数理解析研究所講究録
第 1600 巻 2008 年 48-67

Fermionic renormalization group method based on the smooth Feshbach map

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1 Introduction

In this paer, we illustrate that the renormalization group method, which is originally proposed in [1, 2] and largely improved in [3], is also useful to analyze the spectrum of the Hamiltonian for the fermion system.

We consider a system which a fermion field coupled to a quantum system S. The Hilbert space of the total system is given by

$$\mathcal{H}=\mathcal{H}_{\mathrm{S}}\otimes\mathcal{F}, \tag{1.1}$$

where $\mathcal{H}_{\mathrm{S}}$ denotes the Hilbert space for the quantum system S which is a separable Hilbert space, and $\mathcal{F}$ denotes the fermion Fock space:

$$\mathcal{F}=\bigoplus_{n=0}^{\infty}\bigwedge^{n}L^{2}(\mathbf{M}),$$

where $\wedge^{n}L^{2}(\mathbf{M})$ denotes the n-fold antisymmetric tensor product of $L^{2}(\mathbf{M})$ with $\wedge^{0}L^{2}(\mathbf{M})=\mathbb{C}$, $\mathbf{M}:=\mathbb{R}^{d}\times\mathbf{L}$ is the momentum-spin arguments of a single fermion with $\mathbb{L}:=\{-s,-s+1,\ldots,s-1,s\}$ and s denotes a non-negative half-integer. The Hamiltonian of the system S is denoted by H_{S} which is a given self-adjoint operator on $\mathcal{H}_{\mathrm{S}}$ and bounded from below. Let $b^{*}(k),b(k),\ k\in\mathbb{M}$ be the kernels of the fermion creation and annihilation operators, which obey the canonical anticommutation relations:

$$\{b(k),b^{*}(\tilde{k})\}=\delta_{l,\tilde{l}}\delta(\mathbf{k}-\tilde{\mathbf{k}}),\quad \{b(k),b(\tilde{k})\}=\{b^{*}(k),b^{*}(\tilde{k})\}=0, \tag{1.2}$$
$$k=(\mathbf{k},l),\ \ k=(\tilde{\mathbf{k}},\tilde{l})\in\mathbf{M}.$$

Let $\Omega=(1,0,0,\ldots)\in\mathcal{F}$ be the vacuum vector. The vacuum vector is specified by the condition

$$b(k)\Omega=0,\quad k\in\mathbf{M}. \tag{1.3}$$

The free Hamiltonian of the fermion field H_{f} is defined by

$$H_{\mathrm{f}}=\int_{\mathbb{R}^{d}}\sum_{l\in\mathbf{L}}\omega(\mathbf{k},l)b^{*}(\mathbf{k},l)b(\mathbf{k},l)d\mathbf{k},$$

with the single free fermion energy $\omega(k)=c|\mathbf{k}|^{\nu},\ k=(\mathbf{k},l)\in\mathbf{M}$.

The operator for the coupled system is defined by

$$H_{g}(\theta)=H_{\mathrm{S}}\otimes\mathbf{1}+e^{\theta\nu}\mathbf{1}\otimes H_{\mathrm{f}}+W_{g}(\theta). \tag{1.4}$$

Here, the operator $W_{g}(\theta)$ is the interaction Hamiltonian between the system S and the fermion field, and $\theta\in\mathbb{C}$ is a complex scaling parameter. We suppose that the interaction $W_{g}(\theta)$ has the form

$$W_{g}(\theta)=\sum_{M+N=1}^{\infty}g^{M+N}W_{M,N}(\theta), \tag{1.5}$$

$$W_{M,N}(\theta)=\int_{\mathbf{M}^{M+N}}dK^{(M,N)}G_{M,N}^{(\theta)}(K^{(M,N)})\otimes b^{*}(k_{1})\cdots b^{*}(k_{M})b(\tilde{k}_{1})\cdots b(\tilde{k}_{N}), \tag{1.6}$$

[1]This work was supported by JSPS Fellowship (DC2)
[2]This work was supported by JSPS Fellowship (DC2)

where $g \in \mathbb{R}$ is the coupling constant and

$$K^{(M,N)} = (k_1, \cdots, k_M, \tilde{k}_1, \cdots, \tilde{k}_N) \in \mathbf{M}^{M+N},$$

$$\int_{\mathbf{M}^{M+N}} dK^{(M,N)} := \int_{\mathbf{R}^{d(M+N)}} \sum_{\substack{(l_1,\dots,l_M)\in\mathbf{L}^M,\\ (\tilde{l}_1,\dots,\tilde{l}_N)\in\mathbf{L}^N}} d\mathbf{k}_1 \cdots d\mathbf{k}_M d\tilde{\mathbf{k}}_1 \cdots d\tilde{\mathbf{k}}_N, \tag{1.7}$$

and $G^{(\theta)}_{M,N}$ are functions with values in operators on $\mathcal{H}_S$. The precise conditions for $G^{(\theta)}_{M,N}$ are written in the next section. Suppose that H_S has a non-degenerate discrete eigenvalue $E \in \sigma_d(H_S)$. Since the vacuum vector Ω is an eigenvector of H_f with eigenvalue 0, $H_0(\theta)$ has an eigenvalue E. We are interested in the fate of the eigenvalue E under influence of the perturbation $W_g(\theta)$.

The fermionic renormalization group is constructed for the operator (1.4), and under suitable conditions, it is proved that $H_g(\theta)$ has an eigenvalue $E_g(\theta)$ closed to E for small $g \in \mathbf{R}$. The eigenvalue $E_g(\theta)$ and the corresponding eigenvector $\Psi_g(\theta)$ is constructed by the same process as in [3].

The (bosonic) operator theoretic renormalization group was invented by V. Bach, J. Fröhlich, and I. M. Sigal [2, 1]. In [1], the operator of the similar form (1.4)-(1.6) is considered, but boson is treated instead of fermion and $M+N \leq 2$ is assumed. They proved the existence of an eigenvalue of the (complex scaled) Hamiltonian, and constructed the eigenvalue and the corresponding eigenvector. Moreover, they gave the range of the continuous spectrum which extended from the eigenvalue. In the paper [3], V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal introduced the smooth Feshbach map and largely improved the proof of the convergence of the renormalization group.

Our paper is based on the smooth Feshbach map and the improved renormalization group method [3]. Our construction for the fermionic operator theoretic renormalization group is similar as in [3] without the Wick ordering and its related estimate. The feature of this paper is that we can treat a large class of interactions. In particular, the interaction Hamiltonian $W_g(\theta)$ includes arbitrary order of the creation and annihilation operators.

The paper is organized as follows. The precise definitions of $H_g(\theta)$ is given in the Section 2, where we explain the problem in detail. We review the smooth Feshbach map in Section 3 for reader convenience. The main originality of this paper is to obtain the Wick ordering formula for fermion. The Wick ordering formula for fermion and related formulas are given in the Section 4. In the last section we sketch the proof of our main result.

2 Hypotheses and Main Results

Through this paper, we denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|$ respectively, where we use the convention that the inner product is antilinear (respectively linear) in the first (respectively second) variable. If there is no danger of confusion, then we omit the subscript $\mathcal{X}$ in $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|$. For a linear operator T on a Hilbert space, we denote its domain, spectrum and resolvent by dom(T), $\sigma(T)$ and Res(T), respectively. If T is densely defined, then the adjoint of T is denoted by T^*.

One can identify a vector $\Psi \in \mathcal{F}$ with a sequence $(\Psi^{(n)})_{n=0}^{\infty}$ of n-fermion state $\Psi^{(n)} \in \wedge^n L^2(\mathbf{M}) \subset L^2(\mathbf{M}^n)$. We observe that, for all $\psi \in \wedge^n L^2(\mathbf{M})$ and $\pi \in \mathcal{S}_n$,

$$\psi(k_{\pi(1)}, \cdots, k_{\pi(n)}) = \mathrm{sgn}(\pi)\psi(k_1, \cdots, k_n), \quad \text{a.e.} \tag{2.1}$$

where $\mathcal{S}_n$ is the group of permutations of n elements and sgn(π) the sign of the permutation π. The inner product of $\mathcal{F}$ is defined by

$$\langle \Psi, \Phi \rangle = \sum_{n=0}^{\infty} \langle \Psi^{(n)}, \Phi^{(n)} \rangle_{\wedge^n L^2(\mathbf{M})} \tag{2.2}$$

for $\Psi, \Phi \in \mathcal{F}$, where

$$\langle \Psi^{(n)}, \Phi^{(n)} \rangle_{\wedge^n L^2(\mathbf{M})} = \int_{\mathbf{M}^n} \prod_{j=1}^{n} dk_j \Psi^{(n)}(k_1, \cdots, k_n)^* \Phi^{(n)}(k_1, \cdots, k_n). \tag{2.3}$$

We define the free Hamiltonian of the fermion field H_f by

$$\mathrm{dom}(H_f) := \left\{ \Psi \in \mathcal{F} \middle| \sum_{n=0}^{\infty} \|(H_f\Psi)^{(n)}\|^2 < \infty \right\}, \tag{2.4}$$

$$(H_f\Psi)^{(n)}(k_1, \cdots, k_n) = \left(\sum_{j=1}^{n} \omega(k_j) \right) \Psi^{(n)}(k_1, \cdots, k_n), \quad n \in \mathbb{N} \tag{2.5}$$

$$(H_f\Psi)^{(0)} = 0, \tag{2.6}$$

where

$$\omega(k) := c|\mathbf{k}|^{\nu}, \quad k = (\mathbf{k}, l) \in \mathbf{M},$$

with a positive constant $c, \nu > 0$. For a nonrelativistic fermion, the choice of the constants c, ν are $c = 1/2m$ and $\nu = 2$, where m denotes the mass of the fermion. In this paper, for any $\Psi \in \mathcal{F}$, $b(k)\Psi$ is regarded as a $\times_{n=0}^{\infty} \wedge^n L^2(\mathbf{M})$-valued function:

$$b(k) : \mathbf{M} \ni k \longmapsto b(k)\Psi \in \mathop{\times}_{n=0}^{\infty} \wedge^n L^2(\mathbf{M}), \quad \text{a.e.}, \tag{2.7}$$

$$(b(k)\Psi)^{(n)}(k_1, \cdots, k_n) = \sqrt{n+1}\Psi^{(n+1)}(k, k_1, \cdots, k_n), \tag{2.8}$$

where the symbol "$\times$" denotes the Cartesian product. We set

$$\mathrm{dom}(b(k)) := \{\Psi \in \mathcal{F} | b(k')\Psi \in \mathcal{F} \text{ a.e.} k' \in \mathbf{M}\}.$$

Note that $\mathrm{dom}(b(k))$ is independent of $k \in \mathbf{M}$. We observe that, for all $\Psi \in \mathcal{F}$ and $\Phi \in \mathrm{dom}(H_f)$,

$$\begin{aligned} \langle \Psi, H_f\Phi \rangle &= \sum_{n=0}^{\infty} \int_{\mathbf{M}^{(n+1)}} \prod_{j=1}^{n+1} dk_j \Psi^{(n+1)}(k_1, \cdots, k_{n+1})^* \\ &\quad \times \left(\sum_{j=1}^{n+1} \omega(k_j) \right) \Psi^{(n+1)}(k_1, \cdots, k_{n+1}) \\ &= \sum_{n=0}^{\infty} \int_{\mathbf{M} \times \mathbf{M}^n} dk \prod_{j=1}^{n} dk_j (b(k)\Psi)^{(n)}(k_1, \cdots, k_n)^* \\ &\quad \times \omega(k)(b(k)\Psi)^{(n)}(k_1, \cdots, k_n) \end{aligned} \tag{2.9}$$

where we have used the antisymmetry (2.1). Hence we have

$$\langle \Psi, H_f\Phi \rangle = \int_{\mathbf{M}} dk \omega(k) \langle b(k)\Psi, b(k)\Phi \rangle \tag{2.10}$$

and, in this sense, write symbolically

$$H_f = \int_{\mathbf{M}} dk \omega(k) b^*(k) b(k). \tag{2.11}$$

In the same way as (2.11), the number operator, N_f, is defined by

$$N_f = \int_{\mathbf{M}} dk b^*(k) b(k). \tag{2.12}$$

We remark that

$$\mathrm{dom}(H_f^{1/2}),\ \mathrm{dom}(N_f^{1/2}) \subset \mathrm{dom}(b(k)), \tag{2.13}$$

since, for all $\Psi \in \mathrm{dom}(H_f^{1/2})$ and $\Phi \in \mathrm{dom}(N_f^{1/2})$,

$$\|H_f^{1/2}\Psi\|^2 = \int_{\mathbf{M}} dk\omega(k)\|b(k)\Psi\|^2 < \infty,$$
$$\|N_f^{1/2}\Phi\|^2 = \int_{\mathbf{M}} dk\|b(k)\Phi\|^2 < \infty.$$

The (smeared) annihilation operator $b(f)$ ($f \in L^2(\mathbf{M})$) defined by

$$b(f) = \int_{\mathbf{M}} f(k)^* b(k) dk, \tag{2.14}$$

and the adjoint $b^*(f)$, called the (smeared) creation operator, obey the canonical anti-commutation relations (CAR):

$$\{b(f), b(g)\} = \langle f, g\rangle, \quad \{b(f), b(g)\} = \{b^*(f), b^*(g)\} = 0 \tag{2.15}$$

for all $f, g \in L^2(\mathbf{M})$, where $\{X, Y\} = XY + YX$.

The Hamiltonian of the total system is defined by

$$H_g := H_{\mathrm{S}} \otimes \mathbf{1} + \mathbf{1} \otimes H_{\mathrm{f}} + W_g,$$

where the symmetric operator W_g is of the form:

$$W_g = \sum_{M+N=1}^{\infty} g^{M+N} W_{M,N}, \tag{2.16}$$

$$W_{M,N} = \int_{\mathbf{M}^{M+N}} dK^{(M,N)} G_{M,N}(K^{(M,N)}) \otimes b^*(k_1) \cdots b^*(k_M) b(\tilde{k}_1) \cdots b(\tilde{k}_N), \tag{2.17}$$

and

$$K^{(M,N)} = (k_1, \cdots, k_M, \tilde{k}_1, \cdots, \tilde{k}_N) \in \mathbf{M}^{M+N},$$

$$\int_{\mathbf{M}^{M+N}} dK^{(M,N)} := \int_{\mathbf{R}^{d(M+N)}} \sum_{\substack{(l_1,\ldots,l_M)\in \mathbf{L}^M,\\ (\tilde{l}_1,\ldots,\tilde{l}_N)\in \mathbf{L}^N}} d\mathbf{k}_1 \cdots d\mathbf{k}_M d\tilde{\mathbf{k}}_1 \cdots d\tilde{\mathbf{k}}_N. \tag{2.18}$$

Here, for almost every $K^{(M,N)} \in \mathbf{M}^{M+N}$, $G_{M,N}(K^{(M,N)})$ is a densely defined closable operator on $\mathcal{H}_{\mathrm{S}}$. $H_0 := H_{\mathrm{S}} \otimes \mathbf{1} + \mathbf{1} \otimes H_{\mathrm{f}}$ is regarded to the unperturbed Hamiltonian, and W_g is regarded to the perturbation Hamiltonian.

In what follows we formulate hypotheses of main theorem and introduce some objects.

Hypothesis 1. (spectrum) *Assume that H_{S} has a non-degenerate isolate eigenvalue $E \in \sigma_{\mathrm{d}}(H_{\mathrm{S}})$ such that*

$$\mathrm{dist}(E, \sigma(H_{\mathrm{S}}))\backslash\{E\}) \geq 1. \tag{2.19}$$

In general, if the operator H_{S} has a discrete eigenvalue E, it holds that $c_1 := \mathrm{dist}(E, \sigma(H_{\mathrm{S}})\backslash\{E\}) > 0$ and $\mathrm{dist}(c_1^{-1}E, \sigma(c_1^{-1}H_{\mathrm{S}}))\backslash\{c_1^{-1}E\} \geq 1$. We can assume (2.19) without loss of generality.

Since $\sigma(H_{\mathrm{f}}) = [0, \infty)$, the spectrum of the unperturbed Hamiltonian is $\sigma(H_0) = [E_0, \infty)$ with $E_0 := \inf \sigma(H_{\mathrm{S}})$. The vector Ω is an eigenvector of H_0 with eigenvalue 0. Hence, H_0 has an embedded eigenvalue E. In this paper, we study the fate of E under the perturbation $W_g(\theta)$. To analyze the perturbed Hamiltonian H_g, for $\theta \in \mathbb{R}$, we introduce the family of operators $H_g(\theta)$ of the form

$$H_g(\theta) \equiv (\mathbf{1} \otimes \Gamma_\theta) H_g (\mathbf{1} \otimes \Gamma_\theta^*) = H_0(\theta) + W_g(\theta), \tag{2.20}$$

where Γ_ρ is the dilation operator, i.e.,

$$\Gamma_\rho b(\mathbf{k}, l) \Gamma_\rho^* = \rho^{-d/2} b(\rho^{-1}\mathbf{k}, l), \tag{2.21}$$

and

$$H_0(\theta) \equiv H_S \otimes \mathbf{1} + e^{\theta\nu}\mathbf{1} \otimes H_f \tag{2.22}$$

$$W_g(\theta) \equiv (\mathbf{1} \otimes \Gamma_{e^\theta}) W_g (\mathbf{1} \otimes \Gamma_{e^\theta}^*) = \sum_{M+N=1}^{\infty} g^{M+N} W_{M,N}(\theta), \tag{2.23}$$

$$\begin{aligned} W_{M,N}(\theta) &\equiv \Gamma_{e^\theta} W_{M,N} \Gamma_{e^\theta}^* \\ &= \int_{\mathbf{M}^{M+N}} dK^{(M,N)} G_{M,N}^{(\theta)}(K^{(M,N)}) \otimes b^*(k_1)\cdots b^*(k_M) b(\tilde{k}_1)\cdots b(\tilde{k}_N), \end{aligned} \tag{2.24}$$

$$G_{M,N}^{(\theta)}(K^{(M,N)}) := e^{d(M+N)\theta/2} G_{M,N}(e^\theta K^{(M,N)}), \tag{2.25}$$

$$e^\theta K^{(M,N)} := (e^\theta \mathbf{k}_1, l_1; \dots; e^\theta \mathbf{k}_M, l_M; e^\theta \tilde{\mathbf{k}}_1 \tilde{l}_1; \dots; e^\theta \tilde{\mathbf{k}}_N, \tilde{l}_N). \tag{2.26}$$

Hypothesis 2. *Assume that, for every θ in some complex neighborhood of* 0*, the following hold:*

(i) The operator $G_{M,N}(e^\theta K^{(M,N)})$ is defined on $\mathrm{dom}(G_{M,N})$ *that contains* $\mathrm{dom}(H_0(\theta))$ *and the map $\theta \mapsto G_{M,N}(e^\theta K^{(M,N)})(H_S + i)^{-1}$ is extended to a bounded operator-valued analytic function on some complex neighborhood of $\theta = 0$.*

(ii) For all $M + N \geq 1$, $W_{M,N}(\theta)$ is relatively bounded with respect to $H_0(\theta)$ and

$$\sum_{M+N=1}^{\infty} g^{M+M} \|W_{M,N}(\theta)\Psi\| \leq a_g(\theta)\|H_0(\theta)\Psi\| + b_g(\theta)\|\Psi\|, \tag{2.27}$$

for all $\Psi \in \mathrm{dom}(H_0(\theta))$, with some constants $a_g(\theta), b_g(\theta) \geq 0$,

(iii) $\lim_{g\to 0} a_g(\theta) = 0$ *and* $\lim_{g\to 0} b_g(\theta) = 0$.

(iv) There exists a constant $\gamma > 1/2$ such that

$$\int_{\mathbf{M}^{M+N}} \frac{dK^{(M,N)}}{\left[\prod_{j=1}^{M}\omega(k_j)\prod_{j=1}^{N}\omega(k_j)\right]^{1+2\gamma}} \|G_{M,N}^{(\theta)}(K^{(M,N)})(H_S + i)^{-1}\|_{\mathrm{op}}^2 < \infty,$$

holds for all $M + N \geq 1$.

By the hypothesis above, one can show that, $H_g(\theta)$ is closed operator with the domain $\mathrm{dom}(H_g(\theta)) = \mathrm{dom}(H_0)$. In particular, H_g is a self-adjoint operator on $\mathrm{dom}(H_0)$.

By Hypothesis 2, we can consider the case $\theta = -i\vartheta/\nu$ $(0 < \vartheta < \pi/2)$. In what follows, we set $\theta = -i\vartheta/\nu$ and fix the parameter $\vartheta \in (0, \pi/2)$ so that Hypothesis 2 holds. Then, the spectrum $\sigma(H_0(-i\vartheta/\nu))$ contains separate rays of continuous spectrum and the eigenvalue E of $H_0(-i\vartheta/\nu)$ are located at tip of a branch of a continuous spectrum. Indeed, we observe

$$\begin{aligned} \sigma(H_0(-i\vartheta/\nu)) &= \{\lambda_1 + e^{-i\vartheta}\lambda_2 | \lambda_1 \in \sigma(H_S), \lambda_2 \in \sigma(H_f)\} \\ &\supset \left\{E + e^{-i\vartheta}\lambda \mid \lambda \in [0,\infty)\right\}. \end{aligned}$$

In order to study the fate of E under the perturbation of W_g, we introduce a spectral parameter $z \in \mathbb{C}$, and define a family of operators $H[z]$ by

$$H[z] = H_g(-i\vartheta/\nu) - E - z, \tag{2.28}$$

where $0 < \vartheta < \pi/2$. By using the fermionic renormalization group method, we will construct a constant e_g and a vector $\Psi_g \in \mathrm{dom}(H_g(-i\vartheta/\nu)) \setminus \{0\}$ such that

$$H[e_g]\Psi_g = 0,$$

which implies that $E_g := E + e_g$ is an eigenvalue of $H_g(-i\vartheta/\nu)$ and Ψ_g is the corresponding eigenvector.

The following theorem is our main result:

Theorem 2.1. *Fix $\theta = -i\vartheta/\nu$ as above. There exists a constant $g_0 > 0$ such that, for all g with $|g| \le g_0$, $H_g(\theta)$ has an eigenvalue E_g and the corresponding eignevector Ψ_g with the property*

$$\lim_{g\to 0} E_g = E, \quad \lim_{g\to 0} \Psi_g = \varphi_S \otimes \Omega, \tag{2.29}$$

where φ_S is the normalized eigenvector of H_S.

3 Smooth Feshbach map

In this section we review the smooth Feshbach map [3]. The smooth Feshbach map is the main ingredient to construct the operator theoretic renormalization group. Let χ be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$ such that $0 \le \chi \le 1$. We set

$$\bar{\chi} := \sqrt{1-\chi^2}.$$

Suppose that χ and $\bar{\chi}$ are non-zero operators. Let T be a closed operator on $\mathcal{H}$. We assume that

$$\chi T \subset T\chi,$$

and hence $\bar{\chi}T \subset T\bar{\chi}$, which mean that χ and $\bar{\chi}$ leave $\mathrm{dom}(T)$ invariant and commute with T. Let H be a closed operator on $\mathcal{H}$ such that $\mathrm{dom}(H) = \mathrm{dom}(T)$ and we set

$$H_\chi := T + \chi W \chi, \qquad H_{\bar{\chi}} := T + \bar{\chi} W \bar{\chi},$$

where $W := H - T$. We observe that, by the assumptions, the operators W, H_χ and $H_{\bar{\chi}}$ are defined on $\mathrm{dom}(T)$ and H_χ (resp. $H_{\bar{\chi}}$) is reduced by $\overline{\mathrm{Ran}\,\chi}$ (resp. $\overline{\mathrm{Ran}\,\bar{\chi}}$). We denote the projection onto $\overline{\mathrm{Ran}\,\chi}$ (resp. $\overline{\mathrm{Ran}\,\bar{\chi}}$) by P (resp. $\bar{P}$) and have

$$H_\chi \subset PH_\chi P + P^\perp T P^\perp, \qquad H_{\bar{\chi}} \subset \bar{P} H_{\bar{\chi}} \bar{P} + \bar{P}^\perp T \bar{P}^\perp,$$

where $P^\perp := 1 - P$ (resp. $\bar{P}^\perp := 1 - \bar{P}$) is the projection on $\ker\chi$ (resp. $\ker\bar{\chi}$).

We now introduce the *Feshbach triple* $\langle \chi, T, H\rangle$ as follows:

Definition 3.1. *Let χ, T and H as above. Then, we call $\langle\chi, H, T\rangle$ a Feshbach triple if $H_{\bar{\chi}}$ is bounded invertible on $\overline{\mathrm{Ran}\,\bar{\chi}}$ and the following conditions hold: the operators $\chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi}$ and $\chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi$ extend to bounded operators from $\mathcal{H}$ to $\overline{\mathrm{Ran}\,\chi}$ and $\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi$ to bounded operators from $\mathcal{H}$ to $\overline{\mathrm{Ran}\,\bar{\chi}}$, where $H_{\bar{\chi}}^{-1}$ denotes the inverse operator of $\bar{P} H_{\bar{\chi}} \bar{P}$.*

We remark that, if $H_{\bar{\chi}}$ is bounded invertible on $\overline{\mathrm{Ran}\,\bar{\chi}}$, then the operators $\chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi}$, $\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi$ and $\chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi$ are defined on $\mathrm{dom}(T)$.

For a Feshbach triple $\langle\chi, H, T\rangle$, we denote the closures of the operators $\chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi}$, $\chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi$ and $\bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi$ by the same symbols.

The definition of the Feshbach triple as above implies

$$\chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi},\ \chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi \in \mathcal{B}(\mathcal{H}; \overline{\mathrm{Ran}\,\chi}),\ \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi \in \mathcal{B}(\mathcal{H}; \overline{\mathrm{Ran}\,\bar{\chi}}). \tag{3.1}$$

For a Feshbach triple $\langle\chi, H, T\rangle$, we define the operator

$$F_\chi(H, T) := H_\chi - \chi W \bar{\chi} H_{\bar{\chi}}^{-1} \bar{\chi} W \chi, \tag{3.2}$$

acting on $\mathcal{H}$. We observe, by the definition of the Feshbach triple, that $F_\chi(H,T)$ is defined on $\mathrm{dom}(T)$.

The map from Feshbach pairs to operators on $\mathcal{H}$

$$\langle\chi, H, T\rangle \longmapsto F_\chi(H, T) \tag{3.3}$$

is called the *smooth Feshbach map (SFM)*. We remark that $F_\chi(H,T)$ is reduced by $\overline{\mathrm{Ran}\,\chi}$ and

$$F_\chi(H,T) \subset PF_\chi(H,T)P + P^\perp T P^\perp.$$

The SFM is an isospectral map in the sense of the following theorem.

Theorem 3.2. (SFM [3]) *Let $\langle \chi, H, T\rangle$ be a Feshbach triple. Then the following (i)-(v) hold:*

(i) If T is bounded invertible on $\overline{\mathrm{Ran}\,\bar\chi}$ and H is bounded invertible on $\mathcal{H}$ then $F_\chi(H,T)$ is bounded invertible on $\mathcal{H}$. In this case,

$$F_\chi(H,T)^{-1} = \chi H^{-1}\chi + \bar\chi T^{-1}\bar\chi. \tag{3.4}$$

If $F_\chi(H,T)$ is bounded invertible on $\overline{\mathrm{Ran}\,\chi}$, then H is bounded invertible on $\mathcal{H}$. In this case,

$$H^{-1} = Q_\chi(H,T)F_\chi(H,T)^{-1}Q_\chi^{\#}(H,T) + \bar\chi H_{\bar\chi}^{-1}\bar\chi, \tag{3.5}$$

where we set

$$Q_\chi(H,T) := \chi - \bar\chi H_{\bar\chi}^{-1}\bar\chi W\chi \in \mathcal{B}(\overline{\mathrm{Ran}\,\chi}, \mathcal{H}), \tag{3.6}$$

$$Q_\chi^{\#}(H,T) := \chi - \chi W\bar\chi H_{\bar\chi}^{-1}\bar\chi \in \mathcal{B}(\mathcal{H}, \overline{\mathrm{Ran}\,\chi}). \tag{3.7}$$

(ii) If $\psi \in \ker H\setminus\{0\}$, then $\chi\psi \in \ker F_\chi(H,T)\setminus\{0\}$:

$$F_\chi(H,T)\chi\psi = 0. \tag{3.8}$$

(iii) If $\phi \in \ker F_\chi(H,T)\setminus\{0\}$, then $Q_\chi(H,T)\phi \in \ker H$:

$$HQ_\chi(H,T)\phi = 0. \tag{3.9}$$

Assume, in addition that, T is bounded invertible on $\overline{\mathrm{Ran}\,\bar\chi}$. Then, $\phi \in \overline{\mathrm{Ran}\,\chi}\setminus\{0\}$ and

$$Q_\chi(H,T)\phi \neq 0.$$

4 Wick ordering

In this section, we give the Wick's theorem for fermion. Let $b^+(k)$, $b^-(k)$, $k \in \mathbb{M}$ be the kernels of the fermion creation and annihilation operators, respectively.

For $\mathcal{N} := \{1,\dots,N\}$ and $(\sigma_1,\sigma_2,\dots,\sigma_N) \in \{-1,+1\}^N$, we denote

$$\prod_{j\in\mathcal{N}} b^{\sigma_j}(k_j) := b^{\sigma_1}(k_1)b^{\sigma_2}(k_2)\cdots b^{\sigma_N}(k_N). \tag{4.1}$$

For any subset $\mathcal{I} \subseteq \mathcal{N}$, we denote

$$\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j) := \prod_{j\in\mathcal{N}} \chi(j\in\mathcal{I})b^{\sigma_j}(k_j),$$

where $\chi(j\in\mathcal{I})$ is the characteristic function of $\mathcal{I}$. For $\mathcal{I} \subseteq \mathcal{N}$, we set $\mathcal{I}_\pm := \{j \in \mathcal{I} | \sigma_j = \pm 1\}$. The Wick-ordered product of $\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j)$ is defined by

$$:\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j): := \left(\prod_{j\in\mathcal{I}_+} b^+(k_j)\right)\left(\prod_{j\in\mathcal{I}_-} b^-(k_j)\right).$$

For $(\sigma_1,\dots,\sigma_N) \in \{-1,1\}^N$ and any subset $\mathcal{I} \in \mathcal{N}$, we define

$$\begin{aligned}\mathrm{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-) &:= \begin{pmatrix} 1 & \cdots & N \\ \mathcal{N}\setminus\mathcal{I} & \mathcal{I}_+ & \mathcal{I}_-\end{pmatrix} \\ &:= \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & K & K+1 & \cdots & K+L & K+L+1 & \cdots & N \\ j_1 & j_2 & \cdots & j_K & j_{K+1} & \cdots & j_{K+L} & j_{K+L+1} & \cdots & j_N\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned} \{j_1,j_2,\dots,j_K\} &:= \mathcal{N}\setminus\mathcal{I}, &&\text{with } j_1<j_2<\cdots<j_N,\\ \{j_{K+1},\dots,j_{K+L}\} &:= \mathcal{I}_+, &&\text{with } j_{K+1}<j_{K+2}\cdots<j_{K+L},\\ \{j_{K+L+1},\dots,j_N\} &:= \mathcal{I}_-, &&\text{with } j_{K+L+1}<j_{K+L+2}<\cdots<j_N. \end{aligned}$$

The Wick-ordering of the Fermion product (4.1) is given by the following Theorem:

Theorem 4.1. *For any $(\sigma_1,\ldots,\sigma_N)\in\{+1,-1\}^N$, the formula*

$$\prod_{j\in\mathcal{N}} b^{\sigma_j}(k_j)=\sum_{\mathcal{I}\subseteq\mathcal{N}}\operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega,\prod_{j\in\mathcal{N}\setminus\mathcal{I}} b^{\sigma_j}(k_j)\Omega\right\rangle:\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j): \tag{4.2}$$

holds.

Proof. We prove the theorem by induction with respect to $N\in\mathbb{N}$. For $N=1$, (4.2) is trivial. Assume that (4.2) is true for all products with up to N factors, for some $N\geq 1$, and consider the product of $N+1$-factors. We set $\mathcal{N}+1:=\mathcal{N}\cup\{N+1\}$. For simplicity we write $b_j^{\sigma_j}:=b^{\sigma_j}(k_j)$. In the case $\sigma_{N+1}=-1$, we have

$$\begin{aligned}\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j}&=\prod_{j\in\mathcal{N}} b_j^{\sigma_j}b_{N+1}^-\\&=\sum_{\mathcal{I}\subseteq\mathcal{N}}\operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega,\prod_{j\in\mathcal{N}\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle:\prod_{j\in\mathcal{I}} b_j^{\sigma_j}:b_{N+1}^-\\&=\sum_{\mathcal{I}\subseteq\mathcal{N}}\operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega,\prod_{j\in\mathcal{N}\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle:\prod_{j\in\mathcal{I}} b_j^{\sigma_j}b_{N+1}^-:.\end{aligned}$$

On the other hand, for $\mathcal{I}'\subseteq\mathcal{N}+1$,

$$\operatorname{sgn}((\mathcal{N}+1)\setminus\mathcal{I}';\mathcal{I}'_+;\mathcal{I}'_-)\left\langle\Omega,\prod_{j\in(\mathcal{N}+1)\setminus\mathcal{I}'} b_j^{\sigma_j}\Omega\right\rangle:\prod_{j\in\mathcal{I}'} b_j^{\sigma_j}b_{N+1}^-: \tag{4.3}$$

vanishes if $N+1\in(\mathcal{N}+1)\setminus\mathcal{I}'$. In the case $N+1\in\mathcal{I}'$, we have

$$(4.3)=\operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega,\prod_{j\in\mathcal{N}\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle:\prod_{j\in\mathcal{I}} b_j^{\sigma_j}b_{N+1}^-:,$$

with $\mathcal{I}=\mathcal{I}'\setminus\{N+1\}$, where we use the fact that $\operatorname{sgn}((\mathcal{N}+1)\setminus\mathcal{I}';\mathcal{I}'_+;\mathcal{I}'_-)=\operatorname{sgn}(\mathcal{N}\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)$. Hence, we obtain

$$\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j}=\sum_{\mathcal{I}\subseteq\mathcal{N}+1}\operatorname{sgn}((\mathcal{N}+1)\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega,\prod_{j\in(\mathcal{N}+1)\setminus\mathcal{I}} b^{\sigma_j}(k_j)\Omega\right\rangle:\prod_{j\in\mathcal{I}} b^{\sigma_j}(k_j):.$$

Next we consider the case $\sigma_{N+1}=+1$. By the CAR, we have

$$\{b_i^{\sigma_i},b_j^{\sigma_j}\}=\langle\Omega,b_i^{\sigma_i}b_j^{\sigma_j}\Omega\rangle.$$

By using this relation and the induction hypothesis, we have

$$\begin{aligned}\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j}&=\sum_{k=1}^{N}(-1)^{N-k}\langle\Omega,b_k^{\sigma_k}b_{N+1}^+\Omega\rangle\prod_{j\in\mathcal{N}\setminus\{k\}} b_j^{\sigma_j}+(-1)^N b_{N+1}^+\prod_{j\in\mathcal{N}} b_j^{\sigma_j}\\&=\sum_{k=1}^{N}(-1)^{N-k}\langle\Omega,b_k^{\sigma_k}b_{N+1}^+\Omega\rangle\sum_{\mathcal{I}\subseteq\mathcal{N}\setminus\{k\}}\operatorname{sgn}((\mathcal{N}\setminus\{k\})\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\\&\quad\times\left\langle\Omega,\prod_{j\in(\mathcal{N}\setminus\{k\})\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle:\prod_{j\in\mathcal{I}} b_j^{\sigma_j}:\\&\quad+(-1)^N b_{N+1}^+\prod_{j\in\mathcal{N}} b_j^{\sigma_j}.\end{aligned}$$

We note that

$$\sum_{k=1}^{N}\sum_{\mathcal{I}\subseteq\mathcal{N}\setminus\{k\}} F(k,\mathcal{I}) = \sum_{\mathcal{I}\subseteq\mathcal{N}}\sum_{k\in\mathcal{N}\setminus\mathcal{I}} F(k,\mathcal{I}), \tag{4.4}$$

for any function $F(k,\mathcal{I})$. By using (4.4), we observe

$$\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j} = \sum_{\mathcal{I}\subseteq\mathcal{N}}\sum_{k\in\mathcal{N}\setminus\mathcal{I}} (-1)^{N-k}\langle\Omega, b_k^{\sigma_k} b_{N+1}^{+}\Omega\rangle \operatorname{sgn}((\mathcal{N}\setminus\{k\})\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)$$

$$\times\left\langle\Omega, \prod_{j\in(\mathcal{N}\setminus\{k\})\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}} b_j^{\sigma_j} : \tag{4.5}$$

$$+ (-1)^N b_{N+1}^{+}\prod_{j\in\mathcal{N}} b_j^{\sigma_j}. \tag{4.6}$$

For $\mathcal{I}\subseteq\mathcal{N}\setminus\{k\}$, we set

$$K-1 := |(\mathcal{N}\setminus\{k\})\setminus\mathcal{I}|,$$
$$\{\ell_1,\ldots,\ell_{K-1}\} := (\mathcal{N}\setminus\{k\})\setminus\mathcal{I}, \quad \text{with } \ell_1<\cdots<\ell_{K-1}.$$

Let $\{j_{K+1},\ldots,j_N\}$ be indexes such that

$$j_{K+1}<\cdots<j_N,$$

and

$$:\prod_{j\in\mathcal{I}} b_j^{\sigma_j}: \;=\; \prod_{s=K+1}^{N} b_{j_s}^{\sigma_{j_s}},$$

namely,

$$\left\langle\Omega, \prod_{j\in(\mathcal{N}\setminus\{k\})\setminus\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}} b_j^{\sigma_j} : = \left\langle\Omega, \prod_{j=1}^{K-1} b_{\ell_j}^{\sigma_{\ell_j}}\Omega\right\rangle : \prod_{s=K+1}^{N} b_{j_s}^{\sigma_{j_s}} :. \tag{4.7}$$

The sign in Eq. (4.6) can be written as

$$\operatorname{sgn}((\mathcal{N}\setminus\{k\})\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)$$
$$=\operatorname{sgn}\begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & K-1 & K & K+1 & \cdots & N \\ \ell_1 & \cdots & \ell_{k-1} & k & \ell_k & \cdots & \ell_{K-2} & \ell_{K-1} & j_{K+1} & \cdots & j_N \end{pmatrix}$$

For each fixed $k\in\mathcal{N}\setminus\mathcal{I}$, we set

$$n := \max\{s\in\{1,\ldots,K-1\} | \ell_s<k\}$$

Then we have

$$(-1)^{k-n}\operatorname{sgn}((\mathcal{N}\setminus\{k\})\setminus\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)$$
$$=\operatorname{sgn}\begin{pmatrix} 1\cdots n-1 & n & n+1 & \cdots & k & k+1\cdots & K & K+1\cdots N \\ \ell_1\ldots\ell_{n-1} & k & \ell_n & \cdots & \ell_{k-1} & \ell_k & \ldots\ell_{K-1} & j_{K+1}\cdots j_N \end{pmatrix}. \tag{4.8}$$

Note that

$$\ell_1<\cdots<\ell_{n-1}<k<\ell_n<\cdots<\ell_{K-1}.$$

By changing the names

$$(\ell_1,\ldots,\ell_{n-1},k,\ell_n,\ldots,\ell_{k-1},\ldots,\ell_{K-1}) \to (j_1,\ldots,j_{n-1},j_n,j_{n+1},\ldots,j_k,\ldots,j_{K-1}), \tag{4.9}$$

we obtain that

$$\begin{aligned}\operatorname{sgn}((\mathcal{N}\backslash\{k\})\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-) =&(-1)^{k-n}\operatorname{sgn}\begin{pmatrix}1 & \cdots & N\\ j_1 & \cdots & j_N\end{pmatrix}\\ =&(-1)^{k-n}\operatorname{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-).\end{aligned} \tag{4.10}$$

By (4.7),(4.8), and (4.10), we have

$$\begin{aligned}(4.5) =& \sum_{\mathcal{I}\subseteq\mathcal{N}}\sum_{k\in\mathcal{N}\backslash\mathcal{I}}(-1)^{N-k}(-1)^{k-n}\operatorname{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega, b_k^{\sigma_k}b_{N+1}^+\Omega\right\rangle\left\langle\Omega,\prod_{\substack{l=1\\ l\neq n}}^K b_{j_l}^{\sigma_{j_l}}\Omega\right\rangle : \prod_{l=K+1}^N b_{j_l}^{\sigma_{j_l}} :\\
=& \sum_{\mathcal{I}\subseteq\mathcal{N}}\operatorname{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\sum_{n=1}^K(-1)^{N-n}\left\langle\Omega, b_{j_n}^{\sigma_{j_n}}b_{N+1}^+\Omega\right\rangle\left\langle\Omega,\prod_{\substack{l=1\\ l\neq n}}^K b_{j_l}^{\sigma_{j_l}}\Omega\right\rangle : \prod_{l=K+1}^N b_{j_l}^{\sigma_{j_l}} :\\
=& \sum_{\mathcal{I}\subseteq\mathcal{N}}\operatorname{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)(-1)^N\left\langle\Omega,\prod_{l=1}^K b_{j_l}^{\sigma_{j_l}}b_{N+1}^+\Omega\right\rangle : \prod_{l=K+1}^N b_{j_l}^{\sigma_{j_l}} :\\
=& \sum_{\mathcal{I}\subseteq\mathcal{N}}\operatorname{sgn}((\mathcal{N}+1)\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega,\prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}} b_j^{\sigma_j} :,\end{aligned} \tag{4.11}$$

where we use the equation

$$\begin{aligned}&\sum_{n=1}^K(-1)^{N-n}\left\langle\Omega, b_{j_n}^{\sigma_{j_n}}b_{N+1}^+\Omega\right\rangle\left\langle\Omega,\prod_{\substack{l=1\\ l\neq n}}^K b_{j_l}^{\sigma_{j_l}}\Omega\right\rangle\\
&=\begin{cases}\left\langle\Omega,\prod_{l=1}^K b_{j_l}^{\sigma_{j_l}}b_{N+1}^+\Omega\right\rangle, & K \text{ is odd},\\ 0 & K \text{ is even}.\end{cases}\end{aligned}$$

Similarly, we have

$$(4.6) = \sum_{\mathcal{I}\subseteq\mathcal{N}}\operatorname{sgn}((\mathcal{N}+1)\backslash\mathcal{I}';\mathcal{I}'_+;\mathcal{I}'_-)\left\langle\Omega,\prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}'} b_j^{\sigma_j} :, \tag{4.12}$$

where $\mathcal{I}' := \mathcal{I}\cup\{N+1\}$. By (4.11), (4.12), we obtain the desired result:

$$\prod_{j\in\mathcal{N}+1} b_j^{\sigma_j} = \sum_{\mathcal{I}\subseteq\mathcal{N}+1}\operatorname{sgn}(\mathcal{N}\backslash\mathcal{I};\mathcal{I}_+;\mathcal{I}_-)\left\langle\Omega,\prod_{j\in(\mathcal{N}+1)\backslash\mathcal{I}} b_j^{\sigma_j}\Omega\right\rangle : \prod_{j\in\mathcal{I}} b_j^{\sigma_j} :$$

□

Lemma 4.2. *Let $f_j[r] : \mathbf{M}\to\mathbb{R}_+$, $j = 1,\ldots,N$ be Borel measurable functions. Then*

$$\begin{aligned}&\prod_{j=1}^N\{b^{\sigma_j}(k_j)f_j[H_f]\}\\
&=\sum_{\mathcal{I}\subset\mathcal{N}}\operatorname{sgn}(\mathcal{N}\backslash\mathcal{I}, :\mathcal{I}:)\prod_{j\in I_+} b^+(k_j)\\
&\times\left\langle\Omega,\prod_{j=1}^N\left\{[b^{\sigma_j}(k_j)]^{\chi[j\notin\mathcal{I}]}f_j\left[H_f + r + \sum_{\substack{i=1\\ i\in\mathcal{I}_-}}^j\omega(k_i) + \sum_{\substack{i=j+1\\ i\in\mathcal{I}_+}}^N\omega(k_i)\right]\right\}\Omega\right\rangle\Bigg|_{r=H_f}\\
&\times\prod_{j\in\mathcal{I}_-} b^-(k_j),\end{aligned}$$

where $[b^{\sigma_j(k_j)}]^{\chi[j\notin\mathcal{I}]} = b^{\sigma_j}(k_j)$ for $j\notin\mathcal{I}$ and $[b^{\sigma_j}(k_j)]^{\chi[j\notin\mathcal{I}]} = 1$ for $j\in\mathcal{I}$.

Proof. Similar to the proof of [1, Lemma A.3]. □

Let

$$w_{m,n} : (\mathbb{R}_+) \times \mathbf{M}^m \times \mathbf{M}^n \to \mathbb{C}, \quad m,n \in \mathbb{N}_0, \tag{4.13}$$

be measurable functions. In the following, we use the notations

$$k^{(m)} := (k_1, \dots, k_m) \in \mathbf{M}^m, \quad \tilde{k}^{(n)} := (\tilde{k}_1, \dots, \tilde{k}_n) \in \mathbf{M}^n.$$

We assume that each function $w_{m,n}[r; k^{(m)}; \tilde{k}^{(n)}]$ is antisymmetric with respect to $k^{(m)} \in \mathbf{M}^m$, $\tilde{k}^{(n)} \in \mathbf{M}^n$, respectively, i.e.,

$$\begin{aligned} w_{m,n}[r; k^{(m)}; \tilde{k}^{(n)}] =& \{w_{m,n}[r; k^{(m)}; \tilde{k}^{(n)}]\}^{\text{asym}}_{m,n} \\ :=& \frac{1}{m!n!} \sum_{\pi \in S_m} \sum_{\tilde{\pi} \in S_n} \text{sgn}(\pi)\text{sgn}(\tilde{\pi}) w_{m,n}[r; k^{(m)}_\pi; \tilde{k}^{(n)}_{\tilde{\pi}}], \end{aligned} \tag{4.14}$$

where

$$k^{(m)}_\pi := (k_{\pi(1)}, \dots, k_{\pi(m)}), \quad \tilde{k}^{(n)}_{\tilde{\pi}} := (\tilde{k}_{\tilde{\pi}(1)}, \dots, \tilde{k}_{\tilde{\pi}(n)}).$$

For $L \in \mathbb{N}_0$, we consider the operator

$$f_0[H_f] W_{M_1,N_1} f_1[H_f] W_{M_2,N_2} \cdots f_{L-1}[H_f] W_{M_L,N_L} f_L[H_f], \tag{4.15}$$

where the operators $W_{m,n}$ is given by

$$\begin{aligned} W_{m,n} &\equiv W_{m,n}[w_{m,n}] \\ &= \int_{\mathbf{M}^{m+n}} dK^{(m,n)} b^*(k^{(m)}) w_{m,n}[H_f; K^{(m,n)}] b(\tilde{k}^{(n)}) \end{aligned} \tag{4.16}$$

We set

$$\begin{aligned} K &:= M + N, \\ M &:= \sum_{\ell=1}^{L} M_\ell, \quad N := \sum_{\ell=1}^{L} N_\ell. \end{aligned} \tag{4.17}$$

Corresponding to (4.17), we set

$$\begin{aligned} k^{(M)} :=& (k^{(M_\ell)}_\ell)_{\ell=1}^L \in \mathbf{M}^{M_1} \times \cdots \times \mathbf{M}^{M_L} \\ =& (k_{1,1}, \dots, k_{1,M_1}; k_{2,1}, \dots, k_{2,M_2}; \cdots ; k_{L,1}, \dots, k_{L,M_L}), \\ \tilde{k}^{(N)} :=& (\tilde{k}^{(N_\ell)}_\ell)_{\ell=1}^L \in \mathbf{M}^{N_1} \times \cdots \times \mathbf{M}^{N_L} \\ =& (\tilde{k}_{1,1}, \dots, \tilde{k}_{1,N_1}; \tilde{k}_{2,1}, \dots, \tilde{k}_{2,N_2}; \cdots ; \tilde{k}_{L,1}, \dots, \tilde{k}_{L,N_L}) \end{aligned}$$

We define

$$\begin{aligned} \mathcal{K} :=& \{1, \dots, K\}, \\ \mathcal{K}_{M,\ell} :=& \left\{ \sum_{j=1}^{\ell-1} (M_j + N_j) + 1, \dots, \sum_{j=1}^{\ell-1} (M_j + N_j) + M_\ell \right\}, \\ \mathcal{K}_{N,\ell} :=& \left\{ \sum_{j=1}^{\ell-1} (M_j + N_j) + M_j + 1, \dots, \sum_{j=1}^{\ell} (N_j + M_j) \right\}, \quad \ell = 1, \dots, L. \end{aligned}$$

Clearly,

$$\mathcal{K} = \bigcup_{\ell=1}^{L} \bigcup_{\mu=M,N} \mathcal{K}_{\mu,\ell} = \{\mathcal{K}_{M,1}, \mathcal{K}_{N,1}, \mathcal{K}_{M,2}, \mathcal{K}_{N,2}, \cdots, \mathcal{K}_{M,L}, \mathcal{K}_{N,L}\}.$$

For $m, n, p, q \in \mathbb{N}_0$ with $m+n+p+q \geq 1$, we define

$$W_{p,q}^{m,n}[r; k^{(m)}; \tilde{k}^{(n)}] := \int_{\mathbf{M}^{p+q}} dx^{(p)} d\tilde{x}^{(q)} b^+(x^{(p)}) w_{m+p,n+q}[r; k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)}] b^-(\tilde{x}^{(q)}).$$

The Wick ordering formula for the operator (4.15) is given by the following result:

Theorem 4.3. *Let $L \in \mathbb{N}$ be a number. Suppose that $M_\ell \in \mathbb{N}_0$, $N_\ell \in \mathbb{N}_0$ are numbers such that $M_\ell + N_\ell \geq 1$. Let $\{w_{M_\ell,N_\ell}\}_{\ell=1}^{L}$ be functions defined in (4.13). Then,*

$$\begin{aligned}
& f_0[H_f] W_{M_1,N_1} f_1[H_f] W_{M_2,N_2} \cdots f_{L-1}[H_f] W_{M_L,N_L} f_L[H_f] \\
&= \sum_{\substack{\mathcal{I}_{M,\ell} \subseteq \mathcal{K}_{M,\ell} \\ \ell=1,\ldots,L}} \sum_{\substack{\mathcal{I}_{N,\ell} \subseteq \mathcal{K}_{N,\ell} \\ \ell=1,\ldots,L}} \operatorname{sgn}(\mathcal{K}\backslash\mathcal{I}, : \mathcal{I} :) \prod_{\ell=1}^{L} \operatorname{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell} \\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell} \end{pmatrix} \\
&\times \operatorname{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell} \\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell} \end{pmatrix} \int_{\mathbf{M}^{m+n}} \prod_{\ell=1}^{L} \left\{ dk_\ell^{(m_\ell)} d\tilde{k}_\ell^{(n_\ell)} \right\} \prod_{\ell=1}^{L} b^+(k_\ell^{(m_\ell)}) \\
&\times \left\{ D_L[H_f; \{W_{M_\ell - m_\ell, N_\ell - n_\ell}^{m_\ell,n_\ell}; k_\ell^{(m_\ell)}; \tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L}; \{f_\ell\}_{\ell=0}^{L}] \right\}_{m,n}^{\mathrm{asym}} \prod_{\ell=1}^{L} b^-(\tilde{k}_\ell^{(n_\ell)}),
\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}
& D_L[r; \{W_{M_\ell - m_\ell, N_\ell - n_\ell}^{m_\ell,n_\ell}; k_\ell^{(m_\ell)}; \tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L}; \{f_\ell\}_{\ell=1}^{L}] \\
&:= f_0[r + \tilde{r}_0] \Big\langle \Omega, \Big\{ \prod_{\ell=1}^{L-1} W_{M_\ell - m_\ell, N_\ell - n_\ell}^{m_\ell,n_\ell}[H_f + r + r_\ell; k_\ell^{(m_\ell)}; \tilde{k}_\ell^{(n_\ell)}] f_\ell[H_f + r + \tilde{r}_\ell] \Big\} \\
&\times W_{M_L - m_L, N_L - n_L}^{m_L,n_L}[r + r_L; k_L^{(m_L)}; \tilde{k}_L^{(n_L)}] \Omega \Big\rangle f_L[r + \tilde{r}_L],
\end{aligned}$$

and

$$\operatorname{sgn}(\mathcal{K}\backslash\mathcal{I}, : \mathcal{I} :) := \operatorname{sgn}\begin{pmatrix} & \mathcal{K} & \\ \mathcal{K}\backslash\mathcal{I} & \bigcup_{\ell=1}^{L} \mathcal{I}_{M,\ell} & \bigcup_{\ell=1}^{L} \mathcal{I}_{N,\ell} \end{pmatrix} \tag{4.19}$$

$$r_\ell := \sum_{l=1}^{\ell-1} \Sigma[\tilde{k}_l^{(n_l)}] + \sum_{l=\ell+1}^{L} \Sigma[k_l^{(m_l)}], \quad \ell = 2, 3, \ldots, L-1, \tag{4.20}$$

$$r_0 := \sum_{l=1}^{L} \Sigma[k_l^{(m_l)}], \quad r_1 := \sum_{l=2}^{L} \Sigma[k_l^{(m_l)}], \quad r_L := \sum_{l=1}^{L-1} \Sigma[\tilde{k}_l^{(n_l)}], \tag{4.21}$$

$$\tilde{r}_\ell := \sum_{l=1}^{\ell} \Sigma[\tilde{k}_l^{(n_l)}] + \sum_{l=\ell+1}^{L} \Sigma[k_l^{(m_l)}], \quad \ell = 1, \ldots, L-1. \tag{4.22}$$

$$\tilde{r}_0 := \sum_{l=1}^{L} \Sigma[k_l^{(m_l)}], \quad \tilde{r}_L := \sum_{l=1}^{L} \Sigma[\tilde{k}_l^{(n_l)}], \tag{4.23}$$

$$m_\ell := |\mathcal{I}_{M,\ell}|, \quad n_\ell := |\mathcal{I}_{N,\ell}|, \quad m := \sum_{\ell=1}^{L} m_\ell, \quad n := \sum_{\ell=1}^{L} n_\ell. \tag{4.24}$$

(4.25)

Here, $\Sigma[\kappa^{(n)}] := \sum_{j=1}^{n} \omega(\kappa_j)$, $(\kappa = k_l, \tilde{k}_l)$.

Proof. By the definition of W_{M_ℓ,N_ℓ}, we have

$$
\begin{aligned}
(\text{L.H.S. of } (4.18)) = &\int_{\mathbf{M}^K} \prod_{\ell=1}^{L} \left\{ \prod_{j=1}^{M_\ell} dk_{\ell,j} \prod_{j=1}^{N_\ell} d\tilde{k}_{\ell,j} \right\} f_0[H_f] \\
&\times b^+(k_1^{(M_1)}) w_{M_1,N_1}[H_f; k_1^{(M_1)}; \tilde{k}_1^{(N_1)}] b^-(\tilde{k}_1^{(N_1)}) f_1[H_f] \\
&\times b^+(k_2^{(M_2)}) w_{M_2,N_2}[H_f; k_2^{(M_2)}; \tilde{k}_2^{(N_2)}] b^-(\tilde{k}_2^{(N_2)}) f_2[H_f] \\
&\times \cdots \\
&\times b^+(k_{L-1}^{(M_{L-1})}) w_{M_{L-1},N_{L-1}}[H_f; k_{L-1}^{(M_{L-1})}; \tilde{k}_{L-1}^{(N_{L-1})}] b^-(\tilde{k}_{L-1}^{(N_{L-1})}) f_{L-1}[H_f] \\
&\times b^+(k_L^{(M_L)}) w_{M_L,N_L}[H_f; k_L^{(M_L)}; \tilde{k}_L^{(N_L)}] b(\tilde{k}_L^{(N_L)}) f_L[H_f].
\end{aligned}
$$

By using Lemma (4.2), we have

$$
\begin{aligned}
&(\text{L.H.S. of } (4.18)) \\
&= \int_{\mathbf{M}^K} \prod_{\ell=1}^{L} \left\{ \prod_{j=1}^{M_\ell} dk_{\ell,j} \prod_{j=1}^{N_\ell} d\tilde{k}_{\ell,j} \right\} \sum_{\substack{\mathcal{I}_{M_\ell} \subseteq \mathcal{K}_{M,\ell} \\ \ell=1,\ldots,L}} \sum_{\substack{\mathcal{I}_{N_\ell} \subseteq \mathcal{K}_{N,\ell} \\ \ell=1,\ldots,L}} \operatorname{sgn}(\mathcal{K}\backslash\mathcal{I}, : \mathcal{I} :) \left[\prod_{\ell=1}^{L} \prod_{j\in\mathcal{I}_{M,\ell}} b^+(k_{\ell,j}) \right] \\
&\times f_0[r+\Lambda_0] \Bigg\langle \Omega, \Bigg\{ \prod_{\ell=1}^{L-1} \Bigg(\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}} b^+(k_{\ell,j}) \Bigg) w_{M_\ell,N_\ell}\Big[H_f + r + \Lambda_\ell; k_\ell^{(M_\ell)}; \tilde{k}_\ell^{(N_\ell)}\Big] \\
&\times \Bigg(\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}} b^-(\tilde{k}_{\ell,j}) \Bigg) f_\ell\Big[H_f + r + \Lambda_\ell + \sum_{j\in\mathcal{I}_{N,\ell}} \omega(\tilde{k}_{\ell,j})\Big] \Bigg\} \\
&\times \Bigg(\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}} b^+(k_{L,j}) \Bigg) w_{M_L,N_L}\Big[H_f + r + \Lambda_L; k_L^{(M_L)}; \tilde{k}_L^{(N_L)}\Big] \Bigg(\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}} b^-(\tilde{k}_{L,j}) \Bigg) \Omega \Bigg\rangle \Bigg|_{r=H_f} \\
&\times f_L\Bigg[r + \Lambda_L + \sum_{j\in\mathcal{I}_{N,L}} \omega(\tilde{k}_{L,j}) \Bigg] \Bigg[\prod_{\ell=1}^{L} \prod_{j\in\mathcal{I}_{N,\ell}} b^-(k_{\ell,j}) \Bigg]
\end{aligned}
\tag{4.26}
$$

where

$$
\Lambda_\ell := \sum_{l=1}^{\ell-1} \sum_{j\in\mathcal{I}_{N,l}} \omega(\tilde{k}_{l,j}) + \sum_{l=\ell+1}^{L} \sum_{j\in\mathcal{I}_{M,l}} \omega(k_{l,j}), \quad \ell = 2,3,\ldots,L-1,
$$

$$
\Lambda_0 := \sum_{l=1}^{L} \sum_{j\in\mathcal{I}_{M,l}} \omega(k_{l,j}), \quad \Lambda_1 := \sum_{l=2}^{L} \sum_{j\in\mathcal{I}_{M,l}} \omega(k_{l,j}), \quad \Lambda_L := \sum_{l=1}^{L-1} \sum_{j\in\mathcal{I}_{M,l}} \omega(\tilde{k}_{l,j}).
$$

Next, we move the integral in the variables $\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}$, $\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}$ to the inside of the inner product $\langle\Omega, \cdots \Omega\rangle$:

$$
\begin{aligned}
&(\text{L.H.S. of } (4.18)) \\
&= \sum_{\substack{\mathcal{I}_{M_\ell} \subseteq \mathcal{K}_{M,\ell} \\ \ell=1,\ldots,L}} \sum_{\substack{\mathcal{I}_{N_\ell} \subseteq \mathcal{K}_{N,\ell} \\ \ell=1,\ldots,L}} \operatorname{sgn}(\mathcal{K}\backslash\mathcal{I}, : \mathcal{I} :) \int_{\mathbf{M}^{m+n}} \prod_{\ell=1}^{L} \left\{ \prod_{j\in\mathcal{I}_{M,\ell}} dk_{\ell,j} \prod_{j\in\mathcal{I}_{N_\ell}} d\tilde{k}_{\ell,j} \right\} \\
&\times \left[\prod_{\ell=1}^{L} \prod_{j\in\mathcal{I}_{M,\ell}} b^+(k_{\ell,j}) \right] G\Big[r; \Big\{ \{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}}, \{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}} \Big\}_{\ell=1}^{L} \Big] \Bigg|_{r=H_f} \\
&\times \left[\prod_{\ell=1}^{L} \prod_{j\in\mathcal{I}_{N,\ell}} b^-(\tilde{k}_{\ell,j}) \right],
\end{aligned}
\tag{4.27}
$$

where

$$G\left[r;\left\{\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}},\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\right\}_{\ell=1}^{L}\right]$$

$$= f_0[r+\Lambda_0]\Big\langle\Omega,\Big\{\prod_{\ell=1}^{L-1}\int\Big[\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}} dk_{\ell,j}\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}} d\tilde{k}_{\ell,j}\Big]$$

$$\times\Big(\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}} b^+(k_{\ell,j})\Big)w_{M_\ell,N_\ell}\Big[H_f+r+\Lambda_\ell;k_\ell^{(M_\ell)};\tilde{k}_\ell^{(N_\ell)}\Big]\Big(\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}} b^-(\tilde{k}_{\ell,j})\Big)$$

$$\times f_\ell\Big[H_f+r+\Lambda_\ell+\sum_{j\in\mathcal{I}_{N,\ell}}\omega(\tilde{k}_{\ell,j})\Big]\Big\}$$

$$\times\int\Big[\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}} dk_{L,j}\prod_{j\in\mathcal{K}_{N,L}\backslash\mathcal{I}_{N,L}} d\tilde{k}_{L,j}\Big]$$

$$\times\Big(\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}} b^+(k_{L,j})\Big)w_{M_L,N_L}\Big[H_f+r+\Lambda_L;k_L^{(M_L)};\tilde{k}_L^{(N_L)}\Big]\Big(\prod_{j\in\mathcal{K}_{M,L}\backslash\mathcal{I}_{M,L}} b^-(\tilde{k}_{L,j})\Big)\Omega\Big\rangle$$

$$\times f_L\Big[r+\Lambda_L+\sum_{j\in\mathcal{I}_{N,L}}\omega(\tilde{k}_{L,j})\Big]$$

Here we used the fact that Λ_ℓ, $\ell=1,\dots,L$ and $\sum_{j\in\mathcal{I}_{N,\ell}}\omega(\tilde{k}_{\ell,j})$ are independent of $k_{\ell,j}(j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell})$, $\tilde{k}_{\ell,j}(j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell})$. We rename the variables in (4.26) as follows

$$k_{\ell,j}\to x_{\ell,j},\quad j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell},$$
$$\tilde{k}_{\ell,j}\to\tilde{x}_{\ell,j},\quad j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}.$$

Then we have

$$w_{M_\ell,N_\ell}[r;k_\ell^{(M_\ell)};\tilde{k}_\ell^{(N_\ell)}]\Big|_{\substack{k_{\ell,j}=x_{\ell,j},\,j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\\ \tilde{k}_{\ell,j}=\tilde{x}_{\ell,j},\,j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}}$$

$$=\operatorname{sgn}\begin{pmatrix}\mathcal{K}_{M,\ell}\\ \mathcal{I}_{M,\ell}\ \ \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\operatorname{sgn}\begin{pmatrix}\mathcal{K}_{N,\ell}\\ \mathcal{I}_{N,\ell}\ \ \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}$$

$$\times\, w_{M_\ell,N_\ell}\Big[r;\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}},\{x_{\ell,j}\}_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}}\Big|\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}},\{\tilde{x}_{\ell,j}\}_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}}\Big],$$

and

$$\int\Big[\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}} dk_{\ell,j}\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}} d\tilde{k}_{\ell,j}\Big]\Big(\prod_{j\in\mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}} b^+(k_{\ell,j})\Big)$$

$$\times\, w_{M_\ell,N_\ell}\Big[H_f+r+\Lambda_\ell;k_\ell^{(M_\ell)};\tilde{k}_\ell^{(N_\ell)}\Big]\Big(\prod_{j\in\mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}} b^-(\tilde{k}_{\ell,j})\Big)$$

$$=\operatorname{sgn}\begin{pmatrix}\mathcal{K}_{M,\ell}\\ \mathcal{I}_{M,\ell}\ \ \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\operatorname{sgn}\begin{pmatrix}\mathcal{K}_{N,\ell}\\ \mathcal{I}_{N,\ell}\ \ \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}$$

$$\times\, W^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell}\Big[H_f+r+\Lambda_\ell;\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}};\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\Big],$$

where

$$m_\ell:=|\mathcal{I}_{M,\ell}|,\quad |n_\ell|:=|\mathcal{I}_{N,\ell}|,\quad \ell=1,\dots,L.$$

Hence we have

$$
\begin{aligned}
&G\Big[r;\Big\{\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}},\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\Big\}_{\ell=1}^{L}\Big]\\
&=\left[\prod_{\ell=1}^{L}\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell}\\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell}\\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}\right]f_0[r+\Lambda_0]\\
&\times\Big\langle\Omega,\prod_{\ell=1}^{L-1}\Big[W^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell}\Big[H_f+r+\Lambda_\ell;\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}};\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\Big]\\
&\times f_\ell\Big[H_f+r+\Lambda_\ell+\sum_{j\in\mathcal{I}_{N,\ell}}\omega(\tilde{k}_{\ell,j})\Big]\Big]W^{m_L,n_L}_{M_L-m_L,N_L-n_L}\Big[r+\Lambda_L;\{k_{L,j}\}_{j\in\mathcal{I}_{M,L}};\{\tilde{k}_{L,j}\}_{j\in\mathcal{I}_{N,L}}\Big]\Omega\Big\rangle\\
&\times f_L\left[r+\Lambda_L+\sum_{j\in\mathcal{I}_{N,L}}\omega(\tilde{k}_{L,j})\right].
\end{aligned}
\tag{4.28}
$$

By changing the names of the variables $\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}}$, $\{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}$ in (4.27) with (4.28):

$$\{k_{\ell,j}\}_{j\in\mathcal{I}_{M,\ell}}\to k_\ell^{(m_\ell)},\quad \{\tilde{k}_{\ell,j}\}_{j\in\mathcal{I}_{N,\ell}}\to \tilde{k}_\ell^{(n_\ell)},$$

we have

$$
\begin{aligned}
(\text{L.H.S. of (4.18)}) = &\sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell}\\ \ell=1,\dots,L}}\ \sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell}\\ \ell=1,\dots,L}}\mathrm{sgn}(\mathcal{K}\backslash\mathcal{I},:\mathcal{I}:)\left[\prod_{\ell=1}^{L}\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell}\\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\right.\\
&\left.\times\,\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell}\\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}\right]\int_{\mathbf{M}^{m+n}}\prod_{\ell=1}^{L}\Big\{dk_\ell^{(m_\ell)}d\tilde{k}_\ell^{(n_\ell)}\Big\}\prod_{\ell=1}^{L}b^+(k_\ell^{(m_\ell)})\\
&\times D_L\Big[H_f;\{W^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell};k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L};\{f_\ell\}_{\ell=1}^{L-1}\Big]\prod_{\ell=1}^{L}b^-(\tilde{k}^{(n)}).
\end{aligned}
$$

Finally, by using this fact and the anticommutativity of b^-, b^+, we obtain the formula (4.18). □

We set

$$W:=\sum_{N+M\geq 1}W_{M,N}.$$

Theorem 4.4. *Let W be a operator defined above. We write as*

$$f_0Wf_1W\cdots Wf_L=H[\underline{\tilde{w}}],\tag{4.29}$$

where $\underline{\tilde{w}}=(\tilde{w}_{m,n})_{m+n\geq 0}$. *Then*

$$
\begin{aligned}
\tilde{w}_{m,n}(r;K^{(m,n)}) = &\sum_{\substack{m_1+\cdots+m_L=m\\ n_1+\cdots+n_L=n}}\ \sum_{\substack{p_\ell,q_\ell\geq 0\\ m_\ell+p_\ell+n_\ell+q_\ell\geq 1\\ \ell=1,\dots,L}}\mathrm{sgn}(\{m_\ell\}_{\ell=1}^{L};\{n_\ell\}_{\ell=1}^{L})\\
&\int_{\mathbf{M}^{m+n}}\prod_{\ell=1}^{L}\Big\{dk_\ell^{(m_\ell)}d\tilde{k}_\ell^{(n_\ell)}\Big\}\prod_{\ell=1}^{L}b^+(k_\ell^{(m_\ell)})\\
&\times\Big\{D_L[H_f;\{W^{m_\ell,n_\ell}_{p_\ell,q_\ell};k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L};\{f_\ell\}_{\ell=0}^{L}]\Big\}^{\mathrm{asym}}_{m,n}\prod_{\ell=1}^{L}b^-(\tilde{k}_\ell^{(n_\ell)}),
\end{aligned}
$$

where $D_L[\cdots]$ *is the function defined in Theorem 4.3,*

$$\operatorname{sgn}(\{m_\ell\}_{\ell=1}^L;\{n_\ell\}_{\ell=1}^L) := \sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell}\\ m_\ell=|\mathcal{I}_{M,\ell}|\\ \ell=1,\dots,L}} \sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell}\\ n_\ell=|\mathcal{I}_{N,\ell}|\\ \ell=1,\dots,L}} \operatorname{sgn}(\mathcal{K}\backslash\mathcal{I},:\mathcal{I}:)$$
$$\times\prod_{\ell=1}^L \operatorname{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell} \\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix} \operatorname{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell} \\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix}, \tag{4.30}$$

and $\operatorname{sgn}(\mathcal{K}\backslash\mathcal{I},:\mathcal{I}:)$ *is a constant defined in Theorem 4.3.*

Proof. Note that

$$(\text{L. H. S. of (4.29)}) = \sum_{M_1+N_1\geq 1}\cdots\sum_{N_L+M_L\geq 1}(4.18). \tag{4.31}$$

It is easy to see that, for all $\ell = 1,\dots,L$,

$$\sum_{M_\ell+N_\ell\geq 1}\sum_{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell}}\sum_{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell}} = \sum_{M_\ell+N_\ell\geq 1}\sum_{m_\ell=0}^{M_\ell}\sum_{n_\ell=0}^{N_\ell}\sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell}\\ |\mathcal{I}_{M,\ell}|=m_\ell}}\sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell}\\ |\mathcal{I}_{N,\ell}|=n_\ell}}. \tag{4.32}$$

Furthermore, for any function $X[\cdots]$, we have

$$\sum_{M_\ell+N_\ell\geq 1}\sum_{m=0}^{M_\ell}\sum_{n=0}^{N_\ell} X(M_\ell,N_\ell,m_\ell,n_\ell) = \sum_{\substack{(M_\ell,N_\ell,m_\ell,n_\ell)\in\mathbb{N}_0^4\\ M_\ell\geq m_\ell\geq 0;\ N_\ell\geq n_\ell\geq 0\\ M_\ell+N_\ell\geq 1}} X(M_\ell,N_\ell,m_\ell,n_\ell)$$
$$= \sum_{\substack{(p_\ell,q_\ell,m_\ell,n_\ell)\in\mathbb{N}_0^4\\ p_\ell+q_\ell+m_\ell+n_\ell\geq 1}} X(m_\ell+p_\ell,n_\ell+q_\ell,m_\ell,n_\ell). \tag{4.33}$$

By connecting (4.31)-(4.33) with Theorem 4.3, one can obtain the desired result. □

5 Sketch of proof

We hereafter assume Hypotheses 1-2. By using the smooth Feshbach map, we eliminate the degree of high energy fermion, and restrict the degree of the system S to the normalized eigenvector φ_S. Let

$$\chi := P\otimes\sin\left[\frac{\pi}{2}\Xi(H_f)\right], \tag{5.1}$$

where P is the orthogonal projection onto the eigenspace $\ker(H_S-E)$ and the function $\Xi:\mathbb{R}\to[0,1]$ is smooth in $(0,1)$ and obeys

$$\Xi(r) = \begin{cases}1 & (0\leq r<\frac{3}{4}),\\ 0 & (r<0,\tau\leq r),\end{cases} \tag{5.2}$$

where $3/4<\tau<1$. Then we have

$$\bar\chi := \sqrt{1-\chi^2} = P\otimes\cos\left[\frac{\pi}{2}\Xi(H_f)\right] + P^\perp\otimes\mathbf{1}. \tag{5.3}$$

Let

$$T[z] := H_0(-i\vartheta/\nu) - E - z \tag{5.4}$$

and

$$W := H[z]-T[z] = W_g(-i\vartheta/\nu). \tag{5.5}$$

It is evident that $T[z]$ is closed, commuting with χ. Furthermore, we have the following lemma.

Lemma 5.1. *$T[z]$ is bounded invertible on* $\operatorname{Ran}\bar{\chi}$ *for all z with*

$$|z| < \min\{3/4, \sin(\vartheta/\nu)\}.$$

Proof. Let us first note that the orthogonal projection $P_{\bar{\chi}}$ onto $\overline{\operatorname{Ran}\bar{\chi}}$ is of the following form

$$P_{\bar{\chi}} = P \otimes \mathbf{1}_{[H_f > \frac{3}{4}]} + P^{\perp} \otimes \mathbf{1}, \tag{5.6}$$

and hence

$$P_{\bar{\chi}} T[z] P_{\bar{\chi}} = L_1 + L_2, \tag{5.7}$$

where the function $\mathbf{1}_A$ is the indicator of a set A and

$$L_1 = P \otimes \mathbf{1}_{[H_f > \frac{3}{4}]} \left(e^{-i\vartheta} H_f - z\right) \mathbf{1}_{[H_f > \frac{3}{4}]}, \tag{5.8}$$

$$L_2 = P^{\perp} (H_S - E) P^{\perp} \otimes \mathbf{1} + P^{\perp} \otimes \left(e^{-i\vartheta} H_f - z\right). \tag{5.9}$$

We need only to prove L_1 and L_2 are bounded invertible, i.e., $z \in \operatorname{Res}(L_1) \cap \operatorname{Res}(L_2)$, since, by (5.7), (5.8) and (5.9), $P_{\bar{\chi}} T[z] P_{\bar{\chi}}$ is reduced by $\operatorname{Ran} P \otimes \mathbf{1}_{[H_f > \frac{3}{4}]}$ and $\operatorname{Ran} P^{\perp} \otimes 1$. Indeed, we observe $z \in \operatorname{Res}(L_1)$ and $z \in \operatorname{Res}(L_2)$ provided $|z| < 3/4$ and $|z| < \sin(\vartheta/\nu)$, respectively. □

Let $T^{-1}[z]$ be the inverse of $P_{\bar{\chi}} T[z] P_{\bar{\chi}}$ for all z with $|z| < \rho_0$:

$$T^{-1}[z] := \left(P_{\bar{\chi}} T[z] P_{\bar{\chi}}\right)^{-1}, \tag{5.10}$$

where we set

$$\rho_0 := \min\left\{\frac{3}{4}, \sin(\vartheta/\nu)\right\}. \tag{5.11}$$

Then, we have, for all z with $|z| < \rho_0/2$,

$$\operatorname{Res}(P_{\bar{\chi}} T[z] P_{\bar{\chi}}) \supset D_{\rho_0/2}, \tag{5.12}$$

where

$$D_\epsilon := \{z \in \mathbb{C} \mid |z| \le \epsilon\} \tag{5.13}$$

for all $\epsilon > 0$. Let

$$H_{\bar{\chi}}[z] := T[z] + \bar{\chi} W \bar{\chi}. \tag{5.14}$$

We have the following lemma.

Lemma 5.2. *For all $z \in D_{\rho_0/2}$, $\langle H[z], T[z], \chi\rangle$ is a Feshbach triple and*

$$F_\chi(H[z], T[z]) = T[z] + \sum_{L=1}^{\infty} (-1)^{L-1} \chi W \left(\bar{\chi} T^{-1}[z] \bar{\chi} W\right)^{L-1} \chi. \tag{5.15}$$

Proof. By Hypothesis 2, we have

$$\begin{aligned} \|W\bar{\chi}T^{-1}[z]\bar{\chi}\Psi\| &\le a_g(-i\vartheta/\nu)\,\|H_0(-i\vartheta/\nu)\,\bar{\chi}T^{-1}[z]\bar{\chi}\Psi\| + b_g(-i\vartheta/\nu)\,\|\bar{\chi}T^{-1}[z]\bar{\chi}\Psi\| \\ &\le \left\{a_g(-i\vartheta/\nu) + (a_g(-i\vartheta/\nu)|E+z| + b_g(-i\vartheta/\nu))\,\|T^{-1}[z]\|\right\}\|\bar{\chi}\Psi\|, \end{aligned} \tag{5.16}$$

where $a_g(-i\vartheta/\nu)$ and $b_g(-i\vartheta/\nu)$ are defined by (2.27). Since, for $g \in \mathbb{R}$ with $|g|$ sufficiently small,

$$2a_g(-i\vartheta/\nu) + \frac{2}{\rho_0}\left(|E| a_g(-i\vartheta/\nu) + b_g(-i\vartheta/\nu)\right) < 1, \tag{5.17}$$

we observe that

$$\sup_{z \in D_{\rho_0/2}} \|W\bar{\chi}T^{-1}[z]\|_{\mathcal{B}(\operatorname{Ran}\bar{\chi};\mathcal{F})} < 1, \tag{5.18}$$

which implies that $H_{\bar{\chi}}[z]$ is bounded invertible on Ran $\bar{\chi}$ and that the Neumann series expansion of the inverse

$$H_{\bar{\chi}}^{-1}[z] = \sum_{L=0}^{\infty}(-1)^L T^{-1}[z]\left(\bar{\chi}W\bar{\chi}T^{-1}[z]\right)^L \tag{5.19}$$

is norm convergent. It is easy to see, from (5.19) and Hypothesis 2, that $\langle H[z], T[z], \chi\rangle$ is a Feshbach triple. By the definition of the Feshbach map (3.2) and the equation (5.19), we obtain

$$\begin{aligned} F_\chi(H[z],T[z]) &= T[z] + \chi W\chi - \chi W\bar{\chi}H_{\bar{\chi}}^{-1}[z]\bar{\chi}W\chi \\ &= T[z] + \chi W\chi + \sum_{L=0}^{\infty}(-1)^{L+1}\chi W\bar{\chi}T^{-1}[z]\left(\bar{\chi}W\bar{\chi}T^{-1}[z]\right)^L\bar{\chi}W\chi \\ &= T[z] + \chi W\chi + \sum_{L=0}^{\infty}(-1)^{L+1}\chi W\left(\bar{\chi}T^{-1}[z]\bar{\chi}W\right)^{L+1}\chi, \end{aligned}$$

which is equivalent to (5.15). □

Let P_χ be the orthogonal projection onto Ran χ:

$$P_\chi = P\otimes \mathbf{1}_{[H_f<\tau]}, \tag{5.20}$$

where the constant $3/4 < \tau < 1$ is defined in (5.2). According to Theorem 3.2 (iii), we need only to consider the spectrum of $P_\chi F_\chi(H[z],T[z])P_\chi$ since $T^{-1}[z]$ is bounded invertible on Ran $\bar{\chi}$ with $z\in D_{\rho_0/2}$. We note that the operator $H_{(0)}[z]$ on Ran $\mathbf{1}_{[H_f<\tau]}$ can be defined by

$$P\otimes H_{(0)}[z] = P_\chi F_\chi(H[z],T[z])P_\chi \tag{5.21}$$

since, by Hypothesis 1, the eigenvalue E is simple.

Let us next derive $H_{(0)}$ from (5.21) and arrange the annihilation and creation operators in order. We observe, from Lemma 5.2 and (5.1), that

$$\begin{aligned} P_\chi F_\chi(H[z],T[z])P_\chi &= P_\chi T[z]P_\chi + \sum_{L=1}^{\infty}(-1)^{L-1}P_\chi\chi W\left(\bar{\chi}T^{-1}[z]\bar{\chi}W\right)^{L-1}\chi P_\chi \\ &= P\otimes\mathbf{1}_{[H_f<\tau]}\left(e^{-i\vartheta}H_f - z\right)\mathbf{1}_{[H_f<\tau]} + \sum_{L=1}^{\infty}(-1)^{L-1}\sum_{M_l+N_l\geq 1; l=1,\cdots,L} g^{\sum_{l=1}^{L}(M_l+N_l)} \\ &\quad \times P\otimes\mathbf{1}_{[H_f<\tau]}K(-i\vartheta/\nu;\{M_l,N_l\}_{l=1}^{L})P\otimes\mathbf{1}_{[H_f<\tau]}, \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} K(-i\vartheta/\nu;\{M_l,N_l\}_{l=1}^{L}) &= P\otimes\sin\left[\frac{\pi}{2}\Xi(H_f)\right]W_{M_1,N_1}(-i\vartheta/\nu)RW_{M_2,N_2}(-i\vartheta/\nu)R\cdots \\ &\quad\times RW_{M_{l-1},N_{l-1}}(-i\vartheta/\nu)RW_{M_l,N_l}(-i\vartheta/\nu)P\otimes\sin\left[\frac{\pi}{2}\Xi(H_f)\right] \end{aligned} \tag{5.23}$$

and

$$R := \bar{\chi}T^{-1}[z]\bar{\chi}. \tag{5.24}$$

Lemma 5.3. (Wick ordering) *Let φ be the normalized eigenvector of P. Let* sgn$(\cdots)$, $\mathcal{K}_{M,\ell}$, $\mathcal{K}_{N,\ell}$, r_ℓ, $\Sigma(\tilde{k}_\ell^{(n_\ell)})$ *be symbols defined in Theorem 4.3. Then*

$$\begin{aligned} &K\left(-i\vartheta/\nu;\{M_\ell,N_\ell\}_{\ell=1}^{L}\right) \\ &= \sum_{\substack{\mathcal{I}_{M,\ell}\subseteq\mathcal{K}_{M,\ell}\\ \ell=1,\ldots,L}}\sum_{\substack{\mathcal{I}_{N,\ell}\subseteq\mathcal{K}_{N,\ell}\\ \ell=1,\ldots,L}} \mathrm{sgn}(\mathcal{K}\backslash\mathcal{I},:\mathcal{I}:)\prod_{\ell=1}^{L}\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{M,\ell} \\ \mathcal{I}_{M,\ell} & \mathcal{K}_{M,\ell}\backslash\mathcal{I}_{M,\ell}\end{pmatrix}\mathrm{sgn}\begin{pmatrix} & \mathcal{K}_{N,\ell} \\ \mathcal{I}_{N,\ell} & \mathcal{K}_{N,\ell}\backslash\mathcal{I}_{N,\ell}\end{pmatrix} \\ &\quad\times P\otimes\int_{\mathbf{M}^{m+n}}\prod_{\ell=1}^{L}\left\{dk_\ell^{(m_\ell)}d\tilde{k}_\ell^{(n_\ell)}\right\}\prod_{\ell=1}^{L}b^*(k_\ell^{(m_\ell)}) \\ &\quad\times\left\{\hat{D}_L[H_f;\{\hat{W}^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell};k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^{L};R]\right\}_{m,n}^{\mathrm{asym}}\prod_{\ell=1}^{L}b(\tilde{k}_\ell^{(n_\ell)}), \end{aligned}$$

where

$$\hat{D}_L[r;\{\hat{W}^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell};k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}\}_{\ell=1}^L;R]$$

$$:= \sin\left[\frac{\pi}{2}\Xi(r+\tilde{r}_0)\right]\left\langle \varphi\otimes\Omega, \left\{\prod_{\ell=1}^{L-1}\hat{W}^{m_\ell,n_\ell}_{M_\ell-m_\ell,N_\ell-n_\ell}[k_\ell^{(m_\ell)};\tilde{k}_\ell^{(n_\ell)}]R[H_f+r+\tilde{r}_\ell+\Sigma(\tilde{k}_\ell^{(m_\ell)})]\right\}\right.$$

$$\left.\times\hat{W}^{m_L,n_L}_{M_L-m_L,N_L-n_L}[k_L^{(m_L)};\tilde{k}_L^{(n_L)}]\varphi\otimes\Omega\right\rangle\sin\left[\frac{\pi}{2}\Xi(r+r_L)\right],$$

and

$$\hat{W}^{m,n}_{p,q}[k_\ell^{(m)};\tilde{k}_\ell^{(n)}] := \int_{\mathbf{M}^{m+n}} dx^{(m)}d\tilde{x}^{(n)}b^+(x^{(p)})G^{(\vartheta)}_{m+p,n+q}[K^{(m+p,n+q)}]b^-(\tilde{x}^{(q)}),$$

$$G^{(\vartheta)}_{M,N}[K^{(M,N)}] := e^{-i\frac{d(M+N)}{2\nu}}G_{M,N}\left(e^{-i\vartheta/2\nu}K^{(M,N)}\right),$$

$$R[r] := \bar{\chi}[r](H_S+e^{-i\vartheta/\nu}r-E-z)^{-1}\bar{\chi}[r]\otimes\mathbf{1}.$$

Proof. Similar to the proof of Theorem 4.3. □

Let $\mathcal{H}_{\text{red}}$ be the closed subspace of $\mathcal{F}$ given by

$$\mathcal{H}_{\text{red}} := \operatorname{Ran}\mathbf{1}_{[H_f<1]} = \mathbf{1}_{[H_f<1]}\mathcal{F}. \tag{5.25}$$

Similar to the proof of Theorem 4.4, we observe that the operator $H_{(0)}[z]$ is a bounded operator on $\mathcal{H}_{\text{red}}$ of the form

$$H_{(0)}[z] = T_{(0)}[z;H_f] - E_{(0)}[z] + \sum_{m+Nn\geq 1}\mathbf{1}_{[H_f<1]}W_{m,n}[w^{(0)}_{m,n}[z]]\mathbf{1}_{[H_f<1]}, \quad z\in D_{\rho_0/2},$$

where $E_{(0)}[z]\in\mathbb{C}$, $T_{(0)}[z;\cdot]\in C^1([0,1])$ with $T_{(0)}[z;0]=0$ and the operator $T_{(0)}[z;H_f]$ is defined by functional calculus. Here the operators $W_{m,n}[w^{(0)}_{m,n}[z]]$ is defined by (4.16) and functions $w^{(0)}_{m,n}[z]:[0,1]\times\mathbb{R}^{d(m+n)}\mapsto\mathbb{C}$ are antisymmetric in the sense (4.14). By Hypothesis 2, we observe that the functions $w^{(0)}_{m,n}[z]$ obey the following norm bound:

$$\sup_{z\in D_{\rho_0/2}}\|w^{(0)}_{m,n}[z]\|_\gamma + \sup_{z\in D_{\rho_0/2}}\|\partial_r w^{(0)}_{m,n}[z]\|_\gamma < \infty,$$

where

$$\|w^{(0)}_{m,n}[z]\|_\gamma := \left(\int_{(B_1\times\mathbf{L})^{m+n}} dK^{(m,n)}\frac{\sup_{r\in[0,1]}|w^{(0)}_{m,n}[z][r;K^{(m,n)}]|^2}{\left[\prod_{j=1}^m w(k_j)\prod_{j=1}^n w(\tilde{k}_j)\right]^{1+2\gamma}}\right)^{1/2}.$$

Here we note that the above constant $\gamma>0$, which is given in Hypothesis 2, makes our renormalization group contractive. With a little modefication of the (bosonic) renormalization group method [3] one can prove that there exists a complex number $e_g\in\mathbb{C}$ such that $H_{(0)}[e_g]$ has the eigenvalue 0. Moreover, one can construct the corresponding eigenvector ψ_g:

$$H_{(0)}[e_g]\psi_g = 0.$$

By Theorem 3.2 and the simplicity of the eigenvalue E, we observe that $H[z]$ has the eigenvalue 0 if $H_{(0)}[z]$ has the eigenvalue 0. Hence, the eigenvalue E_g of the Hamiltonian $H_g(\theta)$ is given by $E_g = E+e_g$, and, thanks to Theorem 3.2, the eigenvector by

$$\Psi_g = Q_\chi(\varphi_S\otimes\psi_g),$$

where

$$Q_\chi = \chi - \bar{\chi}H_{\bar{\chi}}^{-1}[e_g]\bar{\chi}W\chi.$$

It is easy to see, from the constructions of e_g and ψ_g (see [3] for details), that E_g and Ψ_g have the desired property (2.29).

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