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Some remark on weak dividing

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Abstract

Weak dividing has been characterized variously in simple theory. We try to argue about the restricted notions of it.

1. Weak dividing

We recall some definitions.

Definition 1 Let $\varphi(x_0, x_1, \cdots\cdots, x_{n-1})$ be a formula and $p(x)$ be a type. We denote the type $\{\varphi(x_0, x_1, \cdots\cdots, x_{n-1})\} \cup p(x_0) \cup p(x_1) \cup \cdots\cdots \cup p(x_{n-1})$ by $[p]^{\varphi}$.

Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *divides overA* if there are a formula $\varphi(x, b) \in p(x)$ and an infinite sequence $\{b_i : i < \omega\}$ with $b \equiv b_i\,(A)$ such that $\{\varphi(x, b_i) : i < \omega\}$ is $k-$inconsistent for some $k < \omega$.

$p(x)$ *weakly divides over* A if there is a formula $\varphi(\bar{x}) \in L_n(A)$ such that $[p\lceil A]^{\varphi}$ is consistent, while $[p]^{\varphi}$ is inconsistent.

In this note, we call such formula "$\varphi(\bar{x})$" in the definition the *witness formula* of weak dividing for the sake of convenience.

I introduce examples from [2], [3].

Example 2 Let T be the theory of dense linear order and $p(x) =$ "$a < x < b$". Then $p(x)$ does not weakly divdie over $\emptyset$. And let $q(x, y) =$ "$x < c < y$". Then $q(x, y)$ weakly divides over $\emptyset$ by the formula $\varphi(x_1, y_1\,; x_2, y_2) =$ "$y_1 < x_2$".

Example 3 Let T be the theory of an equivalence relation with two infinite classes of the language $L = \{$a binary relation $E(x, y)\}$. And let $\models \neg E(a, b)$. Then the type $\mathrm{tp}(a/b)$ does not divide over $\emptyset$, while $\mathrm{tp}(a/b)$ weakly divides over $\emptyset$ by the formula $\neg E(x, y)$.

Example 4 Let $(V, <, >)$ be a vector space V over a finite field equipped with an inner product giving orthogonality between two independent vectors. Let a, b, c be independent vectors in V such that $a \perp b$, while $b \not\perp c$ and $a \not\perp c$. Then $\text{tp}(a/bc)$ does not weakly divide over $\emptyset$. But $\text{tp}(a/bc)$ weakly divides over c by the formula $\varphi(x, y) =$ "x is a linear combination of y and c".

In various characterizations, one of the most important results is the next theorem.

Theorem 5 (Kim [3])
T is stable if and only if weak dividing is symmetric in T.

2. Restricted notions of weak dividing

In examples above, we can see that witness formulas have different properties in the sense of stability theory. So I considered that we can divides witness formulas into some classes according to the properties.

Definition 6 Let $A \subset B$ and $p(x) \in S(B)$.
$p(x)$ *$\mathcal{M}$−weakly divides over A* if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p\lceil A$ such that $\models \varphi(a_0, a_1, \cdots\cdots, a_{n-1})$, while the type $[p]^{\varphi}$ is inconsistent.
$p(x)$ *M−weakly divides over A* if there are a formula $\varphi(\bar{x}) \in L_n(A)$ and a Morley sequence $I = \{a_i : i < n+1\}$ of $p\lceil A$ such that $\models \varphi(a_0, a_1, \cdots\cdots, a_{n-1})$, while there is no Morley sequence $J = \{b_i : i < n+1\}$ of p over A such that $\models \varphi(b_0, b_1, \cdots\cdots, b_{n-1})$.

If we set the sequence I indiscernible over A in the definition above, we can define $\mathcal{I}$−weak dividing and I−weak dividing in the same way.
I proved the next result.

Theorem 7 *Let T be simple.*
Then T is stable if and only if $\mathcal{M}$−weak dividing is symmetric in T.

Example 3 is easy. But some of examples of $\mathcal{M}$−weak dividing may be variants of it.

Fact 8 *Let T be simple. And let $A \subset B$ and $p(x) \in S(B)$.*
$p(x)$ does not divide over A, while $p(x)$ $\mathcal{M}$−weakly divides over A by a formula $\varphi(x, y) \in L_2(A)$. Then for any realization ab of φ with $a \perp_A b$, $Lstp(a/A) \neq Lstp(b/A)$.

I showed Fact 8 for 2−variable witness formulas. But the same Fact holds for n−variable witness formulas under the assumption that T has n−amalgamation property. (see [4],[5]) Once I told about some weak dividing for n−dividing in n−simple theory.

In recent years another variant of dividing, "*thorn*"−dividing has been characterized in rosy theory. (see e.g. [6]) I tried to define weak notion of $\mathfrak{p}$−dividing (thorn-dividing). We recall some definitions first.

Definition 9 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *strongly divides over* A if there is a formula $\varphi(x, b) \in p(x)$ such that $b \notin acl(A)$ and $\{\varphi(x, b_i) : b_i \models \mathrm{tp}(b/A)\}$ is k−inconsistent for some $k < \omega$.

$p(x)$ $\mathfrak{p}$*−divides over* A if $p(x)$ strongly divides over Ac for some parameter c.

Weak notions of $\mathfrak{p}$−dividing could be defined in many ways. By the definition, $\mathfrak{p}$−dividing implies dividing. So we expect that weak $\mathfrak{p}$−dividing implies weak dividing.

Definition 10 Let $A \subset B$ and $p(x) \in S(B)$.

$p(x)$ *weakly* $\mathfrak{p}$*−divides over* A if there is a formula $\varphi(\bar{x}) = \exists y \bigwedge_{i<n} \psi(x_i, y) \in L_n(A)$ such that $[p\lceil A]^{\varphi}$ is consistent, while $[p]^{\varphi}$ is inconsistent.

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