数理解析研究所講究録
第 1602 巻 2008 年 26-32

A SHORT PROOF OF NUBLING'S RESULT

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ABSTRACT. Nubling shows that CM-triviality (=non-2-amplenss) is preserved under reducts in finite U-rank theories. We give a short proof.

1. REDUCTION AND INDEPENDENCE

Let T^- be a reduct of T. Let $\mathcal{M} \models T$, $\mathcal{M}^- \models T^-$ be big models. $a, b, c, \ldots \bar{a}, \bar{b}, \bar{c}, \ldots$ denote finite tuples, and $A, B, C, \ldots$ denote small sets. Let $A \subset \mathcal{M}^{\mathrm{eq}}$. $\mathrm{ACL}^{\mathrm{eq}}(A)$ denotes the algebraically closure of A in T, and $\mathrm{acl}^{\mathrm{eq}}(A)$ denotes the algebraically closure of $A \cap (\mathcal{M}^-)^{\mathrm{eq}}$. Let $\bar{a} \in (\mathcal{M}^-)^{\mathrm{eq}}$. $\mathrm{TP}(\bar{a}/A)$ denotes the type of $\bar{a}$ over A in T, and $\mathrm{tp}(\bar{a}/A)$ denotes the type of $\bar{a}$ over A in T^-. SU denotes Lascar rank in T, and su denotes Lascar rank in T^-. We show the following fact in the last section.

Fact 1.1. *Let T be a simple theory having EHI such that T^- also has EHI, where T^- be a reduct of T. Let $a, C \subset (\mathcal{M}^-)^{\mathrm{eq}}$ and $B \subset \mathcal{M}^{\mathrm{eq}}$. If $a \mathop{\smile\!\!\!\!\!\!|}_B C$, then $a \mathop{\smile\!\!\!\!\!\!|}^-_{B^-} C$, where $B^- = \mathrm{ACL}^{\mathrm{eq}}(B) \cap (\mathcal{M}^-)^{\mathrm{eq}}$ and $\mathop{\smile\!\!\!\!\!\!|}^-$ is the non-forking relation in T^-.*

Proposition 1.2. *If* $\mathrm{SU}(T) < \omega$, *then* $\mathrm{su}(T^-) < \omega$.

Proof. Let $a \in (\mathcal{M}^-)^{\mathrm{eq}}, A \subset \mathcal{M}^{\mathrm{eq}}$. Put $A^- = \mathrm{ACL}^{\mathrm{eq}}(A) \cap \mathcal{M}^{\mathrm{eq}}$. We will show that there exists $\bar{a}' \models \mathrm{tp}(a/A^-)$ such that $\mathrm{SU}(a'/A) \geq \mathrm{su}(a'/A^-)$ by induction on $n = \mathrm{su}(a/A^-)$.
If $n = 0$, it is clear. Let $\mathrm{su}(a/A^-) = n+1$. So, there exists $A^- \subset B \subset (\mathcal{M}^-)^{\mathrm{eq}}$ such that $\mathrm{su}(a/B) = n$. So, $a \not\mathop{\smile\!\!\!\!\!\!|}^-_{A^-} B$. Put $B^- = \mathrm{ACL}^{\mathrm{eq}}(B) \cap (\mathcal{M}^-)^{\mathrm{eq}}$. So, we have $A^- \subset B \subseteq B^-$. Take $a_1 \models \mathrm{tp}(a/B)$ such that $\bar{a}_1 \mathop{\smile\!\!\!\!\!\!|}^-_B B^-$. As $\mathrm{su}(a_1/B) = \mathrm{su}(a_1/B^-) = n$, by induction hypothesis, there exists $a_1' \models \mathrm{tp}(a_1/B^-)$ such that $\mathrm{SU}(a_1'/B) \geq \mathrm{su}(a_1'/B^-) = n$. As $\mathrm{tp}(a_1'/A^-) = \mathrm{tp}(a/A^-)$, we see $a_1' \not\mathop{\smile\!\!\!\!\!\!|}^-_{A^-} B^-$. As $B^- \subseteq B$ and $a \not\mathop{\smile\!\!\!\!\!\!|}^-_{A^-} B$, by Fact 1.1, we see $\bar{a}_1' \not\mathop{\smile\!\!\!\!\!\!|}_A B$. Therefore we have
$\mathrm{SU}(a_1'/A) \geq \mathrm{SU}(a_1'/B)+1 \geq \mathrm{su}(a_1'/B^-)+1 = n+1 = \mathrm{su}(a/A^-) = \mathrm{su}(a_1'/A^-)$, as desired. $\square$

Date: January 31, 2008.
1991 *Mathematics Subject Classification.* 03C45.
Key words and phrases. CM-triviality.

Lemma 1.3. *Suppose that* $\mathrm{U}(T) < \omega$. *Let* T^- *be a reduct of* T. u *denotes the Lascar rank in* T^-.*(Then* $\mathrm{u}(T^-) < \omega$.*) Let* $a, b, c \in (\mathcal{M}^-)^{\mathrm{eq}}$ *be algebraically independent in* T^- *such that* $\mathrm{u}(a/b) = 1$ *(So,* $a \mathop{\smile\!\!\!\!\!\!\mid}_b^- c$, *because* $a \notin \mathrm{acl}^{\mathrm{eq}}(bc)$.*) Then there exist* $a', b', c' \in \mathcal{M}^{\mathrm{eq}}$ *such that* a', b', c' *are algebraically independent in* T, *a realization of* $\mathrm{tp}(abc)$ *with* $a' \mathop{\smile\!\!\!\!\!\!\mid}_{b'} c'$.

Proof. Let $a'b'c' \models \mathrm{tp}(abc)$ be such that $\mathrm{U}(a'b'c')$ is maximal.

Claim. *a', b', c' are algebraically independent in T.*

As $a' \notin \mathrm{acl}^{\mathrm{eq}}(b'c')$, we can find $a'' \models \mathrm{tp}(a'/b'c')$ such that $a'' \notin \mathrm{ACL}^{\mathrm{eq}}(b'c')$. So, if $a' \in \mathrm{ACL}^{\mathrm{eq}}(b'c')$, then $\mathrm{SU}(a''b'c') > \mathrm{SU}(a'b'c')$, a contradiction. Similarly, we see $b' \notin \mathrm{ACL}^{\mathrm{eq}}(a'c')$ and $c' \notin \mathrm{ACL}^{\mathrm{eq}}(a'b')$.

Claim. $a' \mathop{\smile\!\!\!\!\!\!\mid}_{b'} c'$.

By way of contradiction, suppose that $a' \not\mathop{\smile\!\!\!\!\!\!\mid}_{b'} c'$. Let $a'_0 \models \mathrm{TP}(a'/\mathrm{ACL}^{\mathrm{eq}}(b'))$ such that $a'_0 \mathop{\smile\!\!\!\!\!\!\mid}_{b'} c'$. As $1 = \mathrm{u}(a'/b')$, $\mathrm{stp}(a'/b') = \mathrm{stp}(a'_0/b')$ and $a'_0 \notin \mathrm{ACL}^{\mathrm{eq}}(b'c') \supseteq \mathrm{acl}^{\mathrm{eq}}(b'c')$, we see $1 = \mathrm{u}(a'_0/b') \geq \mathrm{u}(a'_0/b'c') \geq 1$. So we see $a'_0 \mathop{\smile\!\!\!\!\!\!\mid}_{b'} c'$. By STATIONARITY of strong types, we see $\mathrm{stp}(a'_0/b'c') = \mathrm{stp}(a'/b'c')$. In particular, $a'_0b'c' \models \mathrm{tp}(a'b'c')$. Now, we have

$$\begin{aligned}
\mathrm{U}(a'_0b'c') &= \mathrm{U}(a'_0/b'c') + \mathrm{U}(b'c') \\
&= \mathrm{U}(a'_0/b') + \mathrm{U}(b'c') \\
&= \mathrm{U}(a'/b') + \mathrm{U}(b'c') \\
&> \mathrm{U}(a'/b'c') + \mathrm{U}(b'c') = \mathrm{U}(a'b'c')
\end{aligned}$$

□

2. A SHORT PROOF

We begin with basics of supersimple theories.

Fact 2.1. *Let T be a supersimple theory.*

(1) *Let* $a \in \mathcal{M}^{eq}, A \subseteq \mathcal{M}^{eq}$. *Then there exists finite tuple* $\bar{b} \subset \mathcal{M}^{eq}$ *such that* $\mathrm{acl}^{eq}(\mathrm{Cb}(a/A)) = \mathrm{acl}^{eq}(\bar{b}) = \mathrm{acl}^{eq}(\mathrm{Cb}(a/\bar{b}))$.
(2) *Let* $A \subset \mathcal{M}$ *be finitely generated algebraically closed set, and* $B = \mathrm{acl}(B) \subset A$. *Then* B *is finitely generated algebraically closed.*
(3) *Let* $\mathrm{SU}(T) < \omega$ *and* p *be a non-algebraic type. Then there exists a minimal type, non-orthogonal to* p. *(Coordinatization Theorem)*

Proof. (1): Let $B = \mathrm{Cb}(a/A)$. Take a finite tuple $\bar{b} \subset B \subset \mathcal{M}^{eq}$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{b}} B$. Then $B = \mathrm{Cb}(a/A) = \mathrm{Cb}(a/\bar{b})$ and $\mathrm{acl}^{eq}(\bar{b}) = \mathrm{acl}^{eq}(B)$.
(2): By way of contradiction, suppose that there exist $C_0 \subset C_1 \subset \cdots C_n \subset \cdots B \subset A = \mathrm{acl}(\bar{a})$, where C_i are f.g. algebraically closed. Let $\bar{a}_n$ be such that $\bar{a}_n \equiv_{C_n} \bar{a}$ and $\bar{a}_n \mathop{\smile\!\!\!\!\!\!\mid}_{C_n} \bar{a}$. As $C_n \subset \mathrm{acl}(\bar{a})$, we see that $C_n = \mathrm{acl}(\bar{a}_n) \cap \mathrm{acl}(\bar{a})$.

As $C_n \subset C_{n+1}$, so $\bar{a}_{n+1} \not\perp_{C_n} \bar{a}$. So, $\bar{a} \not\perp_{C_n} C_{n+1}$, because $\bar{a} \perp_{C_n} C_{n+1}$ and $\bar{a} \perp_{C_{n+1}} \bar{a}_{n+1}$ imply $\bar{a} \perp_{C_n} \bar{a}_{n+1}$. This contradicts supersimplicity.
(3): We may assume that $p = \mathrm{tp}(a)$. Let $n = \mathrm{SU}(p)$. Take B such that $\mathrm{SU}(a/B) = n-1$. Let $b \in \mathcal{M}^{eq}$ be such that $\mathrm{acl}^{eq}(\mathrm{Cb}(a/B)) = \mathrm{acl}^{eq}(b) = \mathrm{acl}^{eq}(\mathrm{Cb}(a/b))$ by (1). As $a \not\perp b$, $b \notin \mathrm{acl}^{eq}(\emptyset)$. Take C be such that $\mathrm{SU}(b/C) = 1$. We may assume $C \perp_b a$. Then we have $a \not\perp_C b$, otherwise $\mathrm{Cb}(a/bC) = \mathrm{Cb}(a/b) \subseteq \mathrm{acl}(C)$, so $b \in \mathrm{acl}(C)$ would follow. On the other hand, as $n = \mathrm{SU}(a) \geq \mathrm{SU}(a/C) > \mathrm{SU}(a/Cb) = \mathrm{SU}(a/b) = n-1$, we see $\bar{a} \perp C$. □

Notation 2.2. $A \wedge B$ denotes $\mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)$. $a \leftharpoondown A$ denotes $a \in \mathrm{acl}^{eq}(A)$.

Definition 2.3. (1) We say that a sequence (a_0, a_1, a_2) is 2-ample over A, if $a_0 A \wedge a_1 A = A, a_0 a_1 A \wedge a_0 a_1 A = A, a_2 \perp_{a_1 A} a_0$ and $a_2 \not\perp_A a_0$.

(2) We say that a sequence (a_0, a_1, a_2) is weakly 2-ample over A, if $a_2 \perp_{a_1, A} a_0$ and $a_2 \not\perp_{a_1 \wedge a_0 a_2, A} a_0$.
(3) A complete simple theory T with EHI is (weakly) 2-ample, if there exist (weakly) 2-ample sequence over some parameters.

Remark 2.4. (1) T is 2-ample if and only if T is weak 2-ample.
(2) If (a_0, a_1, a_2) is weakly 2-ample, then so is (a_2, a_1, a_0).
(3) If (a_0, a_1, a_2) is weakly 2-ample, then (a_0, a_1, a_2) are algebraically independent.

Proof. (1): Clearly, any 2-ample sequence is weakly 2-ample. Let (a_0, a_1, a_2) be weakly 2-ample and let a_0' be such that $\mathrm{acl}^{eq}(a_0') = a_0 a_1 \wedge a_0 a_2$. Then we have $a_0' a_1 \wedge a_0' a_2 = \mathrm{acl}^{eq}(a_0')$ and $a_0' \wedge a_1 = a_1 \wedge a_0 a_2$. Then we see that (a_0', a_1, a_2) is 2-ample over $a_1 \wedge a_0 a_2$. (2):Clear. (3): If a_0 or a_2 were algebraic over a_1, then it would be algebraic over $a_1 \wedge a_0 a_2$. If a_1 were algebraic over $a_0 a_2$, then $\mathrm{acl}^{eq}(a_1) = a_1 \wedge a_0 a_2$ would follow. As $a_2 \perp_{a_1} a_0$, we see a_0, a_1, a_2 are algebraically independent. □

From now on, we work in a finite SU-rank theory.

Lemma 2.5. *Let* (a_0, a_1, a_2) *be weakly 2-ample.*
(1) *There exist* a_0' *and* B *such that* $a_0' \leftharpoondown a_0 B, \mathrm{SU}(a_0'/B) = 1$ *and* (a_0, a_1, a_2) *is weakly 2-ample over* B.
(2) *Fixing* a_1, *after adding some parameters, we can retake* a_0, a_2 *such that* $\mathrm{SU}(a_0/a_1) = \mathrm{SU}(a_2/a_1) = 1$.

Proof. (1): By coordinatization theorem, there exist a_0' and B such that $a_0' \not\perp_B a_0$, $\mathrm{SU}(a_0'/B) = 1$ and $a_0 \perp B$. We may assume $a_1 a_2 \perp_{a_0} B a_0'$. Since $a_0 a_1 a_2 \perp B$, (a_0, a_1, a_2) is weakly 2-ample over B, as desired.
(2): By remark 2.4 (2), we have only to retake a_0 such that $\mathrm{SU}(a_0/a_1) = 1$.

Let a_0 be minimal of SU-rank such that (a_0, a_1, a_2) is weakly 2-ample. Suppose that $\mathrm{SU}(a_0/a_1) > 1$. By (1) take a_0' such that $a_0' \leftharpoondown a_0, \mathrm{SU}(a_0') = 1$. By Fact 2.1 (2), take a be such that $\mathrm{acl}^{\mathrm{eq}}(a) = a_0 \wedge a_0'a_1$. Then $\mathrm{SU}(a_0) > \mathrm{SU}(a), \mathrm{SU}(a_0/a)$, because $\mathrm{SU}(a_0) = \mathrm{SU}(a_0/a) + \mathrm{SU}(a)$ and $\mathrm{SU}(a), \mathrm{SU}(a_0/a) \geq 1$. (If $a_0 \leftharpoondown a$, then a_0, a_0' are interalgebraic over a_1, a contradiction.) If $a_0 \not\perp_{a,a_1 \wedge a_0 a_2} a_2$, then (a_0, a_1, a_2) is weakly 2-ample over a, which contradicts the minimality of $\mathrm{SU}(a_1)$. If $a_0 \perp_{a,a_1 \wedge a_0 a_2} a_2$, then $a \not\perp_{a_1 \wedge a_0 a_2} a_2$, so we see (a, a_1, a_2) is weakly 2-ample over $a_1 \wedge a_0 a_2$, a contradiction. □

Proposition 2.6. *Let (a_0, a_1, a_2) be weakly 2-ample. Then, after adding some parameters, we can retake a_0, a_1, a_2 such that*

$$\mathrm{SU}(a_0/a_1) = \mathrm{SU}(a_2/a_1) = \mathrm{SU}(a_1/a_0 a_2) = 1.$$

Proof. By Lemma 2.5, take a_1 be minimal of SU-rank such that (a_0, a_1, a_2) is weakly 2-ample and $\mathrm{SU}(a_0/a_1) = \mathrm{SU}(a_2/a_1) = 1$. Suppose that $\mathrm{SU}(a_1/a_0 a_2) > 1$. Take $a_1' \leftharpoondown a_1$ be such that $\mathrm{SU}(a_1') = 1$ after possibly adding parameters. Let a, b be such that $\mathrm{acl}^{\mathrm{eq}}(a) = a_0 a_1 \wedge a_0 a_1' a_2$ and $\mathrm{acl}^{\mathrm{eq}}(b) = a \wedge a_1$. Then $\mathrm{SU}(a_1) > \mathrm{SU}(b), \mathrm{SU}(a_1/b)$. (If $a_1 \leftharpoondown a_0 a_1' a_2$, then a_1, a_1' would be interalgebraic over $a_0 a_2$. So we see $\mathrm{SU}(a_1/b) \geq 1$. Clearly $\mathrm{SU}(b) \geq 1$. The above follows from $\mathrm{SU}(a_1) = \mathrm{SU}(a_1/b) + \mathrm{SU}(b)$.)
If $a \not\perp_b a_2$, then (a, a_1, a_2) is weakly 2-ample over b, because $b \subseteq (a_1 \wedge a a_2) b \subseteq a_1 \wedge a_0 a_1' a_2 = b$. As $a \leftharpoondown a_0 a_1$ and $b \leftharpoondown a_1$, we have $\mathrm{SU}(a/a_1 b) = \mathrm{SU}(a_2/a_1 b) = 1$. This contradicts the minimality of $\mathrm{SU}(a_1)$.
If $a \perp_b a_2$, then $a_0 \perp_b a_2$. Then (a_0, b, a_2) is weakly 2-ample over $a_1 \wedge a_0 a_2$. By Lemma 2.5, we may assume $\mathrm{SU}(a_0/b) = \mathrm{SU}(a_2/b) = 1$. This also contradicts the minimality of $\mathrm{SU}(a_1)$. □

Now, we prove the Nubling's theorem.

Theorem 2.7. *Suppose that $\mathrm{U}(T) < \omega$. If a reduct T^- of T is 2-ample, then so is T.*

Proof. By Proposition 2.6, let (a_0, a_1, a_2) be weakly 2-ample such that $\mathrm{u}(a_0/a_1) = \mathrm{u}(a_2/a_1) = \mathrm{u}(a_1/a_0 a_2) = 1$. As a_0, a_1, a_2 are algebraically independent in T^-, by Lemma 1.3, there exist $abc \models \mathrm{tp}(a_0 a_2 a_3)$ such that a, b, c are algebraically independent in T and $a \perp_b c$.

Claim. *$a \not\perp_{\mathrm{ACL}^{\mathrm{eq}}(b) \cap \mathrm{ACL}^{\mathrm{eq}}(ac)} c$. So, (a, b, c) is weakly 2-ample.*

Put $A = \mathrm{ACL}^{\mathrm{eq}}(b) \cap \mathrm{ACL}^{\mathrm{eq}}(ac)$, and $A^- = A \cap (\mathcal{M}^-)^{\mathrm{eq}}$. By way of contradiction, suppose that $a \perp_A c$. Then we have $a \perp^-_{A^-} c$ by Fact 1.1. As $a \notin \mathrm{ACL}^{\mathrm{eq}}(b) = \mathrm{ACL}^{\mathrm{eq}}(bA) \supseteq \mathrm{acl}^{\mathrm{eq}}(bA^-)$, we see $1 = \mathrm{u}(a/b) \geq \mathrm{u}(a/bA^-)$, so $a \perp^-_b A^-$ follows. Moreover, as $c \notin \mathrm{ACL}^{\mathrm{eq}}(ab) = \mathrm{ACL}^{\mathrm{eq}}(abA) \supseteq \mathrm{acl}^{\mathrm{eq}}(abA^-)$, we see $1 = \mathrm{u}(c/b) \geq \mathrm{u}(c/abA^-) \geq 1$, so $c \mathop{\smile\!\!\!\!\!\mid}^{-}_{b} aA^-$ holds. So, we have $A^- \mathop{\smile\!\!\!\!\!\mid}^{-}_{b} ac$. On the other hand, $b \notin \mathrm{ACL}^{\mathrm{eq}}(ac) = \mathrm{ACL}^{\mathrm{eq}}(acA) \supseteq \mathrm{acl}^{\mathrm{eq}}(acA^-)$, we see $1 = \mathrm{u}(b/ac) \geq \mathrm{u}(b/acA^-) \geq 1$, we have $b \mathop{\smile\!\!\!\!\!\mid}^{-}_{ac} A^-$. So, we have $\mathrm{Cb}(\mathrm{tp}(A^-/abc)) \subseteq b \wedge^- ac := \mathrm{acl}^{\mathrm{eq}}(b) \cap \mathrm{acl}^{\mathrm{eq}}(ac)$, $A^- \mathop{\smile\!\!\!\!\!\mid}^{-}_{b\wedge^- ac} abc$ holds. Since $a \mathop{\smile\!\!\!\!\!\mid}^{-}_{A^-} c$ and $a \mathop{\smile\!\!\!\!\!\mid}^{-}_{b\wedge^- ac} A^-$, so $a \mathop{\smile\!\!\!\!\!\mid}^{-}_{b\wedge^- ac} c$, (a,b,c) is not weakly 2-ample in T^-, a contradiction. □

Remark 2.8. There is a modular O-minimal theory which has a non-CM-trivial reduct [Y]. Nubling theorem can not be extended to finite U$^\bullet$-rank theories.

3. Indiscernible sequences and the proof of Fact 1.1

We work in a complete theory and consider imaginary elements.

Let $(a_i : i \in I)$ be a sequence and $I_0 \subseteq I$. a_{I_0} denotes $(a_i : i \in I_0)$. When I is an partially ordered set, $a_{<i}$ denotes $(a_j : j < i)$. Similarly for $a_{>i}$. We write $I_0 < I_1$, if $I_0, I_1 \subseteq I$ and $i_1 < i_2$ holds for any $i_1 \in I_1, i_2 \in I_2$.

Definition 3.1. Let $X = (a_i : i \in I)$ be a B-indiscernible sequence and $A \subseteq B$.

(1) Put $\ker_A(X) := \bigcup_{|I_0|=|J_0|=k<\omega, I_0<J_0} (\mathrm{acl}^{\mathrm{eq}}(a_{I_0}A) \cap \mathrm{acl}^{\mathrm{eq}}(a_{J_0}A))$. We call it the kernel of X over A.

(2) We say that X is algebraically independent over A, if $\mathrm{acl}^{\mathrm{eq}}(Aa_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(A_{I_1}) = \mathrm{acl}^{\mathrm{eq}}(A)$ for any $I_0 < I_1 \subseteq I$.

Lemma 3.2. *Let $X = (a_i : i \in I)$ be a B-indiscernible sequence.*

(1) *For infinite subsets $I_1 < I_2$, $\ker_B(X) = \mathrm{acl}^{\mathrm{eq}}(a_{I_1}B) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_2}B)$.*

(2) *$\ker_B(X)$ is the smallest algebraically closed set (containing B) over which X is algebraically independent.*

(3) *X is indiscernible over $\ker_B(X)$.*

(4) *$\ker_B(X)$ is the biggest subset (containing B) of $\mathrm{acl}^{\mathrm{eq}}(XB)$ over which X is indiscernible.*

Proof. For ease of notation, we assume $B = \emptyset$.

(1): Suppose that I_0, I_1, J are finite with the same size and $I_0, I_1 < J$. As $a_{I_0} \equiv_{\mathrm{acl}^{\mathrm{eq}}(a_J)} a_{I_1}$, we see

$$\mathrm{acl}^{\mathrm{eq}}(a_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_J) = \mathrm{acl}^{\mathrm{eq}}(a_{I_1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_J).$$

By the same argument, we see that

$$\mathrm{acl}^{\mathrm{eq}}(a_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{J_0}) = \mathrm{acl}^{\mathrm{eq}}(a_{I_1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{J_1}).$$

for any $I_0 < J_0, I_1 < J_1, |I_0| = |I_1| = |J_0| = |J_1|$. Therefore, we see $\ker(X) \subseteq \mathrm{acl}^{\mathrm{eq}}(a_{I_1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_2})$ for any infinite $I_1 < I_2$. We show the converse inclusion. Let $a \in \mathrm{acl}^{\mathrm{eq}}(a_{I_1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_2})$. Then there exist $J_1 \subset I_1, J_2 \subset I_2$ such that

$|J_1| = |J_2| < \omega$ such that $a \in \mathrm{acl}^{\mathrm{eq}}(a_{J_1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{J_2})$. By the above argument, we see $\mathrm{acl}^{\mathrm{eq}}(a_{I_1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_2}) \subseteq \ker(X)$.
(2): Let C be such that X is algebraically independent over C. Then, for any infinite $I_0 < J_0$, $\ker(X) = \mathrm{acl}^{\mathrm{eq}}(a_{I_1}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{J_1}) \subseteq \mathrm{acl}^{\mathrm{eq}}(Ca_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(Ca_{J_0}) = \mathrm{acl}^{\mathrm{eq}}(C)$, as desired.
(3): By (1), we see that if X' is an extended indiscernible sequence of X, then $\ker(X) = \ker(X')$. It suffices to show that, if I_0, J_0 are finite sets with the same size, then $a_{I_0} \equiv_{\ker(X)} a_{J_0}$. Take an infinite set $J \subseteq I$ such that $I_0, J_0 < J$, if necessarily, extend X. As $a_{I_0} \equiv_{\mathrm{acl}^{\mathrm{eq}}(a_J)} a_{J_0}$, we see the conclusion.
(4): Let $C \subset \mathrm{acl}(X)$ be such that X is indiscernible over C. Let $c \in C$. Then there exists a finite I_1 such that $c \in \mathrm{acl}^{\mathrm{eq}}(a_{I_1})$. For any $I_0 < I_1, |I_0| = |I_1|$, we have $c \in \mathrm{acl}^{\mathrm{eq}}(a_{I_0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_1})$, since $a_{I_0} \equiv_c a_{I_1}$. Now, we see that $C \subseteq \ker(X)$. □

From now on, we work in a simple theory T with EHI.

Lemma 3.3. *Let $X = (a_i : i \in I)$ be a B-indiscernible sequence and $A \subseteq B$.*

1. *If X is sufficiently long and independent over A, then $X \mathop{\smile\!\!\!\!\!\!\mid}_A B$.*
2. *If X is sufficiently long, then* $\mathrm{Cb}(B/(a_i : i \in I)A) \subseteq \ker_A(X)$.
3. *If X is a Morley sequence over B, then* $\ker_A(X) \subseteq \mathrm{acl}^{\mathrm{eq}}(B)$.

Proof. (1): By simplicity, take $B_0 \subseteq a_{<|T|^+}$ such that $B \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} a_{<|T|^+}$ and $|B_0| \leq |T|$. So there exists $\lambda < |T|^+$ such that $B_0 \subseteq a_{<\lambda}$. We have $a_{<|T|^+} \mathop{\smile\!\!\!\!\!\!\mid}_{a_{<\lambda}} B$. By B-indiscerniblity of X, we have $a_{\geq\lambda} \mathop{\smile\!\!\!\!\!\!\mid}_{a_{<\lambda}} B$. So, $a_{\geq\lambda} \mathop{\smile\!\!\!\!\!\!\mid}_{a_{<\lambda}A} B$. As X is independent over A, $a_{\geq\lambda} \mathop{\smile\!\!\!\!\!\!\mid}_A a_{<\lambda}B$ follows. By A-independence of X again, we see the conclusion.
(2): Let $I_0 \subseteq I$ be such that $|I_0| = |T|^+$. Then there exists $B_0 \subseteq a_{I_0}$ such that $B \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} a_{I_0}$ and $|B_0| \leq |T|$. As there exists $\lambda \in I_0$ such that $B_0 \subseteq a_{<\lambda}$, we see $a_{I_0} \mathop{\smile\!\!\!\!\!\!\mid}_{a_{<\lambda}} B$. By B-indscernibility and finite character, we have $a_{\geq\lambda} \mathop{\smile\!\!\!\!\!\!\mid}_{a_{<\lambda}} B$. Therefore we have $a_I \mathop{\smile\!\!\!\!\!\!\mid}_{a_{I_0}A} B$. As we assume EHI, $\mathrm{Cb}(B/Aa_I) \subseteq \mathrm{acl}^{\mathrm{eq}}(a_{I_0}A)$. Let I_1 be such that $I_0 < I_1$ and $|I_1| = |T|^+$. By the same argument, we see $\mathrm{Cb}(B/Aa_I) \subseteq \mathrm{acl}^{\mathrm{eq}}(a_{I_0}A) \cap \mathrm{acl}^{\mathrm{eq}}(a_{I_1}A) = \ker_A(X)$.
(3): By our statement, we may assume X is sufficiently long. By 3.2 (4), we have $\ker_A(X)B \subseteq \ker_B(X)$. So, X is $\ker_A(X)B$-indiscernible and independent over B. By (1), we see $X \mathop{\smile\!\!\!\!\!\!\mid}_B \ker_A(X)$. Since $\ker_A(X) \subseteq \mathrm{acl}(BX)$, we see $\ker_A(X) \subseteq \mathrm{acl}(B)$. □

Proposition 3.4. *Let $X = (a_i : i \in I)$ be an A-indiscernible sequence.*

1. *If X is algebracally independent over A, then X is a Morley sequence over A.*
2. *If X is a Morley sequence over A, then* $\ker(X) = \mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(a_0/A))$.

Proof. (1): By our asumption, we may assume X is sufficiently long. Let a_∞ be such that a_I, a_∞ is an extended A-indisernible sequence, algebraically independent over A. As X is Aa_∞-indicernible and algebraically independent over A, by Lemma 3.3 (1), $\mathrm{Cb}(Aa_\infty/AX) \subseteq \ker_A(X) = \mathrm{acl}^{\mathrm{eq}}(A)$. Therefore $a_\infty \mathop{\smile\!\!\!\!\!\!\mid}_A a_I$. By A-indiscernibility of $a_I a_\infty$, we see X is independent over A.
(2): As X is algebraically independent over $\mathrm{Cb}(a_0/A)$, we see $\ker(X) \subseteq \mathrm{acl}^{\mathrm{eq}}(\mathrm{Cb}(a_0/A))$ by Lemma 3.2 (2). By Lemma 3.3 (3), $\ker(X) \subseteq \mathrm{acl}^{\mathrm{eq}}(A)$. As X is $\ker(X)$-indiscernible and algebraically independent over $\ker(X)$, X is a Morley sequence over $\ker(X)$ by (1). Now, by Lemma 3.3 (1), we have $X \mathop{\smile\!\!\!\!\!\!\mid}_{\ker(X)} A$. In particular, $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{\ker(X)} A$ holds. So, we see $\mathrm{Cb}(a_0/A) \subseteq \mathrm{acl}^{\mathrm{eq}}(\ker(X))$. □

FACT 1.1: *Let T be a simple theory having EHI such that T^- also has EHI, where T^- be a reduct of T. Let $a, C \subset (\mathcal{M}^-)^{\mathrm{eq}}$ and $B \subset \mathcal{M}^{\mathrm{eq}}$. If $a \mathop{\smile\!\!\!\!\!\!\mid}_B C$, then $a \mathop{\smile\!\!\!\!\!\!\mid}^-_{B^-} C$, where $B^- = \mathrm{ACL}^{\mathrm{eq}}(B) \cap (\mathcal{M}^-)^{\mathrm{eq}}$ and $\mathop{\smile\!\!\!\!\!\!\mid}^-$ is the non-forking relation in T^-.*
Proof. Let $X = (a_i : i \in \mathbb{Z})$ be a Morley sequence of $\mathrm{TP}(a/BC)$. Then, by Proposition 3.4 (2) and our assumption, we have $\mathrm{ACL}^{\mathrm{eq}}(a_{<0}) \cap \mathrm{ACL}^{\mathrm{eq}}(a_{>0}) = \ker(X) = \mathrm{ACL}^{\mathrm{eq}}(\mathrm{Cb}(a/BC)) \subseteq \mathrm{ACL}^{\mathrm{eq}}(B)$. So, $\mathrm{acl}^{\mathrm{eq}}(a_{<0}) \cap \mathrm{acl}^{\mathrm{eq}}(a_{>0}) \subseteq B^-$. As X is algebraically independent over BC, so is over B^-C in T^-. Since X is B^-C-indiscernible in T^-, by Proposition 3.4 (1), X is a Morley sequence of $\mathrm{tp}(a/B^-C)$. By $\ker^-(X) \subseteq B^-$ and Proposition 3.4 (2), we see $a \mathop{\smile\!\!\!\!\!\!\mid}^-_{B^-} C$. □

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