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Gröbner bases on projective bimodules and the Hochschild cohomology *

Part III. Syzygies

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This is a continuation of the previous papers [2] and [3]. We develop the theory of Gröbner bases on projective modules over an algebra based on a well-ordered semigroup. We discuss syzygy modules on projective modules and construct generators of the modules in terms of Gröbner bases. The results can be used to compute the intersection of given submodules (see [1] for the polynomial algebra case).

9 Derivation graphs

Let $S = B \cup \{0\}$ be a well-ordered reflexive semigroup with 0 and K be a commutative ring with 1. Let $F = K \cdot B$ be the K-algebra based on B and let I be a (two-sided) ideal of F. Let $A = F/I$ be the quotient algebra of F by I and $\rho : F \to A$ be the natural surjection. We fix a reduced Gröbner basis G of I. For $f \in F$, $\hat{f}$ denotes the normal form of f modulo G.

Let X be an left edged set and $F \cdot X$ be the projective left F-module generated by X. Let T be a (not necessarily complete) rewriting system on $F \cdot X$. Set

$$H = H_T = \{ s - t \mid s \to t \in T \}.$$

The set H is assumed to be uniform and is considered to be a left edged set; for an element h in H, $\sigma(h)$ is defined by $\sigma(h) = \sigma(x)$, where $\mathrm{lt}(h) = x \cdot \xi$ $(x \in B, \xi \in X)$. We consider the projective left F-module $F \cdot H$ generated by H. For $h \in H$, $[h]$ denotes the formal generator of $F \cdot H$ corresponding to $h \in H$. An element f of $F \cdot H$ is written as a finite sum

$$f = \sum k_i x_i [h_i] \tag{9.1}$$

with $k_i \in K \backslash \{0\}, h_i \in H, x_i \in B_{\sigma(h_i)}$.

We define a graph $\mathcal{D} = \mathcal{D}(T, G)$ called the *derivation graph* associated to T and G as follows. The set of vertices is the projective F-module $F \cdot X$ and for

*This is a preliminary report and the details appear elsewhere

$f, g \in F \cdot X$ an (positive) edge e from the source $f = \sigma(e)$ to the target $g = \tau(e)$ is a one-step (T, G)-reduction from f to g, that is, f has a term $k \cdot x\xi$ with $k \in K \backslash 0$, $x \in B_{\sigma(\xi)}$, $\xi \in X$, and

(i) $x \to_G x'$ and $g = f + k \cdot (x' - x)\xi$, or

(ii) $x = x'z, h = z\xi - t \in H$ and $g = f + k \cdot x'(t - z\xi)$.

In case (i), e is called a G-edge, and in case (ii), e is called a T-edge (or an H-edge). The *label* of the T-edge e in (ii) is the element $k \cdot x'[h]$ of $F \cdot H$. For an edge e from f to g, we have the reverse (negative) edge e^{-1} from $\sigma(e^{-1}) = g$ to and $\tau(e^{-1}) = f$. The label of the reverse e^{-1} of the T-edge e in case (ii) above is $-k \cdot x'[h]$. A path p in $\mathcal{D}$ is a concatenation

$$p = e_1 \circ e_2 \circ \cdots \circ e_n \tag{9.2}$$

of (positive or negative) edges e_i with $\tau(e_i) = \sigma(e_{i+1})$ for $i = 1, ..., n-1$. The path p is *positive* if all the edges e_i in p are positive. Define the source $\sigma(p)$ and the target $\tau(p)$ of p by $\sigma(p) = \sigma(e_1)$ and $\tau(p) = \tau(e_n)$ respectively. Here, p is closed if $\sigma(p) = \tau(p)$. For two paths p and q such that $\tau(p) = \sigma(q)$, we have a path $p \circ q$ which is a concatenation of p and q at $\tau(p) = \sigma(q)$.

We define a mapping $\int$ from the set of all paths in $\mathcal{D}$ to $F \cdot H$ as follows. Let p be a path given in (9.2). If p is trivial, that is, $n = 0$, then $\int(p) = 0$. If $n \geq 1$, let $p' = e_2 \circ \cdots \circ e_n$. If e_1 is a G-edge, then $\int(p) = \int(p')$. If e_1 is a T-edge with label $k \cdot x'[h]$, then

$$\int(p) = k \cdot x[h] + \int(p').$$

Thus $\int$ sums up all the labels of T-edges in p. We also define a mapping d called the *boundary mapping* from the set of paths to the projective module $F \cdot X$ by

$$d(p) = \sigma(p) - \tau(p).$$

It is easy to see that for paths p and q with $\tau(p) = \sigma(q)$,

$$\int(p \circ q) = \int(p) + \int(q)$$

and

$$d(p \circ q) = d(p) + d(q).$$

Let δ be a morphism of left F-modules from $F \cdot H$ to $F \cdot X$ defined by

$$\delta([h]) = h$$

for $h \in H$.

Proposition 9.1. *We have*

$$\delta \circ \int(p) \equiv d(p) \pmod{G} \tag{9.3}$$

for any path p in $\mathcal{D}$. In particular,

$$\delta \circ \int(p) \equiv 0 \pmod{G},$$

for a closed path p in $\mathcal{D}$.

Let $A \cdot X$ and $A \cdot H$ be the projective left A-modules generated by X and H, respectively. We consider a morphism $\partial : A \cdot H \to A \cdot X$ of left A-modules by $\partial([h]) = \rho_X(h)$ for $h \in H$. Then we have a commutative diagram

$$\begin{array}{ccc} F \cdot H & \xrightarrow{\delta} & F \cdot X \\ \downarrow \rho_H & & \downarrow \rho_X \\ A \cdot H & \xrightarrow{\partial} & A \cdot X, \end{array} \tag{9.4}$$

where ρ_X and ρ_H are the canonical surjections. Clearly we have $\mathrm{Im}(\delta) = L(H)$ and $\mathrm{Im}(\partial) = L_A(H)$. Set $\overline{\int} = \rho_X \circ \int$ and $\overline{d} = \rho_H \circ d$, which are mappings from the set of paths to $A \cdot X$ and to $A \cdot H$ respectively.

Corollary 9.2. *We have*

$$\partial \circ \overline{\int}(p) = \overline{d}(p)$$

in $A \cdot X$ for any path p in $\mathcal{D}$. In particular,

$$\partial \circ \overline{\int}(p) = 0$$

for any closed path p in $\mathcal{D}$.

10 Standard reductions and the linear map $\int$

Suppose that a rule $x \cdot \xi \to t \in T$ is applied to a term $ky \cdot \xi$ of $f \in F \cdot X$, where $y = y'x$, and we have $f \to_T f - ky'(x \cdot \xi - t)$. If $(y', x\xi)$ is an leftmost (resp. rightmost) appearance of $\mathrm{Left}(T)$ in $y \cdot \xi$, the application is *leftmost* (resp. rightmost). Since T is reduced only one rule can be applied to $y \cdot \xi$ at the leftmost (rightmost) position.

A positive path

$$f_1 \to_{T,G} f_2 \to_{T,G} \cdots \to_{T,G} f_n \tag{10.1}$$

in $\mathcal{D}$ is *standard*, if for every $i = 1, ..., n-1$,

(i) when f_i is G-reducible, $f_i \to_{T,G} f_{i+1}$ is a G-edge, and

(ii) when f_i is G-irreducible, the edge $f_i \to_{T,G} f_{i+1}$ is by a leftmost application of a rule from T to the greatest T-reducible term of f_i with respect to $\succ$, that is, $x \cdot \xi \to t \in T$, $x \in \Sigma^*$, $k \in K \setminus \{0\}$, $k \cdot zx \cdot \xi$ is the greatest T-reducible term of f_i and no rule $x' \cdot \xi \to t'$ in T can be applied to $zx \cdot \xi$ so that $x' \cdot \xi$ appears at the left of $x \cdot \xi$.

If f_1 is reduced to f_n through a standard reduction as above, we write as $f_1 \Rightarrow^*_{T,G} f_n$. A standard one-step reduction by a rule from T is denoted by $\Rightarrow_T$, that is, $f \Rightarrow_T g$ if f is G-irreducible and g is obtained by a leftmost application of a rule of T to the greatest T-reducible term of f.

Since $\to_G$ is complete, if f_n is G-irreducible, the standard reduction (10.1) can be rewritten as

$$f_1 = g_1 \to^*_G \hat{g}_1 \Rightarrow_T g_2 \to^*_G \hat{g}_2 \Rightarrow_T \cdots \Rightarrow_T g_m \to_G \hat{g}_m = f_n, \tag{10.2}$$

where $\hat{g}_i$ is the G-normal form of g_i Since T is reduced, in the step $\hat{g}_i \Rightarrow_T g_{i+1}$ in the above reduction sequence, only one rule from T is applicable to the greatest T-reducible term of $\hat{g}_i$ at a unique leftmost position. In this sense, a standard reduction from f to f_n is unique. In particular, if f_n is (T, G)-irreducible, it is unique. This unique element f_n is called the *standard form* of f, denoted by f^s. If T is complete modulo G, f^s coincides with the normal form $\tilde{f}$ of f.

Proposition 10.1. *Let $f, f', g, g' \in F \cdot X \cdot F$, $k, \ell \in K$ and assume that there are standard reductions $f \Rightarrow^*_{T,G} f'$ and $g \Rightarrow^*_{T,G} g'$.*

(1) There is a standard reduction

$$k \cdot f + \ell \cdot g \Rightarrow^*_{T,G} k \cdot f' + \ell \cdot g'.$$

(2) If f' and g' are the standard forms of f and g respectively, then $k \cdot f' + \ell \cdot g'$ is the standard form of $k \cdot f + \ell \cdot g$;

$$(k \cdot f + \ell \cdot g)^s = k \cdot f^s + \ell \cdot g^s.$$

Now we define a K-linear map $\int = \int_H : F \cdot X \to F \cdot H$ by

$$\int(f) = \int(p(f))$$

for $f \in F \cdot X$, where $p(f)$ is a standard path from f to the standard form f^s. The reader should not be confused by using the same symbol $\int$ for the mapping from the module $F \cdot X$ and for the mapping from the set of paths in $\mathcal{D}$. Clearly, $\int(f)$ does not depend on the choice of the standard reduction of f to f^s. So we can choose the standard reduction

$$p : f = g_1 \to^*_G \hat{g}_1 \Rightarrow_T g_2 \to^*_G \hat{g} \Rightarrow_T \cdots \Rightarrow_T g_m \to_G \hat{g}_m = f^s \tag{10.3}$$

like (10.2) to define $\int(f)$. Precisely, for $f \in F \cdot X$ let $\pi(f)$ be the unique standard path p given in (10.3), then

$$\int(f) = \int(\pi(f)).$$

Proposition 10.2. *(1) $\int(f) = \int(\hat{f})$ for $f \in F \cdot X$, where $\hat{f}$ is the normal form of f with respect to G.*

(2) $\int$ is a morphism of K-modules, that is,

$$\int(k_1 f_1 + k_2 f_2) = k_1 \int(f_1) + k_2 \int(f_2)$$

for $k_1, k_2 \in K$ and $f_1, f_2 \in F \cdot X$.

By the definition, we have

$$d(\pi(f)) = f - f^s.$$

Thus, by Proposition 9.1 we obtain

Proposition 10.3. *For $f \in F \cdot X$,*

$$\delta \circ \int(f) \equiv f - f^s \pmod{G},$$

in $F \cdot X$.

The assertion (1) in Proposition 10.2 means that $\int(f) = \int(g)$ follows from $\rho_X(f) = \rho_X(g)$. Thus, $\int$ induced a K-linear map $\int' : A \cdot X \to F \cdot H$ such that $\int = \int' \circ \rho_X$. The composition $\overline{\int} = \overline{\int}_H = \rho_H \circ \int'$ with the surjection ρ_H is a K-linear map from $A \cdot X$ to $A \cdot H$. Thus,

Proposition 10.4. *The K-linear map $\int$ induces a K-linear map $\overline{\int} : A \cdot X \to A \cdot H$ and we have a commutative diagram*

$$\begin{array}{ccc} F \cdot X & \xrightarrow{\int} & F \cdot H \\ \rho_X \downarrow & & \downarrow \rho_H \\ A \cdot X & \xrightarrow{\overline{\int}} & A \cdot H. \end{array}$$

Since $\rho_X(f) = \rho_X(g)$ if and only if $\hat{f} = \hat{g}$, we sometimes regard a G-irreducible element of $F \cdot X$ as an element of $A \cdot X$. Thus, a G-irreducible element f and its standard form f^s, which is also G-irreducible, are considered to be an element of $A \cdot X$. With this convention, Proposition 10.3 means

Corollary 10.5. *For $f \in A \cdot X$ we have*

$$\partial \circ \overline{\int}(f) = f - f^s.$$

11 Cycles made from critical pairs and z-elements

Let $\mathcal{K}$ be the kernel of the morphism $\rho_X \circ \delta = \partial \circ \rho_H$ in (9.4). We are interested in finding generators of $\mathcal{K}$.

Let $h = x \cdot \xi - t$, $h = x' \cdot \xi - t'$ $(t, t' \in F \cdot X,\ \xi \in X,\ x, x' \in_{\tau(\xi)} B)$ be rules in H and $u - v$ $(u \in B,\ v \in F)$ be a rule in G.

First, we consider a critical pair of the first kind. Suppose that $zx = z'x' \neq 0$ for some $z, z' \in B$, where the appearance $(z, x \cdot \xi)$ of $x \cdot \xi$ is at the right of the appearance $(z', x' \cdot \xi)$ of $x' \cdot \xi$ in $zx\xi = z'x'\xi$, and z and z' are left coprime. Then we have critical pair

$$(zx \cdot \xi \to_H zt,\ z'x' \cdot \xi \to_H z't')$$

of reductions. For this critical pair define an element c_1 of $F \cdot H$ by

$$c_1 = z[h] - z'[h'] + \int(z \cdot t) - \int(z't'). \tag{11.1}$$

Next, we consider a critical pair of the second kind. Suppose that $zx = z'uz'' \neq 0$ for some $z, z', z'' \in B$, where z is G-irreducible, the appearance

$(z', u, z' \cdot \xi)$ of u and the appearance $(z, x\xi)$ of $x\xi$ in $zx \cdot \xi$ are rightmost and z and z' are left coprime. Then we have a critical pair

$$(zx \cdot \xi \to_H zt,\ z'uz'' \cdot \xi \to_G z'vz'' \cdot \xi)$$

of reductions. For this critical pair we define an element c_2 of $F \cdot H$ by

$$c_2 = z[h] + \int(z \cdot t) - \int(z'vz''\xi). \tag{11.2}$$

Lemma 11.1. *If the critical pair $(zt, z't')$ (resp. $(zt, z'vz''\xi)$) is resolvable, the elements c_1 (resp. c_2) above is in $\mathcal{K}$.*

Consider a z-pair (z, h), that is, $h = x\xi - t \in H$, $z \in B$ and $zx = 0$. We have an z-element zt and for this z-pair we define an element c_3 of $F \cdot H$ by

$$c_3 = z[h] + \int(z \cdot t). \tag{11.3}$$

Lemma 11.2. *If the z-element zt is resolvable, the element c_3 above is in $\mathcal{K}$.*

Let C be the collection of all the elements c_1, c_2 and c_3 above. If H is a Gröbner basis, then all the critical pairs and the z-elements are resolvable (Theorem 7.2). Hence, C is contained in $\mathcal{K}$ by Lemmas 11.1 and 11.2,. More strongly we have

Theorem 11.3. *If H is a Gröbner basis, C generates $\mathcal{K}$.*

12 Syzygies

Let Y be a left edged set and let $\mathbf{h} = (h_\eta)_{\eta \in Y}$ be a sequence of left uniform elements of $F \cdot X$ indexed by Y with $\sigma(h_\eta) = \sigma(\eta)$. A sequence $\mathbf{f} = (f_\eta)_{\eta \in Y}$ of right uniform elements of F with $\tau(f_\eta) = \sigma(\eta)$ is a *syzygy* of $\mathbf{h}$ modulo G if $f_\eta = 0$ for all but a finite number of η in Y, and

$$\sum_{\eta \in Y} f_\eta h_\eta = 0$$

in $A \cdot X$. The set $\mathrm{Syz}(\mathbf{h})$ of all syzygies of $\mathbf{h}$ forms a submodule of the projective left F-module $F \cdot Y$ generated by Y. We call it the *syzygy module* of $\mathbf{h}$. Let $\delta : F \cdot Y \to F \cdot X$ be the morphism defined by

$$\delta(\eta) = h_\eta$$

for $\eta \in Y$. Then, $\mathrm{Syz}(\mathbf{h})$ is nothing but the kernel of the morphism $\rho_X \circ \delta : F \cdot Y \to A \cdot X$.

Let $\partial : A \cdot Y \to A \cdot X$ be the morphism defined by

$$\partial(\eta) = \rho_X(h_\eta),$$

then we have a commutative diagram

$$\begin{array}{ccc} F\cdot Y & \xrightarrow{\delta} & F\cdot X \\ {\scriptstyle \rho_Y}\downarrow & & \downarrow{\scriptstyle \rho_X} \\ A\cdot Y & \xrightarrow{\partial} & A\cdot X. \end{array}$$

Thus, $\mathrm{Ker}(\partial) = \rho_Y(\mathrm{Syz}(\mathbf{h}))$.

By Theorem 11.3 we have

Theorem 12.1. *If $H = \{h_\eta \mid \eta \in Y\}$ forms a Gröbner basis on $F \cdot X$, the syzygy module Syz(**h**) is generated modulo G by the set C of elements (11.1), (11.2) and (11.3) made from the critical pairs and the z-pairs with respect to H and G.*

If H is not a Gröbner basis, then we take a Gröbner basis $\overline{H} = \{h_{\overline{\eta}} \mid \overline{\eta} \in \overline{Y}\}$ of the submonoid generated by H modulo G. We may apply the completion procedure to obtain $\overline{H}$. We have a morphism $\overline{\delta} : F \cdot \overline{Y} \to F \cdot X$ defined by $\overline{\delta}(\overline{\eta}) = h_{\overline{\eta}}$ for $\overline{\eta} \in \overline{Y}$. Then, $\mathrm{Im}(\delta) = \mathrm{Im}(\overline{\delta})$. Since $F \cdot \overline{Y}$ and $F \cdot Y$ are projective left F-modules we have a morphisms $\phi : F \cdot \overline{Y} \to F \cdot Y$ and $\psi : F \cdot Y \to F \cdot \overline{Y}$ such that $\overline{\delta} = \delta \circ \phi$ and $\delta = \overline{\delta} \circ \psi$;

$$\begin{array}{ccc} F\cdot Y & \xrightarrow{\delta} & F\cdot X \\ {\scriptstyle \phi}\uparrow\downarrow{\scriptstyle \psi} & & \| \\ F\cdot \overline{Y} & \xrightarrow{\overline{\delta}} & F\cdot X. \end{array}$$

Let $C = \{c_\zeta \mid \zeta \in Z\}$ be the set of elements of $F \cdot Y$ made from critical pairs and z-pairs with respect to $\overline{H}$ and G. We have

Theorem 12.2. *The set $\phi(C) \cup \{\eta - \phi(\psi(\eta)) \mid \eta \in Y\}$ generates Syz(**h**) modulo G.*

Suppose that c_ζ $(\zeta \in Z)$, $h_{\overline{\eta}}$ $(\overline{\eta} \in \overline{Y})$ and h_η $(\eta \in Y)$ are written as

$$c_\zeta = \sum_{\overline{\eta}\in\overline{Y}} z_\zeta^{\overline{\eta}} \cdot \overline{\eta}, \quad h_{\overline{\eta}} = \sum_{\eta\in Y} x_{\overline{\eta}}^{\eta} \cdot h_\eta, \quad h_\eta = \sum_{\overline{\eta}\in\overline{Y}} y_\eta^{\overline{\eta}} \cdot h_{\overline{\eta}}$$

with $z_\zeta^{\overline{\eta}}$, $x_{\overline{\eta}}^{\eta}$, $y_\eta^{\overline{\eta}} \in F$. In this situation we have

Corollary 12.3. *Syz(**h**) is generated by the elements*

$$\sum_{\overline{\eta}\in\overline{Y},\eta\in Y} z_\zeta^{\overline{\eta}} x_{\overline{\eta}}^{\eta} \cdot \eta \quad (\zeta \in Z)$$

and

$$\eta - \sum_{\eta'\in Y,\overline{\eta}\in\overline{Y}} y_\eta^{\overline{\eta}} x_{\overline{\eta}}^{\eta'} \cdot \eta' \quad (\eta \in Y).$$

13 Intersections of submodules

Let L_1 and L_2 be a submodules of the projective left A-module $A \cdot X$. Suppose that they are generated by $H_1 = \{h_{\eta_1} \mid \eta_1 \in Y_1\}$ and $H_2 = \{h_{\eta_2} \mid \eta_2 \in Y_2\}$, respectively. Define a morphism

$$\partial : A \cdot X \oplus A \cdot Y_1 \oplus A \cdot Y_2 \to A \cdot X \oplus A \cdot X$$

by

$$\partial(\xi, \eta_1, \eta_2) = (\xi + h_{\eta_1}, \xi + h_{\eta_2})$$

for $\xi \in X$, $\eta_1 \in Y_1$ and $\eta_2 \in Y_2$. Let

$$\pi : A \cdot X \oplus A \cdot Y_1 \oplus A \cdot Y_2 \to A \cdot X$$

be the projection onto the first component. Then, $\pi(\mathrm{Ker}(\delta))$ is equal to the intersection $L_1 \cap L_2$.

Let $H = \{h_\eta \mid \eta \in Y\}$ be a Gröbner basis on $F \cdot X$ of $L_1 + L_2$, which could be obtained by completing $H_1 \cup H_2$. Let $C = \{c_\zeta \mid \zeta \in Z\}$ be the set of elements of $F \cdot Y$ made from critical pairs and z-pairs with respect to H. We see that $H' = \{(\xi, \xi) \mid \xi \in X\} \cup \{(h_\eta, 0) \mid \eta \in Y\}$ is a Gröbner basis of $\mathrm{Im}(\partial)$ on $F \cdot X \oplus F \cdot X$ and $(0, C)$ is the set of elements of $F \cdot X \oplus F \cdot Y$ made from critical pairs and z-pairs with respect to H'. We have a morphism

$$\overline{\partial} : A \cdot X \oplus A \cdot Y \to A \cdot X \oplus A \cdot X$$

defined by

$$\overline{\partial}(\xi, \eta) = (\xi + h_\eta, \xi)$$

for $\xi \in X$ and $\eta \in Y$. Then, $\mathrm{Im}(\partial) = \mathrm{Im}(\overline{\partial})$ and we have morphisms $\phi : A \cdot X \oplus A \cdot Y_1 \oplus A \cdot Y_2 \to A \cdot X \oplus A \cdot Y$ and $\psi : A \cdot X \oplus A \cdot Y X \to A \cdot X \oplus A \cdot Y_1 \oplus A \cdot Y_2$ such that $\partial \circ \phi = \overline{\partial}$ and $\overline{\partial} \circ \psi = \partial$. Thus, We have a commutative diagram

$$\begin{array}{ccccc} A \cdot X & \xleftarrow{\pi} & A \cdot X \oplus A \cdot Y_1 \oplus A \cdot Y_2 & \xrightarrow{\partial} & A \cdot X \oplus A \cdot X \\ & & \phi \uparrow \downarrow \psi & & \| \\ & & A \cdot X \oplus A \cdot Y & \xrightarrow{\overline{\partial}} & A \cdot X \oplus A \cdot X. \end{array}$$

Since H is a Gröbner basis of $L_1 + L_2$, we can write as

$$h_\eta = \sum_{\eta' \in Y_1 \cup Y_2} x_\eta^{\eta'} h_{\eta'}$$

with $x_\eta^{\eta'} \in A$ in $A \cdot Y_1 \oplus A \cdot Y_2$ for $\eta \in Y$, and

$$h_{\eta'} = \sum_{\eta \in Y} y_{\eta'}^{\eta} h_\eta$$

with $y^{\eta}_{\eta'} \in A$ for $\eta' \in Y_1 \cup Y_2$ in $A \cdot Y$. Then, ϕ and ψ are given as

$$\phi(\xi, \eta) = (\xi + \sum_{\eta_2 \in Y_2} x^{\eta_2}_{\eta} h_{\eta_2}, \sum_{\eta_1 \in Y_1} x^{\eta_1}_{\eta} \eta_1, - \sum_{\eta_2 \in Y_2} x^{\eta_2}_{\eta} \eta_2)$$

and

$$\psi(\xi, \eta_1, \eta_2) = (\xi + h_{\eta_2}, \sum_{\eta \in Y} ((y^{\eta}_{\eta_1} - y^{\eta}_{\eta_2})\eta).$$

Theorem 13.1. *Under the above situation, let*

$$c_{\zeta} = \sum_{\eta \in Y} z^{\eta}_{\zeta} \eta$$

with $z^{\eta}_{\zeta} \in A$ for $\zeta \in Z$. Then, $L_1 \cap L_2$ is generated by the elements

$$\sum_{\eta \in Y, \eta_2 \in Y_2} z^{\eta}_{\zeta} x^{\eta_2}_{\eta} \cdot h_{\eta_2} \quad (\zeta \in Z),$$

$$\sum_{\eta \in Y, \eta_2 \in Y_2} y^{\eta}_{\eta_1} x^{\eta_2}_{\eta} \cdot h_{\eta_2} \quad (\eta_1 \in Y_1),$$

and

$$h_{\eta_2} - \sum_{\eta \in Y, \eta'_2 \in Y_2} y^{\eta}_{\eta_2} x^{\eta'_2}_{\eta} \cdot h_{\eta'_2} \quad (\eta_2 \in Y_2).$$

Corollary 13.2. *If $H_1 \cup H_2$ forms a Gröbner basis, then $Y = Y_1 \cup Y_2$ and $L_1 \cap L_2$ is generated by the elements*

$$\sum_{\eta_2 \in Y_2} z^{\eta_2}_{\zeta} \cdot h_{\eta_2} \quad (\zeta \in Z)$$

and

$$h_{\eta} \quad (\eta \in Y_1 \cap Y_2).$$

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