数理解析研究所講究録
第 1604 巻 2008 年 80-89

Place Dependency of a Petri Net Generating a Maximal Prefix Code *

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Abstract In this paper we deal with prefix codes, called CPN languages, defined on Petri nets. And the family **CPN** of all CPN languages is included in the family of all context-sensitive languages. The subclass **mCPN** and **NmCPN** of **CPN** are a family of prefix codes which are maximal prefix codes and a family of prefix codes defined on input-ordinal Petri nets. **NmCPN** is obviously included in **mCPN**. But its converse inclusion is still a open problem. We have already proved under some restricted Petri nets, for example, Petri nets have at most two places or at most one transition. We consider this problem in the case that Petri nets have more than two places.

1 Preliminaries

In this section, we state the definitions and the notations of formal languages and codes in this paper. And we introduce Petri net codes and thier related codes.

Let X be a nonempty finite set called an alphabet, X^* be the free monoid generated by X under the concatenation. An element of X^* is called a word. The identity of X^* is called the empty word, denoted by 1. We denote $X^* \setminus \{1\}$ by X^+, the concatenation of two words x and y by xy, and the length of a word $w \in X^*$ by $|w|$(especially $|1| = 0$).

If for two words $w, u \in X^*$ there exists some word $v \in X^*$ (resp. $v \in X^+$) with $w = uv$, then u is called a prefix (resp. a proper prefix) of w, we represent $u \leq_p w$(resp. $u <_p w$). A language over X is a subset of X^*. The concatenation of two languages L_1 and L_2 is defined by $L_1L_2 = \{w_1w_2 | w_1 \in L_1, w_2 \in L_2\}$. A nonempty language L is a code if for any two integers $n, m \geq 1$ and $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_m \in L$,

$$u_1u_2\cdots u_n = v_1v_2\cdots v_m$$

implies

$$n = m \quad \text{and} \quad u_i = v_i \text{ for } i = 1, \cdots, n.$$

A code L is a prefix code if $u, uv \in L$ implies $v = 1$. A code $C \subset X^+$ is maximal (resp. maximal prefix) in X if C is not included by any other code (resp. prefix code) over X.

Remark A maximal and prefix code is clearly a maximal prefix code because it is not included in any other codes by the maximality. But a maximal prefix code is a prefix code, but is not necessarily a maximal code.

DEFINITION 1.1 (Petri net) *A Petri net PN is a quadruple* (P, X, W, μ_0) *satisfying the following conditions.*

(1) P *and* X *are finite sets with* $P \cap X = \emptyset$ *and* $P \cup X \neq \emptyset$.

(2) W *is a weighting function from* $(P \times X) \cup (X \times P)$ *to the set* $\mathbf{N}$ *of all the nonnegative integers.*

(3) μ_0 *is a function from* P *to* $\mathbf{N}$, *called an initial marking.* □

*This is an abstract and the paper will appear elsewhere.

A marking is called **positive** (or **zero**) if it is a mapping from P to $\boldsymbol{N}\backslash\{0\}$ (or a mapping from P to $\{0\}$, respectively).

And PN is **input-ordinal** if $W(p,\, a) \leq 1$ for any $(p,\, a) \in P \times X$. In the above Petri net PN, we may call $(p,\, a) \in P \times X$ an arc when $W(p, a) > 0$ holds, and then $W(p, a)$ is called the weight of the arc $(p,\, a)$. The similar definition is stated about $(a, p) \in X \times P$.

The transition $a \in X$ is called **enable** under the Petri net PN if $W(p,\, a) \leq \mu(p)$ holds for each place $p \in P$. Then the new marking μ' is defined as follows:

$$\mu'(p) = \mu(p) - W(p,\, a) + W(a,\, p) \ \text{ for } \forall p \in P.$$

The transition function δ_{PN} of PN is defined by $\delta_{PN}(\mu,\, a) = \mu'$. $\delta_{PN}(\mu,\, a)$ is undefined if $a \in X$ is not enable under PN. This function is extended from $P \times X \mapsto \boldsymbol{N}$to $P \times X^* \mapsto \boldsymbol{N}$as follows: $\delta_{PN}(\mu,\, 1) = \mu$ and $\delta_{PN}(\mu,\, ua) = \delta_{PN}(\delta_{PN}(\mu,\, u), a)$. We may denote δ_{PN} by δ if no confusion is possible.

$w \in X^*$ is called a **firing sequence** in PN if $\delta_{PN}(\mu,\, w)$ is defined. $w \in X^*$ is called a **positive firing sequence** in PN if $\delta_{PN}(\mu,\, w)$ is defined and $\delta_{PN}(\mu,\, u)$ is positive for any prefix u of w. We denote the sets of all firing sequences in PN and all positive firing sequences in PN by $\mathbf{FSeq}(PN)$ and $\mathbf{FSeq}^+(PN)$ respectively. We denote $\{\delta(\mu_0, w) | w \in \mathbf{FSeq}(PN)\}$ (or $\{\delta(\mu_0, w) | w \in \mathbf{FSeq}^+(PN)\}$) by $\mathbf{Re}(PN)$ (or $\mathbf{Re}^+(PN)$ resp.).

DEFINITION 1.2 *Let $PN = (P,\, X,\, W,\, \mu_0)$ be a Petri net, μ_0 be a positive marking. Then we define the languages $C(P,\, X,\, W,\, \mu_0)$ and $C_0(P,\, X,\, W,\, \mu_0)$ as follows:*

$$C(P,\, X,\, W,\, \mu_0) = \{w \in \mathbf{FSeq}(PN) |\, \delta(\mu,\, w) \textit{ is not positive}, w = uv, v \in X^+,\, u \in \mathbf{FSeq}^+(PN)\},$$
$$C_0(P,\, X,\, W,\, \mu_0) = \{w \in \mathbf{FSeq}(PN) |\, \delta(\mu,\, w) \textit{ is zero}, w = uv, v \in X^+,\, u \in \mathbf{FSeq}^+(PN)\}.$$

□

If $C(P,\, X,\, W,\, \mu_0)$ and $C_0(P,\, X,\, W,\, \mu_0)$ are not empty, then they are prefix codes. Because both u, uv are thier elements and $v \neq 1$ yield a contradiction since $\delta(\mu,\, u)$ is positive. And we call $C(P,\, X,\, W,\, \mu_0) \neq \emptyset$ a Petri net code, $C_0(P,\, X,\, W,\, \mu_0) \neq \emptyset$ a strict Petri net code. The families of all the Petri net codes and all strict Petri net code are denoted by **CPN** and **CPN0**, respectively. Note that **CPN0** is a subclass of **CPN** . Moreover a Petri net code is said to be **maximal** if it is maximal as a prefix code. The families of all the maximal Petri net codes and all the strict Petri net codes are denoted by **mCPN** and **mCPN0**, respectively..

A Petri net code is said to be **input-ordinal** if it is generated by some input-ordinal Petri net. The family of all the input-ordinal Petri net codes is denoted by **NmCPN**.

Since an input-ordinal Petri net code is clearly a maximal Petri net code, we get the inclusion relation **NmCPN**$\subset$**mCPN**. In this paper, we consider the following problem.

【Problem】 **mCPN**$\subset$**NmCPN**?

Since it is too difficult to solve this problem in general Petri nets, in the next section we prove that the problem is solved affirmatively in a restricted Petri net.

2 Fundamental Properties

In this section we state some fundamental properties about strict Petri net codes and the structure of Petri nets which generate maximal Petri net codes.

2.1 Some Properties of Strict Petri net codes

At first we show that a strict Petri net code is a full uniform code if it is finite and maximal. For a Petri net (P, X, W, μ_0), $p \in P$ and $u = a_1a_2 \ldots a_r \in X^*$, we denote $(W(p, a_1) - W(a_1, p)) + (W(p, a_2) - W(a_2, p)) + \cdots + (W(p, a_r) - W(a_r, p))$ by $p(u)$.

LEMMA 2.1 [2] *Let $C = C(P, X, W, \mu_0)$ be a finite maximal Petri net code over X. For any $u, v \in C$, if there exists a $p \in P$ such that $\mu_0(p) = p(u) = p(v)$, then C is a full uniform code over X, i.e. $C = X^n$ for some $n, n \geq 1$.* □

PROPOSITION 2.1 *If a finite maximal Petri net code over X is strict, then it is a full uniform code over X.* □

LEMMA 2.2 *Let $C = C_0(P, X, W, \mu_0)$ be a maximal strict Petri net code over X. And let p be a place in P. Then there exists a Petri net $(\{p\}, X, W', \mu_0')$ such that $C(\{p\}, X, W', \mu_0') = C$.*

Proof) Let W' be the restriction of W on $\{p\} \times X \cup X \times \{p\}$, μ_0' be the restriction of μ_0 on $\{p\}$. let δ and δ' be transition functions of (P, X, W, μ_0) and $(\{p\}, X, W', \mu_0')$ respectively. Since C is maximal, $\delta(\mu_0, u)(q) > 0$, $\delta'(\mu_0, u)(p) > 0$ and $\delta(\mu_0, w)(q) = \delta'(\mu_0, w)(p) = 0$ for each $q \in P$, $w = uv \in C$, $v \in X^+$. Therefore $C(\{p\}, X, W', \mu_0') = C$. □

PROPOSITION 2.2 *If a maximal Petri net code over X is strict, then it is input-ordinal.* □

A code in this proposition is given the formula (1) in the next chapter. Note that the Petri net code $\{a^3, ab, ba\}$ is strict but not maximal.

2.2 Structure of maximal Petri net codes

DEFINITION 2.1 *Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. For $w \in X^*$, the set $F_w \subset P \times X$ of PN is defined as follows:*

$(p, a) \in F_w \iff$

(i) $W(p, a) > 0$, $(\forall b \in X)(W(p, a) \geq W(p, b))$,

(ii) $w \in \mathbf{FSeq}^+(PN), \mu = \delta(\mu_0, w), (\forall q \in P)(W(q, a) > 0 \Rightarrow \mu(p)/W(p, a) \leq \mu(q)/W(q, a))$.

□

$(p, a) \in F_w$ means that the continuous firing of a makes the number of tokens in p become zero under the marking μ obtained by reading a positive firing sequence w. We denote the set of all such pairs (p, a) by F^*, that is $F^* \stackrel{\text{def}}{=} \cup_{w \in X^*} F_w$. F^* is called the active flow of PN.

A $(p, a) \in F^*$ means that p is a place where the number of tokens first becomes zero when a fires continuously after reading a positive firing sequence w.

LEMMA 2.3 (Fundamental Lemma) *Let F^* be an active flow of a Petri net (P, X, W, μ_0), $C = C(P, X, W, \mu_0)$ be a maximal Petri net code. Let $p \in P$ and $a, b \in X$.*

(i) $(p, a) \in F^* \Rightarrow W(p, a) \geq W(p, b)$,

(ii) $(p, a), (p, b) \in F^* \Rightarrow W(p, a) = W(p, b)$.

(Proof) (i) There exists some non-negative integer n such that $a^{n+1} \in C$ and $\delta(\mu_0, a^n) = W(p, a)$ because $(p, a) \in F^*$. Then by the maximality of C each transition $b \in X$ must be enable. Therefore $W(p, a) \geq W(p, b)$.

(ii) Since $W(p, a) \geq W(p, b)$ and $W(p, b) \geq W(p, a)$ hold by (1), the equality $W(p, a) = W(p, b)$ is true. □

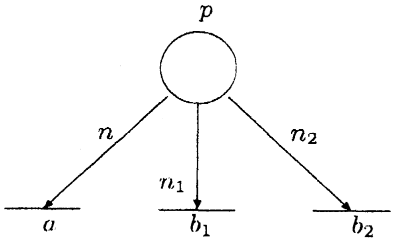

Fig. 1: $(p, a) \in F_* \Rightarrow n \geq n_1, n_2$.

LEMMA 2.4 (Deletion of useless places) *Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. Let $C = C(P, X, W, \mu_0)$ be a maximal Petri net code. Let $p \in P$ be a place such that $\delta(\mu_0, w)(p) \neq 0$ for any $w \in C$. And the Petri net $PN' = (P', X', W', \mu_0')$ is defined as follows, which is obtained by removing the place p and the arcs from p and the arcs to p.*

$P' = P \setminus \{p\}, X' = X$
W' is a restriction of W on $(P' \times X) \cup (X \times P')$,
μ_0' is a restriction of μ_0 on P',

Then,

$$C(P, X, W, \mu_0) = C(P', X', W', \mu_0').$$

□

We called such a place in the lemma a **useless place** in PN. Generally set $P_0 = \{q \in P \mid \exists w \in C, \delta(\mu_0, w)(q) = 0\}$. Applying the above theorem repeatedly, the theorem holds even if we replace P' in the theorem by P_0. The maximality in the theorem is needed as the following example shows.

EXAMPLE 2.1 *Let $P = \{p, q\}$, $X = \{a, b\}$, $W(p, a) = W(p, b) = 1, W(q, b) = 2$, $\mu_0(p) = \mu_0(q) = 1$. The other arcs weigh 0. Then $C = C(P, X, W, \mu_0) = \{a\}$ is not maximal. For any $w \in C$, $\delta(\mu_0, w)(q) \neq 0$, where δ is the transition function of (P, X, W, μ_0). However, Since $P' = P \setminus \{q\} = \{p\}$, $X' = \{a, b\}$, $W'(p, a) = W'(p, b) = 1$, $\mu_0'(p) = 1$, the other arcs weigh 0, $C' = C(P', X', W', \mu_0') = \{a, b\}$. This means that $C = C'$ does not necessarily hold.* □

By the next proposition 2.3, It is decidable whether a place in a given Petri net is a useless place or not. We need the old famous result on the reachability of a Petri net to show this decidability. Next two definitions are old famous decision problems. In case of considering a Petri net code, it is important to judge whether $p \in P$ is one of the places where tokens can be exhausted first. So we suggest the decision problem in the third definition.

DEFINITION 2.2 (The Reachability Problem) *For a given Petri net $PN = (P, X, W, \mu_0)$ and a given marking μ, is $\mu \in$ **Re**(PN) ?* □

DEFINITION 2.3 (The Single-Place Zero-Reachability Problem) *For a given Petri net $PN = (P, X, W, \mu_0)$ and a given place $p \in P$, does there exist $\mu \in$ **Re**(PN) with $\mu(p) = 0$?* □

DEFINITION 2.4 (The Single-Place First Zero-Reachability Problem) *For a give Petri net $PN = (P, T, W, \mu_0)$ and a given place $p \in P$, let δ be the transition function of PN, then does there exist $w \in X^*$ such that $\delta(\mu_0, w)(p) = 0$ and $\delta(\mu_0, w')(q) > 0$ for $\forall w' \in P_r(w) \setminus \{w\}, \forall q \in P$?* □

Fact 2.1 (1) *The reachability problem and the single-place zero-reachability problem are equivalent.* [7]

(2) *The reachability problem is decidable.*[8],[9] *Any algorithm to solve the problem require at least an exponential amount of space for storage and an exponential amount of time.* [10] □

PROPOSITION 2.3 [11] *The single-place first zero-reachability problem and the single-place zero-reachability problem are equivalent, that is, decidable.* □

LEMMA 2.5 (Reduction rule of two-way arcs) *Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. Let $C = C(P, X, W, \mu_0)$ be a maximal Petri net code. Let $p \in P$, $a \in X$ with $W(p, a) > 0$ and $W(a, p) > 0$. Then the Petri net $PN' = (P, X, W', \mu_0)$ is defined as follows, which is obtained by replacing the weights of the two arcs (p, a) and (a, p).*

$$\begin{array}{ll}
W(p,a) > W(a,p) & \Rightarrow W'(p,a) = W(p,a) - W(a,p),\ W'(a,p) = 0 \\
W(p,a) = W(a,p) & \Rightarrow W'(p,a) = W'(a,p) = 0 \\
W(p,a) < W(a,p) & \Rightarrow W'(a,p) = W(a,p) - W(p,a),\ W'(p,a) = 0 \\
q \neq p \text{ or } b \neq a & \Rightarrow W'(b,q) = W(b,q), W'(q,b) = W(q,b)
\end{array}$$

Then

$$C(P, X, W, \mu_0) = C(P, X, W', \mu_0).$$

□

3 Maximal Petri net codes and input-ordinal Petri net code

Here we solve the problem whether **mCPN⊂NmCPN** holds or not under some conditions.

In a Petri net $PN = (P, X, W, \mu_0)$, for a transition $a \in X$, set $I(a) = \{p \in P | W(p, a) > 0\}$ and $O(a) = \{p \in P | W(a, p) > 0\}$. If $I(a) \neq \emptyset$ and $O(a) = \emptyset$, then a is called a consuming transition. If $I(a) \neq \emptyset$ and $O(a) \neq \emptyset$, then a is called a transporting transition. If $I(a) = \emptyset$ and $O(a) \neq \emptyset$, then a is called a supplying transition. If $I(a) = O(a) = \emptyset$, then a is called an isolated transition.

Through this section, without the loss of generality we may assume that a Petri net $PN = (P, X, W, \mu_0)$ with a positive marking μ_0 satisfies the following conditions. Such a Petri net is called a **slim maximal Petri net**.

(i) $C(P, X, W, \mu_0)$ is a maximal Petri net code.

(ii) By lemma 2.4, there is no useless place in PN.

(iii) By lemma 2.5, for any $p \in P$ and any $a \in X$, both the weight of (a, p) and the weight of (p, a) are not positive.

(iv) PN has no isolated transitions.

3.1 Case of $|P| \leq 2$ or $|X| = 1$

At first we consider the case the number $|P|$ of places equals 1 and the case the number $|X|$ of transitions equals 1.

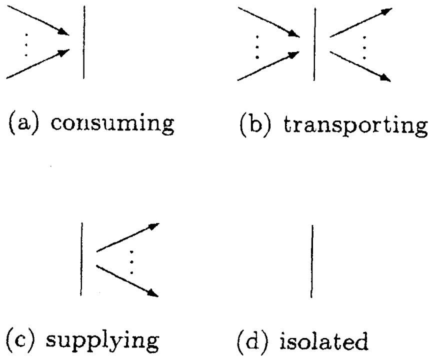

Fig. 2: Classification of transitions

THEOREM 3.1 *Let $PN = (P, X, W, \mu_0)$ be a Petri net and μ_0 be a positive marking. Assume that $|X| = 1$ or $|P| = 1$. If $C = C(P, X, W, \mu_0)$ is a maximal Petri net code, then C is an input-ordinal Petri net code.* □

Assume that $|P| = 1$, that is $P = \{p\}$ in this theorem. Setting $X_1 = \{a \in X | W(p, a) > 0, W(a, p) = 0\}$ and $X_2 = X - X_1$, Then

$$C(P, X, W, \mu_0) = ({X_1}^{n-1} \diamond (\bigcup_{a_i \in X_2} a_i {X_1}^{n_i})^{\diamond}) X_1, \tag{1}$$

where $n_i = W(a_i, p)/n$, $\diamond$ is the shuffle product over two languages L, $K \subset X^*$ defined by $L \diamond K = \cup_{x \in L, y \in K} x \diamond y$, $x \diamond y = \{x_1 y_1 x_2 y_2 \cdots x_n y_n | \, x = x_1 x_2 \cdots x_n, \ y = y_1 y_2 \cdots y_n, \ x_i, y_i \in X^* \text{ for } 1 \le i \le n\}$ for $x, y \in X^*$ and $L^{\diamond}$ is the shuffle closure of a language L, defined by $L^{\diamond} = \cup_{i \ge 0} L^{\diamond\, i}$, $L^{\diamond\, 0} = \{1\}$, $L^{\diamond (i+1)} = L^{\diamond\, i} \diamond L$.

Especially, in the above example setting $X_1 = \emptyset$ and $X_2 = X$, $C = X^k = \{w \in X^* | \, |w| = k\}$. X^k is called a (full) uniform code over X. Therefore a uniform code becomes an input-ordinal Petri net code.

In case that a Petri net has only a place or only a transition, we have proven **NmCPN=mCPN**. we get the following result in the case that a Petri net has two places. The first proposition is for the case without supplying transitions, the second is for the case with with supplying transitions.

PROPOSITION 3.1 *Let $PN = (P, X, W, \mu_0)$ be a Petri net without supplying transitions, μ_0 be positive and $|P| = 2$. If $C = C(P, X, W, \mu_0)$ is a maximal Petri net code, then C is an input-ordinal Petri net code.* □

PROPOSITION 3.2 *Let $PN = (P, X, W, \mu_0)$ be a Petri net with supplying transitions, μ_0 be positive and $|P| = 2$. If $C = C(P, X, W, \mu_0)$ is a maximal Petri net code, then C is an input-ordinal Petri net code.* □

We obtain the final result of this paper from the theorems 3.1 and 3.2.

THEOREM 3.2 *Let $PN = (P, X, W, \mu_0)$ be a Petri net, μ_0 be positive and $|P| = 2$. If $C = C(P, X, W, \mu_0)$ is a maximal Petri net code, then C is an input-ordinal Petri net code.* □

4 Some results in case of $|P| > 2$

In this section, first we introduce some notation and define some dependency of places in a Petri net. Secondly we invastigate the maximality of a specil type of Petri net.

4.1 Place dependency

We introduce the following notations:

DEFINITION 4.1 *Let $a \in X, w \in X^*, U \subset X$,and $\Omega \subset 2^X$. Then, The number $C_\Omega(w)$ is defined as follows:*

$$|w|_U \stackrel{\text{def}}{=} \sum_{a \in U} |w|_a \ (\leq |w|)$$

$$C_\Omega(w) \stackrel{\text{def}}{=} \max\{|w|_U \mid U \in \Omega\},$$

where $|w|_a$ means the number of occurences of a letter a in w. □

Then the following properties hold:

(1) $0 \leq C_\Omega(w) \leq |w|$.

(2) Let $k \geq 1$ and $L_{\Omega,k} = \{w \in X^* \mid C_\Omega(w) = k\} \neq \emptyset$.
$L_{\Omega,k}$ is commutative and regular.
$L_{\Omega,k}$ is finite $\iff X = \bigcup \Omega$.
$X \in \Omega \Rightarrow L_{\Omega,k} = X^k$. The converse is not necessarily true.

(3) $L_{\Omega,k} \cap X^l = \emptyset$ for any $\Omega \subset 2^X$ if $k > l$.

(4) $L_{\Omega,k} \cap X^k = X^k$ iff

$$k \geq |X| \Rightarrow X \in \Omega$$

$$k < |X| \Rightarrow (\forall w \in X^*)(|w| = k \Rightarrow \exists U \in \Omega(\text{alph}(w) \subset U)).$$

alph(w) means the number of the distinct letters in w. The equivalence is due to [1].

EXAMPLE 4.1 *Let $X = \{a, b, c\}$ and $\Omega = \{p^\cdot, q^\cdot, r^\cdot\} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Then $L_{\Omega,2} \cap X^2 = X^2$. But $L_{\Omega,3} \cap X^3 \neq X^3$ because $abc \notin L_{\Omega,3}$.* □

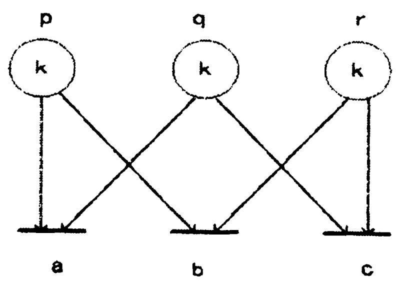

Fig. 3: A Petri net generating $L_{\Omega,k}$.

Now we introduce some notations to prepare to describe the following lemmas.

NOTATIONS Let $PN = (P, X, W, \mu_0)$ be a PN and μ_0 be a positive marking. Let $p \in P$ be a place.

$\alpha(p) \stackrel{\text{def}}{=} \max\{W(p,a) \mid a \in X\}$,
$p\cdot \stackrel{\text{def}}{=} \{a \in X \mid W(p,a) > 0\}$,
$p\star \stackrel{\text{def}}{=} \{a \in X \mid W(p,a) = \alpha(p) > 0\} (\subset p\cdot)$,
$\mathtt{1st}(F_w) \stackrel{\text{def}}{=} \{p \in P \mid (p,a) \in F_w\}$,
$\#_w(p) \stackrel{\text{def}}{=} \mu(p)/\alpha(p)$ and $\mu = \delta(\mu_0, w)$.

In only a slim maximal Petri net, the following fandamental lemmas are true.

LEMMA 4.1 *Let (P, X, W, μ_0) be a slim maximal PN. Let $p \in P$ and $a \in X$. Then,*

$$(p,a) \in F^* \iff W(p,a) = \alpha(p) > 0.$$

□

LEMMA 4.2 *Let (P, X, W, μ_0) be a slim maximal PN. Let $p \in P$ and $b \in X$. If $p \in \mathtt{1st}(F_w)$ for some $w \in X^*$ and $0 < W(p,b) < \alpha(p)$ hold, there exists $q \in \mathtt{1st}(F_w)$ satisfying the following* (i) *or* (ii)*:*

(i) $W(q,b) = \alpha(q) \wedge$
$\quad \#_w(q) = 1$ *for* $\forall w \in X^*$ *with* $p \in \mathtt{1st}(F_w)$,
(ii) $W(q,b) = W(q,a) = \alpha(q) \wedge$
$\quad \#_w(p) = \#_w(q)$ *for* $\forall w \in X^*$ *with* $p \in \mathtt{1st}(F_w)$. □

If (i) (or (ii)) holds, It is said that p **strongly (or weakly) depends** on q, we write $p \rhd_{\mathrm{S}} q$ (or $p \rhd_{\mathrm{W}} q$).

EXAMPLE 4.2 *In the Petri net in Fig. 4, $p \rhd_{\mathrm{S}} q$, $\#_w(p) = \#_w(q) = 1$:*

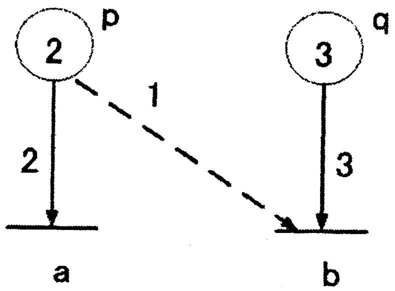

Fig. 4: Explanation of the dependency $\rhd_{\mathrm{S}}$.

LEMMA 4.3 *Let (P, X, W, μ_0) be a slim maximal PN code. Let $p, q, r \in P$.*
(i) *$\rhd_{\mathrm{S}}$ is transitive ($\rhd_{\mathrm{W}}$ is not necessarily transitive).*
(ii) *$p \rhd_{\mathrm{W}} q$ and $q \rhd_{\mathrm{W}} p$ are incompatible.*
(iii) *$p \rhd_{\mathrm{S}} q$ and $q \rhd_{\mathrm{W}} r \Rightarrow q \rhd_{\mathrm{S}} r$.* □

4.2 Petri net of the special type

We define a Petri net of the special type.

DEFINITION 4.2 *A Petri net (P, X, W, μ_0) is called to be* **of type D** *if it satisfies:*
(i) *$P = \{p\} + Q$ and $X = Z + Y$.*

(ii) $W(p,a) = \alpha(p)$, $0 < W(p,b) < \alpha(p)$, $W(q,a) = W(q,b) = \alpha(q)$ *for all* $q \in Q$, $a \in Z$ *and* $b \in Y$.

(iii) $\mu_0(p) = k\alpha(p)$ *and* $\mu_0(q) = k\alpha(q)$ *for all* $q \in Q$.

We denote this Petri net by $(p; Q, Z; Y, W, k)$ *and the code it generates by* $C(p; Q, Z; Y, W, k)$, *where* $Q = \{q_1, q_2, \ldots, q_n\}$, $Z = \{a_1, a_2, \ldots, a_s\}$ *and* $Y = \{b_1, b_2, \ldots, b_t\}$. □

EXAMPLE 4.3 *The following figure is an example of a Petri net* $(p; Q, Z; Y, W, k)$ *of type D. For each* $q_i \in Q = \{q_1, q_2, \ldots, q_n\}$, $p \rhd_W q_i$ *holds there.* □

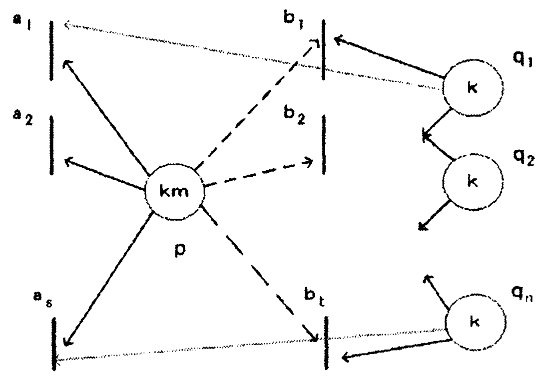

Fig. 5: A Petri net of type D with parameter k.

Let $w = a_1 a_2 \ldots a_n \in X^*$, $a_i \in X (1 \le i \le n)$.

$$\pi_p(w) \stackrel{\text{def}}{=} \sum_{i=1}^{n} W(p, a_i).$$

$\pi_p : (X^*, \cdot) \mapsto (\{0, 1, 2, \ldots\}, +)$ is the monoid morphism. The next lemma is the main result of this paper.

LEMMA 4.4 *Let* $(p; Q, Z; Y, W, k)$ *be a Petri net of type D. If* $C(p; Q, Z; Y, W, k)$ *is a maximal prefix code, the condition:*

$$\text{For all } w \in X^*,\ \pi_p(w) \ge (k-1)m + 1 \Rightarrow C_\Omega(w) \ge k$$

holds, where $m = \alpha(p)$, $X = Y \cup Z$ *and* $\Omega = \{Z\} \cup \{\cdot q \mid q \in Q\}$. *The converse is also true.* □

EXAMPLE 4.4 *Let* C_k *be the Petri net code with the number* k *of tokens in* p *in Fig.6. If* $k = 2$, *then* $C_k = \{a, a', b\}^2$ *is a maximal code. But If* $k = 3$, *then* C_k *is not a maximal prefix code. Because* $\pi_p(aa'b) = 5 > (3-1)2 + 1$ *but* $C_{\cdot p}(aa'b) = C_{\cdot q}(aa'b) = C_{\cdot q'}(aa'b) = 2 < k = 3$ □

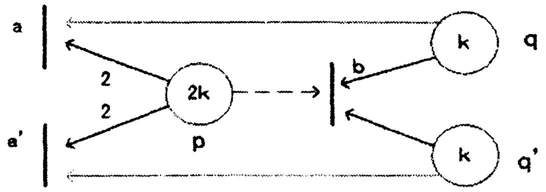

Fig. 6: Petri net of type D with parameter k.

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