[58] V. Rivasseau, "Constructive Matrix Theory," arXiv:0706.1224 [hep-th].

[59] J. Magnen and V. Rivasseau, "Constructive ϕ^4 field theory without tears," arXiv:0706.2457 [math-ph].

[60] A. Abdesselam,A Complete Renormalization Group Trajectory Between Two Fixed Points, `math-ph/0610018`.

[61] J. Magnen and V. Rivasseau, work in progress.

[62] B. Duplantier, Conformal Random Geometry in Les Houches, Session LXXXIII, 2005, Mathematical Statistical Physics, A. Bovier, F. Dunlop, F. den Hollander, A. van Enter and J. Dalibard, eds., pp. 101-217, Elsevier B. V. (2006), `math-ph/0608053`

[63] A.B Zamolodchikov, JETP Letters **43**, 731 (1986)

[64] G. 't Hooft, A planar diagram theory for strong interactions Nuclear Physics B, **72**, 461 (1974)

[65] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, An Intrinsic 1/N Expansion for Many Fermion Systems, *Europhys. Letters* **24**, 437 (1993).

[66] D. Litim, Fixed Points of Quantum Gravity, Phys. Rev. Lett. **92**, (2004) 201301.

[67] Pierre Bieliavsky, Strict Quantization of Solvable Symmetric Spaces, J. Symplectic Geom. Volume 1, Number 2 (2002), 269-320.

[68] R. Gurau, V. Rivasseau and A. Tanasa, in preparation.

[69] V. Barger, Jing Jiang, Paul Langacker and Tianjun Li, Gauge Coupling Unification in the Standard Model, Phys.Lett. **B624** (2005) 233-238, arXiv:hep-ph/0503226