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Noncommutative Integrable Systems and Twistor Geometry

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Abstract

We discuss extension of soliton theory and integrable systems to noncommutative spaces, focusing on integrable aspects of noncommutative anti-self-dual Yang-Mills equations. We give exact soliton solutions by means of Bäcklund transformations and clarify the geometrical origin from the viewpoint of twistor theory. In the construction of noncommutative soliton solutions, one kind of noncommutative determinants, quasideterminants, play crucial roles. This is partially based on collaboration with C. R. Gilson and J. J. C. Nimmo (Glasgow).

1 Introduction

Extension of integrable systems and soliton theories to non-commutative (NC) space-times [2] have been studied by many authors for the last couple of years and various kind of integrable-like properties have been revealed [1]. This is partially motivated by recent developments of NC gauge theories on D-branes. In the NC gauge theories, NC extension corresponds to introduction of background magnetic fields and NC solitons are, in some situations, just lower-dimensional D-branes themselves. Hence exact analysis of NC solitons just leads to that of D-branes and various applications to D-brane dynamics have been successful [2]. In this sense, NC solitons plays important roles in NC gauge theories.

Most of NC integrable equations such as NC KdV equations apparently belong not to gauge theories but to scalar theories. However now, it is proved that they can be derived from NC anti-self-dual (ASD) Yang-Mills (YM) equations by reduction [3, 4], which is first conjectured explicitly by the author and K. Toda [5]. (Original commutative one is proposed by R. Ward [6] and hence this conjecture is sometimes called *NC Ward's conjecture.*) Therefore analysis of exact soliton solutions of NC integrable equations could be applied to the corresponding physical situations in the framework of N=2 string theory [7, 8, 9].

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[2]In the present article, the word "NC" always refers to generalization to noncommutative spaces, not to non-abelian and so on.

Furthermore, integrable aspects of ASDYM equation can be understood from a geometrical framework, the *twistor theory*. Via the Ward's conjecture, the twistor theory gives a new geometrical viewpoint into lower-dimensional integrable equations and some classification can be made in such a way. These results are summarized in the book of Mason and Woodhouse elegantly [10].

In this article, we discuss integrable aspects of NC ASDYM equations from the viewpoint of NC twistor theory. In particular, we present Bäcklund transformations for NC ASDYM equations which yields various exact solutions from a simple seed solution. The generated solutions are just NC version of Atiyah-Ward ansatz solutions which include NC instantons (with finite action) and NC non-linear plane waves (with infinite action) and so on. We have found that a kind of NC determinants, the quasideterminants, play crucial roles in construction of solutions. (Brief introduction of quasideterminants are summarized in Appendix A.) We also clarify an origin of the Bäcklund transformations and the NC Atiyah-Ward ansatz solutions in the framework of NC twistor theory, essentially, a NC Riemann-Hilbert problem. These are based on collaboration with Claire Gilson and Jonathan Nimmo [12, 13].

Finally we also give an example of NC Ward's conjecture, reduction of the NC ASDYM equation into the NC KdV equation. The reduced equation actually has integrable-like properties such as infinite conserved quantities, exact N-soliton solutions and so on. These results would lead to a kind of classification of NC integrable equations from a geometrical viewpoint and to applications to the corresponding physical situations and geometry also.

2 NC ASDYM equation

In this section, we review some aspects of NC ASDYM equation and establish notations, and present Bäcklund transformations for the NC ASDYM equation in Yang's form and generate NC Atiyah-Ward ansatz solutions in terms of quasideterminants.

2.1 NC gauge theory

NC spaces are defined by the noncommutativity of the coordinates:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2.1}$$

where $\theta^{\mu\nu}$ are real constants called the *NC parameters*. The NC parameter is anti-symmetric with respect to μ, ν: $\theta^{\nu\mu} = -\theta^{\mu\nu}$ and the rank is even. This relation looks like the canonical commutation relation in quantum mechanics and leads to "space-space uncertainty relation." Hence the singularity which exists on commutative spaces could resolve on NC spaces. This is one of the prominent features of NC field theories and yields various new physical objects such as $U(1)$ instantons.

NC field theories are given by the exchange of ordinary products in the commutative field theories for the star-products and realized as deformed theories from the commutative ones. The ordering of non-linear terms are determined by some additional requirements such as gauge symmetry. The star-product is defined for ordinary fields on commutative spaces. For Euclidean spaces, it is explicitly given by

$$\begin{aligned} f \star g(x) &:= \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{(x')}\partial_\nu^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2), \end{aligned} \tag{2.2}$$

where $\partial_\mu^{(x')} := \partial/\partial x'^\mu$ and so on. This explicit representation is known as the *Moyal product* [14]. The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$, and returns back to the ordinary product in the commutative limit: $\theta^{\mu\nu} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative," that is, $[x^\mu, x^\nu]_\star := x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual, for example, $\partial_\mu \star \partial_\nu = \partial_\mu\partial_\nu$, and the wedge product of $\omega = \omega_\mu(x)dx^\mu$ and $\lambda = \lambda_\nu(x)dx^\nu$ is $\omega_\mu \star \lambda_\nu dx^\mu \wedge dx^\nu$.

NC gauge theories are defined in this way by imposing NC version of gauge symmetry, where the gauge transformation is defined as follows:

$$A_\mu \to g^{-1} \star A_\mu \star g + g^{-1} \star \partial_\mu g, \tag{2.3}$$

where g is an element of the gauge group G. This is sometimes called the *star gauge transformation.* We note that because of noncommutativity, the commutator terms in field strength are always needed even when the gauge group is abelian in order to preserve the star gauge symmetry. This $U(1)$ part of the gauge group actually plays crucial roles in general. We note that because of noncommutativity of matrix elements, cyclic symmetry of traces is broken in general:

$$\mathrm{Tr}\ A \star B \neq \mathrm{Tr}\ B \star A. \tag{2.4}$$

Therefore, gauge invariant quantities becomes hard to define on NC spaces.

2.2 NC ASDYM equation

Let us consider Yang-Mills theories in 4-dimensional NC spaces whose real coordinates of the space are denoted by (x^0, x^1, x^2, x^3), where the gauge group is $GL(N, \mathbb{C})$. Here, we follow the convention in [10].

First, we introduce double null coordinates of 4-dimensional space as follows

$$ds^2 = 2(dzd\tilde{z} - dwd\tilde{w}), \tag{2.5}$$

We can recover various kind of real spaces by putting the corresponding reality conditions on the double null coordinates $z, \tilde{z}, w, \tilde{w}$ as follows:

- Euclidean Space ($\bar{w} = -\tilde{w}; \bar{z} = \tilde{z}$): An example is

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & -(x^2 - ix^3) \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}. \tag{2.6}$$

- Minkowski Space ($\bar{w} = \tilde{w}$; z and $\tilde{z}$ are real.): An example is

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - x^1 \end{pmatrix}. \tag{2.7}$$

- Ultrahyperbolic Space ($\bar{w} = \tilde{w}; \bar{z} = \tilde{z}$): An example is

$$\begin{pmatrix} \tilde{z} & w \\ \tilde{w} & z \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + ix^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - ix^1 \end{pmatrix}. \tag{2.8}$$

For Euclidean and ultrahyperbolic signatures, Hodge dual operator $*$ satisfies $*^2 = 1$ and hence the space of 2-forms β decomposes into the direct sum of eigenvalues of $*$ with eigenvalues ± 1, that is, self-dual (SD) part ($*\beta = \beta$) and anti-self-dual (ASD) part ($*\beta = -\beta$). From now on, we treat these two signatures.

NC ASDYM equation is derived from compatibility condition of the following linear system:

$$\begin{aligned} L \star \Psi &:= (D_w - \lambda D_{\tilde{z}}) \star \Psi = (\partial_w + A_w - \lambda(\partial_{\tilde{z}} + A_{\tilde{z}})) \star \Psi(x; \lambda) = 0, \\ M \star \Psi &:= (D_z - \lambda D_{\tilde{w}}) \star \Psi = (\partial_z + A_z - \lambda(\partial_{\tilde{w}} + A_{\tilde{w}})) \star \Psi(x; \lambda) = 0, \end{aligned} \tag{2.9}$$

where $A_z, A_w, A_{\tilde{z}}, A_{\tilde{w}}$ and $D_z, D_w, D_{\tilde{z}}, D_{\tilde{w}}$ denote gauge fields and covariant derivatives in the Yang-Mills theory, respectively. The constant λ is called the *spectral parameter*. The compatible condition $[L, M]_\star = 0$ gives rise to a quadratic polynomial of λ and each coefficient yields NC ASDYM equations whose explicit representations are as follows:

$$\begin{aligned} F_{wz} &= \partial_w A_z - \partial_z A_w + [A_w, A_z]_\star = 0, \\ F_{\tilde{w}\tilde{z}} &= \partial_{\tilde{w}} A_{\tilde{z}} - \partial_{\tilde{z}} A_{\tilde{w}} + [A_{\tilde{w}}, A_{\tilde{z}}]_\star = 0, \\ F_{z\tilde{z}} - F_{w\tilde{w}} &= \partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z + \partial_{\tilde{w}} A_w - \partial_w A_{\tilde{w}} + [A_z, A_{\tilde{z}}]_\star - [A_w, A_{\tilde{w}}]_\star = 0. \end{aligned} \tag{2.10}$$

2.3 NC Yang's equation and J, K-matrices

Here we discuss the potential forms of NC ASDYM equations such as NC J, K-matrix formalisms and NC Yang's equation, which is already presented by e.g. K. Takasaki [32].

Let us first discuss the *J-matrix formalism* of NC ASDYM equation. The first equation of NC ASDYM equation (2.10) is the compatible condition of $D_z \star h = 0, D_w \star h = 0$, where h is a $N \times N$ matrix. Hence we get

$$A_z = -h_z \star h^{-1}, \quad A_w = -h_w \star h^{-1}, \tag{2.11}$$

where $h_z := \partial h/\partial z,\ h_w := \partial h/\partial w$. Similarly, the second eq. of NC ASDYM equation (2.10) leads to

$$A_{\tilde{z}} = -\tilde{h}_{\tilde{z}} \star \tilde{h}^{-1}, \quad A_{\tilde{w}} = -\tilde{h}_{\tilde{w}} \star \tilde{h}^{-1}, \tag{2.12}$$

where $\tilde{h}$ is also a $N \times N$ matrix. By defining $J = \tilde{h}^{-1} \star h$, the third eq. of NC ASDYM equation (2.10) becomes NC Yang's equation

$$\partial_z(J^{-1} \star \partial_{\tilde{z}} J) - \partial_w(J^{-1} \star \partial_{\tilde{w}} J) = 0. \tag{2.13}$$

3 NC CFYG transformation and NC Atiyah-Ward Ansatz

Here we present a NC version of the Corrigan-Fairlie-Yates-Goddard (CFYG) transformation [50] which leaves NC Yang's equation for $G = GL(2)$ as it is. The NC CFYG transformation generates some class of exact solutions which belong to NC version of the *Atiyah-Ward ansatz* [16] labeled by a positive integer $m = 1, 2, 3, \cdots$. Origin of these results will be clarified in the next section.

In order to discuss it, we have to rewrite a generic 2×2 matrix J as follows:

$$J = \begin{pmatrix} f - g \star b^{-1} \star e & -g \star b^{-1} \\ b^{-1} \star e & b^{-1} \end{pmatrix}, \tag{3.1}$$

where f and b are non-singular. This parameterization comes from a gauge fixing where h and $\tilde{h}$ are lower-triangular and upper-triangular, respectively, such that,

$$J = \tilde{h}^{-1} \star h = \begin{pmatrix} 1 & g \\ 0 & b \end{pmatrix}^{-1} \star \begin{pmatrix} f & 0 \\ e & 1 \end{pmatrix} = \begin{pmatrix} f - g \star b^{-1} \star e & -g \star b^{-1} \\ b^{-1} \star e & b^{-1} \end{pmatrix}. \tag{3.2}$$

Hence, the inverse of J has a similar form:

$$J^{-1} = h^{-1} \star \tilde{h} = \begin{pmatrix} f & 0 \\ e & 1 \end{pmatrix}^{-1} \star \begin{pmatrix} 1 & g \\ 0 & b \end{pmatrix} = \begin{pmatrix} f^{-1} & f^{-1} \star g \\ -e \star f^{-1} & b - e \star f^{-1} \star g \end{pmatrix}. \tag{3.3}$$

With this decomposition, NC Yang's equation (2.13) becomes

$$\begin{aligned}
\partial_z(f^{-1}\star g_{\tilde{z}}\star b^{-1})-\partial_w(f^{-1}\star g_{\tilde{w}}\star b^{-1}) &= 0,\\
\partial_{\tilde{z}}(b^{-1}\star e_z\star f^{-1})-\partial_{\tilde{w}}(b^{-1}\star e_w\star f^{-1}) &= 0,\\
\partial_z(b_{\tilde{z}}\star b^{-1})-\partial_w(b_{\tilde{w}}\star b^{-1})-e_z\star f^{-1}\star g_{\tilde{z}}\star b^{-1}+e_w\star f^{-1}\star g_{\tilde{w}}\star b^{-1} &= 0,\\
\partial_z(f^{-1}\star f_{\tilde{z}})-\partial_w(f^{-1}\star f_{\tilde{w}})-f^{-1}\star g_{\tilde{z}}\star b^{-1}\star e_z+f^{-1}\star g_{\tilde{w}}\star b^{-1}\star e_w &= 0. \qquad (3.4)
\end{aligned}$$

By using these formulae, we can find the gauge fields and the field strength in terms of b, e, f, g.

3.1 NC CFYG transformation

Now we describe the NC CFYG transformation explicitly. It is a composition of the following two Bäcklund transformations for the decomposed NC Yang's equations (3.4).

- β-transformation [10, 3]:

$$\begin{aligned}
&e_w^{\rm new}=f^{-1}\star g_{\tilde{z}}\star b^{-1},\, e_z^{\rm new}=f^{-1}\star g_{\tilde{w}}\star b^{-1},\\
&g_{\tilde{z}}^{\rm new}=b^{-1}\star e_w\star f^{-1},\, g_{\tilde{w}}^{\rm new}=b^{-1}\star e_z\star f^{-1},\\
&f^{\rm new}=b^{-1},\, b^{\rm new}=f^{-1}.
\end{aligned}$$

  The first four equations can be interpreted as integrability conditions for the first two equations in (3.4). We can easily check that the last two equations in (3.4) are invariant under this transformation. Also, it is clear that β-transformation is *involutive*, that is, $\beta\circ\beta$ is the identity transformation.

- γ_0-transformation:

$$\begin{pmatrix} f^{\rm new} & g^{\rm new}\\ e^{\rm new} & b^{\rm new}\end{pmatrix}=\begin{pmatrix} b & e\\ g & f\end{pmatrix}^{-1}=\begin{pmatrix} (b-e\star f^{-1}\star g)^{-1} & (g-f\star e^{-1}\star b)^{-1}\\ (e-b\star g^{-1}\star f)^{-1} & (f-g\star b^{-1}\star e)^{-1}\end{pmatrix}. \qquad (3.5)$$

  This follows from the fact that the transformation $\gamma_0: J\mapsto J^{\rm new}$ is equivalent to the simple conjugation $J^{\rm new}=C_0^{-1}JC_0$, $C_0=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, which clearly leaves the NC Yang's equation (2.13) invariant. The relation (3.5) is derived by comparing elements in this transformation. It is a trivial fact that γ_0-transformation is also involutive.

3.2 Exact NC Atiyah-Ward ansatz solutions

Now we construct exact solutions by using a chain of Bäcklund transformations from a seed solution. Let us consider $b=e=f=g=\Delta_0^{-1}$, then we can easily find that the decomposed

NC Yang's equation is reduced to a NC linear equation $(\partial_z\partial_{\tilde{z}} - \partial_w\partial_{\tilde{w}})\Delta_0 = 0$. (We note that for the Euclidean space, this is the NC Laplace equation because of the reality condition $\bar{w} = -\tilde{w}$.) Hence we can generate two series of exact solutions R_m and R'_m by iterating the β- and γ_0-transformations one after the other as follows (The seed solution $b = e = f = g = \Delta_0^{-1}$ belongs to R_1.):

$$\begin{array}{ccccccccc} R_1 & \overset{\alpha}{\to} & R_2 & \overset{\alpha}{\to} & R_3 & \overset{\alpha}{\to} & R_4 & \to & \cdots \\ \beta\updownarrow & \gamma_0\nearrow & \beta\updownarrow & \gamma_0\nearrow & \beta\updownarrow & \gamma_0\nearrow & \beta\updownarrow & \nearrow & \cdots \\ R'_1 & \overset{\tilde{\alpha}}{\to} & R'_2 & \overset{\tilde{\alpha}}{\to} & R'_3 & \overset{\tilde{\alpha}}{\to} & R'_4 & \to & \cdots \end{array} \tag{3.6}$$

where $\alpha = \gamma_0 \circ \beta : R_m \to R_{m+1}$ and $\alpha' = \beta \circ \gamma_0 : R'_m \to R'_{m+1}$. These two kind of series of solutions in fact arise from some class of NC Atiyah-Ward ansatz.[3] The explicit form of the solutions R_m or R'_m can be represented in terms of quasideterminants whose elements Δ_r $(r = -m+1, -m+2, \cdots, m-1)$ satisfy

$$\frac{\partial\Delta_r}{\partial z} = \frac{\partial\Delta_{r+1}}{\partial\tilde{w}}, \quad \frac{\partial\Delta_r}{\partial w} = \frac{\partial\Delta_{r+1}}{\partial\tilde{z}}, \qquad -m+1 \leq r \leq m-2 \quad (m \geq 2), \tag{3.7}$$

which implies that every element Δ_r is a solution of the NC linear equation $(\partial_z\partial_{\tilde{z}} - \partial_w\partial_{\tilde{w}})\Delta_r = 0$. The results are as follows. (The definition of quasideterminants are seen in Appendix A.)

- NC Atiyah-Ward ansatz solutions R_m

  NC Atiyah-Ward ansatz solutions R_m are represented by the explicit form of elements b_m, e_m, f_m, g_m in J_m as follows:

$$b_m = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-m} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{m-1} & \Delta_{m-2} & \cdots & \boxed{\Delta_0} \end{vmatrix}^{-1}, \quad f_m = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-m} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{m-1} & \Delta_{m-2} & \cdots & \Delta_0 \end{vmatrix}^{-1},$$

$$e_m = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \boxed{\Delta_{1-m}} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{m-1} & \Delta_{m-2} & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad g_m = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-m} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\Delta_{m-1}} & \Delta_{m-2} & \cdots & \Delta_0 \end{vmatrix}^{-1}.$$

  In the commutative limit, we can easily see that $b_m = f_m$. The ansatz R_1 leads to so called the *Corrigan-Fairlie-'t Hooft-Wilczek* (CFtHW) ansatz [52].

[3] This is discussed in our forthcoming paper in detail [13].

- NC Atiyah-Ward ansatz solutions R'_m

  NC Atiyah-Ward ansatz solutions R'_m are represented by the explicit form of elements b'_m, e'_m, f'_m, g'_m in $\tilde{J}_m$ as follows:

$$b'_m = \begin{vmatrix} \boxed{\Delta_0} & \Delta_{-1} & \cdots & \Delta_{1-m} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{m-1} & \Delta_{m-2} & \cdots & \Delta_0 \end{vmatrix}, \quad f'_m = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-m} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{m-1} & \Delta_{m-2} & \cdots & \boxed{\Delta_0} \end{vmatrix},$$

$$e'_m = \begin{vmatrix} \Delta_{-1} & \Delta_{-2} & \cdots & \boxed{\Delta_{-m}} \\ \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-m} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{m-2} & \Delta_{m-3} & \cdots & \Delta_{-1} \end{vmatrix}, \quad g'_m = \begin{vmatrix} \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} \\ \Delta_2 & \Delta_1 & \cdots & \Delta_{3-m} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\Delta_m} & \Delta_{m-1} & \cdots & \Delta_1 \end{vmatrix}.$$

  In the commutative case, $b'_m = f'_m$ also holds. For $m = 1$, we get $b'_1 = f'_1 = \Delta_0, e'_1 = \Delta_{-1}, g'_1 = \Delta_1$ and then the relation (3.7) implies that $e'_{1,z} = f'_{1,\tilde{w}}$, $e'_{1,w} = f'_{1,\tilde{z}}$, $f'_{1,z} = g'_{1,\tilde{w}}$, $f'_{1,w} = g'_{1,\tilde{z}}$, and leads to the CFtHW ansatz which was first pointed out by Yang [53].

The proof of these results [12] can be made by using identities of quasideterminants, such as, NC Jacobi identity, homological relations, and Gilson-Nimmo's derivative formula as in Appendix A. This implies that *NC Bäcklund transformations are identities of quasideterminants.*

We can also present a compact form of the whole Yang's matrix J in terms of a single quasideterminants expanded by a 2×2 submatrix. For example, the solution in R'_m leads to

$$J'_m = \left|\begin{array}{c:ccc:c:c} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-m} & \Delta_{-m} & -1 \\ \hdashline \Delta_1 & \Delta_0 & \cdots & \Delta_{2-m} & \Delta_{1-m} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \Delta_{m-1} & \Delta_{m-2} & \cdots & \Delta_0 & \Delta_{-1} & 0 \\ \hdashline \Delta_m & \Delta_{m-1} & \cdots & \Delta_1 & \boxed{\Delta_0} & \boxed{0} \\ \hdashline 1 & 0 & \cdots & 0 & \boxed{0} & \boxed{0} \end{array}\right|. \tag{3.8}$$

where

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{33}} & \boxed{a_{34}} \\ a_{41} & a_{42} & \boxed{a_{43}} & \boxed{a_{44}} \end{vmatrix} := \begin{bmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \boxed{a_{33}} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & \boxed{a_{34}} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & \boxed{a_{43}} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & \boxed{a_{44}} \end{vmatrix} \end{bmatrix}.$$

Because J is gauge invariant, this shows that the present Bäcklund transformation is not just a gauge transformation but a non-trivial one. The proof of these representations is given in Appendix A in [12] and in [60].

4 Twistor descriptions of NC ASDYM equations

In this section, we give an origin of the Bäcklund transformations for NC ASDYM equations and NC Atiyah-Ward ansatz solutions from the geometrical viewpoint of NC twistor theory. NC twistor theory has been developed by several authors and the mathematical foundation is established [31]-[38]. Here we just need one-to-one correspondence between a NC ASDYM connection and a Patching matrix P of "NC holomorphic vector bundle on a NC 3-dimensional projective space. The holomorphy implies

$$P = P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta). \tag{4.1}$$

We note that noncommutativity can be introduced into only two variables $\zeta w + \tilde{z}, \zeta z + \tilde{w}$. Then ζ is a commutative variable and steps of solving Riemann-Hilbert problem become similar to commutative one. However the noncommutativity in the first and second coordinates introduce nontrivial effects in the ASDYM connections and give rise to new physical objects, such as $U(1)$ instantons. Original results in this section are due to our forthcoming paper [13].

4.1 Riemann-Hilbert problem for NC Atiyah-Ward Ansatz

The solution ψ ($N \times N$ matrix) of the linear system (2.9) is not regular at $\zeta = \infty$ because of Liouville's theorem. Hence we have to consider another linear system on another local patch whose coordinate $\tilde{\zeta} = 1/\zeta$ as

$$\begin{aligned} \tilde{\zeta} D_w \star \tilde{\psi} - D_{\tilde{z}} \star \tilde{\psi} &= 0, \\ \tilde{\zeta} D_z \star \tilde{\psi} - D_{\tilde{w}} \star \tilde{\psi} &= 0. \end{aligned} \tag{4.2}$$

Then we can prove that if the patching matrix P can be factorized as follows:

$$P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{\psi}^{-1}(x;\zeta) \star \psi(x;\zeta). \tag{4.3}$$

where ψ and $\tilde{\psi}$ are regular near $\zeta = 0$ and $\zeta = \infty$, respectively, then the ψ and $\tilde{\psi}$ are the solutions of the linear systems for NC ASDYM equation.

Hence if we solve the factorization problem (or the *Riemann-Hilbert problem*) (4.3), then we can reproduce ASDYM connections in terms of h and $\tilde{h}$ by using (2.11), (2.12) and the fact $h(x) = \psi(x, \zeta = 0), \tilde{h}(x) = \tilde{\psi}(x, \zeta = \infty)$.

From now on, we restrict ourselves to $G = GL(2)$. In this case, we can take a simple ansatz for the Patching matrix P, which is called the Atiyah-Ward ansatz in commutative case [16]. NC generalization of this ansatz is straightforward and actually leads to a solution of the factorization problem. The l-th order NC Atiyah-Ward ansatz is specified by the following form of the patching matrix ($l = 1, 2, \cdots$):

$$P(x;\zeta) = \begin{pmatrix} 0 & \zeta^{-l} \\ \zeta^{l} & \Delta(x;\zeta) \end{pmatrix}. \tag{4.4}$$

We note that holomorphy of P implies $\Delta(x;\zeta) = (\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta)$, or equivalently, $(\partial_w - \zeta\partial_{\tilde{z}})\Delta = 0$, $(\partial_z - \zeta\partial_{\tilde{w}})\Delta = 0$. Hence, the Laurent expansion of Δ w.r.t. ζ

$$\delta(x;\zeta) = \sum_{i=-\infty}^{\infty} \Delta_i(x)\zeta^{-i}, \tag{4.5}$$

gives rise to the following relations in the coefficients as

$$\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \tilde{z}}, \tag{4.6}$$

which coincide with the chasing relation (3.7). We will soon see that the coefficients $\Delta_i(x)$ are the scalar functions in the solutions generated by the Bäcklund transformations in the previous section.

Now let us solve the factorization problem $\tilde{\psi} \star P = \psi$ for the NC Atiyah-Ward ansatz, which is concretely written down as

$$\begin{pmatrix} \tilde{\psi}_{11} & \tilde{\psi}_{12} \\ \tilde{\psi}_{21} & \tilde{\psi}_{22} \end{pmatrix} \star \begin{pmatrix} 0 & \zeta^{-l} \\ \zeta^{l} & \Delta(x;\zeta) \end{pmatrix} = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}, \tag{4.7}$$

that is,

$$\tilde{\psi}_{12}\zeta^{l} = \psi_{11}, \quad \tilde{\psi}_{22}\zeta^{l} = \psi_{21}, \tag{4.8}$$

$$\tilde{\psi}_{11}\zeta^{-l} + \tilde{\psi}_{12} \star \Delta = \psi_{12}, \quad \tilde{\psi}_{21}\zeta^{-l} + \tilde{\psi}_{22} \star \Delta = \psi_{22}. \tag{4.9}$$

From Eq. (4.8) together with the relation between $\psi, \tilde{\psi}$ and $h, \tilde{h}$, we find that some components become polynomials w.r.t. ζ:

$$\begin{aligned} \psi_{11} &= h_{11} + a_1\zeta + a_2\zeta^2 + \cdots a_{l-1}\zeta^{l-1} + \tilde{h}_{12}\zeta^{l}, \\ \psi_{21} &= h_{21} + b_1\zeta + b_2\zeta^2 + \cdots b_{l-1}\zeta^{l-1} + \tilde{h}_{22}\zeta^{l}, \\ \tilde{\psi}_{12} &= \tilde{h}_{12} + a_{l-1}\zeta^{-1} + a_{l-2}\zeta^{-2} + \cdots a_1\zeta^{1-l} + h_{11}\zeta^{-l}, \\ \tilde{\psi}_{22} &= \tilde{h}_{22} + b_{l-1}\zeta^{-1} + b_{l-2}\zeta^{-2} + \cdots + b_1\zeta^{1-l} + h_{21}\zeta^{-l}, \end{aligned} \tag{4.10}$$

and so on. By substituting these relations into Eq. (4.9), we get sets of equations for h and $\tilde{h}$ in the coefficients of ζ^0, ζ^{-1}, $\cdots$, ζ^{-l}:

$$\begin{aligned} h_{11} &= h_{12} \star |D_{l+1}|^{-1}_{l+1,1} - \tilde{h}_{11} \star |D_{l+1}|^{-1}_{l+1,l+1}, \\ h_{21} &= h_{22} \star |D_{l+1}|^{-1}_{l+1,1} - \tilde{h}_{21} \star |D_{l+1}|^{-1}_{l+1,l+1}, \\ \tilde{h}_{12} &= h_{12} \star |D_{l+1}|^{-1}_{1,1} - \tilde{h}_{11} \star |D_{l+1}|^{-1}_{1,l+1}, \\ \tilde{h}_{22} &= h_{22} \star |D_{l+1}|^{-1}_{1,1} - \tilde{h}_{21} \star |D_{l+1}|^{-1}_{1,l+1}, \end{aligned} \tag{4.11}$$

where

$$D_l := \begin{pmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \end{pmatrix}. \tag{4.12}$$

By taking some gauge condition, Eq. (4.11) can be solved for h and $\tilde{h}$ in terms of quasideterminants of D_{l+1}, which just coincide with the generated solutions by the Bäcklund transformation in the previous section ! This is an origin of the NC Atiyah-Ward solutions.

4.2 Origin of the NC CFYG transformation

Finally let us discuss an origin of the NC CFYG transformations, β-transformation and γ_0-transformation. These transformations can be viewed as adjoint actions for the patching matrix P:

$$\beta : P \mapsto P^{\rm new} = B^{-1}PB, \quad \gamma_0 : P \mapsto P^{\rm new} = C^{-1}PC, \tag{4.13}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ \zeta^{-1} & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.14}$$

together with a singular gauge transformation for β-transformation. It is clear that this leads to γ_0-transformation by considering the action of C for ψ and $\tilde{\psi}$. The action of B is defined at the level of ψ and $\tilde{\psi}$ as follows:

$$\psi^{\rm new} = s \star \psi\, B, \quad \tilde{\psi}^{\rm new} = s \star \tilde{\psi}\, B, \tag{4.15}$$

where

$$s = \begin{pmatrix} 0 & \zeta b^{-1} \\ -f^{-1} & 0 \end{pmatrix}. \tag{4.16}$$

The explicit calculation gives

$$\psi^{\text{new}} = \begin{pmatrix} b^{-1}\psi_{22} & -\zeta b^{-1} \star \psi_{21} \\ -\zeta^{-1} f^{-1} \star \psi_{12} & f^{-1} \star \psi_{11} \end{pmatrix}, \tag{4.17}$$

where ψ_{ij} is the (i,j)-th element of ψ. In the $\zeta \to 0$ limit, this reduces to

$$h^{\text{new}} = \begin{pmatrix} f^{\text{new}} & 0 \\ e^{\text{new}} & 1 \end{pmatrix} = \begin{pmatrix} b^{-1} & 0 \\ -f^{-1} \star j_{12} & 1 \end{pmatrix}, \tag{4.18}$$

where $\psi = h + j\zeta + \mathcal{O}(\zeta^2)$.

Here we note that the linear systems can be represented in terms of b, f, e, g as

$$\begin{aligned} L \star \psi &= (\partial_w - \zeta\partial_{\tilde{z}}) \star \psi + \begin{pmatrix} -f_w \star f^{-1} & \zeta g_{\tilde{z}} \star b^{-1} \\ -e_w \star f^{-1} & \zeta b_{\tilde{z}} \star b^{-1} \end{pmatrix} \star \psi = 0, \\ M \star \psi &= (\partial_z - \zeta\partial_{\tilde{w}}) \star \psi + \begin{pmatrix} -f_z \star f^{-1} & \zeta g_{\tilde{w}} \star b^{-1} \\ -e_z \star f^{-1} & \zeta b_{\tilde{w}} \star b^{-1} \end{pmatrix} \star \psi = 0. \end{aligned} \tag{4.19}$$

By picking the first order term of ζ in the 1-2 component of the first equation, we get

$$\partial_w(f^{-1} \star j_{12}) = -f^{-1} \star g_{\tilde{z}} \star b^{-1}. \tag{4.20}$$

Hence from the 1-1 and 2-1 components of Eq. (4.18), we have

$$f^{\text{new}} = b^{-1}, \quad \partial_w e^{\text{new}} = -\partial_w(f^{-1} \star j_{12}) = f^{-1} \star g_{\tilde{z}} \star b^{-1}, \tag{4.21}$$

which are just parts of the β-transformation (3.5). In similar way, we can get the other ones. Therefore the β-transformation (3.5) can be interpreted as the transformation of the patching matrix $F \mapsto B^{-1}FB$ together with the singular gauge transformation s.

5 NC Ward's Conjecture

Here we briefly discuss reductions of NC ASDYM equation into lower-dimensional NC integrable equations such as the NC KdV equation. let us summarize the strategy for reductions of NC ASDYM equation into lower-dimensions. Reductions are classified by the following ingredients:

- A choice of gauge group
- A choice of symmetry, such as, translational symmetry
- A choice of gauge fixing

- A choice of constants of integrations in the process of reductions

Gauge groups are in general $GL(N)$. We have to take $U(1)$ part into account in NC case. A choice of symmetry reduces NC ASDYM equations to simple forms. We note that non-commutativity must be eliminated in the reduced directions because of compatibility with the symmetry. Hence within the reduced directions, discussion about the symmetry is the same as commutative one. A choice of gauge fixing is the most important ingredient in this paper which is shown explicitly at each subsection. The residual gauge symmetry sometimes shows equivalence of a few reductions. Constants of integrations in the process of reductions sometimes lead to parameter families of NC reduced equations, however, in this paper, we set all integral constants zero for simplicity.

5.1 Reduction to the NC KdV Equation

In this section, we present non-trivial reductions of NC ASDYM equation with $G = GL(2)$ to the NC KdV equation.

First, let us take a dimensional reduction by null translations:

$$X = \partial_w - \partial_{\tilde{w}},\ Y = \partial_{\tilde{z}}. \tag{5.1}$$

Then the gauge field $A_{\tilde{z}}$ becomes a Higgs field which is denoted by $\Phi_{\tilde{z}}$. Here let us take a gauge fixing of $\Phi_{\tilde{z}}$ as follows:

$$\Phi_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the NC ASDYM equation is simply reduced:

$$\begin{aligned} &\Phi'_{\tilde{z}} + [A_{\tilde{w}}, \Phi_{\tilde{z}}]_\star = 0, \\ &\dot{\Phi}_{\tilde{z}} + A'_w - A'_{\tilde{w}} + [A_z, \Phi_{\tilde{z}}]_\star - [A_w, A_{\tilde{w}}]_\star = 0, \\ &A'_z - \dot{A}_w + [A_w, A_z]_\star = 0, \end{aligned} \tag{5.2}$$

where $(t, x) \equiv (z, w + \tilde{w})$ and $\dot{f} := \partial f/\partial t,\ f' := \partial f/\partial x$.

Now let us take the following non-trivial reduction conditions for the gauge fields

$$\begin{aligned} &A_{\tilde{w}} = 0,\ \Phi_{\tilde{z}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ A_w = \begin{pmatrix} q & -1 \\ q' + q\star q & -q \end{pmatrix}, \\ &A_z = \frac{1}{2}\begin{pmatrix} q'' + 2q'\star q & -2q' \\ (1/2)q''' + q'\star q' + q\star q'' + q''\star q + 2q\star q'\star q & -q'' - 2q\star q' \end{pmatrix}. \end{aligned} \tag{5.3}$$

These conditions automatically solve the first and second equations in (5.2) and the third one gives rise to NC potential KdV equation

$$\dot{q} = \frac{1}{4}q''' + \frac{3}{2}(q \star q)', \tag{5.4}$$

which is derived from NC KdV equation with $u = 2q'$

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{4}(u' \star u + u \star u'). \tag{5.5}$$

In this way, NC KdV equation is actually derived. We note that the gauge group is not $SL(2)$ but $GL(2)$ because A_z is not traceless. This implies $U(1)$ part of the gauge group plays a crucial role in the reduction also.

This NC KdV equation has been studied by several authors and proved to possess infinite conserved quantities [58] in terms of Strachan products [40] and exact multi-soliton solutions in terms of quasideterminants also [41, 27]. (See also [56, 57].)

6 Conclusion and Discussion

In this article, we have presented Bäcklund transformations for the NC ASDYM equation with $G = GL(2)$ and constructed from a simple seed solution a series of exact NC Atiyah-Ward ansatz solutions expressed explicitly in terms of quasideterminants. We have also discussed the origin of this transformation in the framework of NC twistor theory.

These results could be applied also to lower-dimensional systems via the results on the NC Ward's conjecture including NC monopoles, NC KdV equations and so on, and might shed light on a profound connection between higher-dimensional integrable systems related to twistor theory and lower-dimensional ones related to Sato's theory.

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A Brief Review of Quasideterminants

In this appendix, we make a brief introduction of quasi-determinants introduced by Gelfand and Retakh in 1992 [11] and present a few properties of them which play important roles in the main sections. Relation between quasi-determinants and noncommutative symmetric functions is seen in e.g. [59].

Quasi-determinants are not just a noncommutative generalization of usual commutative determinants but rather related to inverse matrices.

Let $A = (a_{ij})$ be a $n \times n$ matrix and $B = (b_{ij})$ be the inverse matrix of A. Here all matrix elements are supposed to belong to a (noncommutative) ring with an associative product. This general noncommutative situation includes the Moyal or NC deformation.

Quasi-determinants of A are defined formally as the inverse of the elements of $B = A^{-1}$:

$$|A|_{ij} := b_{ji}^{-1}. \tag{A.1}$$

In the commutative limit, this is reduced to

$$|A|_{ij} \longrightarrow (-1)^{i+j} \frac{\det A}{\det \tilde{A}^{ij}}, \tag{A.2}$$

where $\tilde{A}^{ij}$ is the matrix obtained from A deleting the i-th row and the j-th column.

We can write down more explicit form of quasi-determinants. In order to see it, let us recall the following formula for a square matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}, \tag{A.3}$$

where A and D are square matrices, and all inverses are supposed to exist. We note that any matrix can be decomposed as a 2×2 matrix by block decomposition where the diagonal parts are square matrices, and the above formula can be applied to the decomposed 2×2 matrix. So the explicit forms of quasi-determinants are given iteratively by the following formula:

$$\begin{aligned} |A|_{ij} &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} ((\tilde{A}^{ij})^{-1})_{i'j'} a_{j'j} \\ &= a_{ij} - \sum_{i'(\neq i), j'(\neq j)} a_{ii'} (|\tilde{A}^{ij}|_{j'i'})^{-1} a_{j'j}. \end{aligned} \tag{A.4}$$

It is sometimes convenient to represent the quasi-determinant as follows:

$$|A|_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & & \boxed{a_{ij}} & & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}. \tag{A.5}$$

Examples of quasi-determinants are, for a 1×1 matrix $A = a$

$$|A| = a, \tag{A.6}$$

and for a 2×2 matrix $A = (a_{ij})$

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12}a_{22}^{-1}a_{21}, \quad |A|_{12} = \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11}a_{21}^{-1}a_{22},$$
$$|A|_{21} = \begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = a_{21} - a_{22}a_{12}^{-1}a_{11}, \quad |A|_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & \boxed{a_{22}} \end{vmatrix} = a_{22} - a_{21}a_{11}^{-1}a_{12}, \tag{A.7}$$

and for a 3×3 matrix $A = (a_{ij})$

$$\begin{aligned} |A|_{11} &= \begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - (a_{12}, a_{13}) \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} \\ &= a_{11} - a_{12} \begin{vmatrix} \boxed{a_{22}} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{12} \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} a_{31} \\ &\quad - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ \boxed{a_{32}} & a_{33} \end{vmatrix}^{-1} a_{21} - a_{13} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & \boxed{a_{33}} \end{vmatrix}^{-1} a_{31}, \end{aligned} \tag{A.8}$$

and so on.

Quasideterminants have various interesting properties similar to those of determinants. Among them, the following ones play important roles in this paper. In the block matrices given in these results, lower case letters denote single entries and upper case letters denote matrices of compatible dimensions so that the overall matrix is square.

- NC Jacobi identity [11, 45]

  A simple and useful special case of the NC Sylvester's Theorem [11] is

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}. \tag{A.9}$$

- Homological relations [11]

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix},$$
$$\begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & B & 0 \\ D & f & \boxed{0} \\ E & h & 1 \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \qquad \text{(A.10)}$$

- *Gilson-Nimmo's derivative formula* [45]

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B' \\ C & \boxed{d'} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & (A_k)' \\ C & \boxed{(C_k)'} \end{vmatrix} \begin{vmatrix} A & B \\ e_k^t & \boxed{0} \end{vmatrix}, \qquad \text{(A.11)}$$

where A_k is the kth column of a matrix A and e_k is the column n-vector (δ_{ik}) (i.e. 1 in the kth row and 0 elsewhere).

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