数理解析研究所講究録
第 1605 巻 2008 年 53-67

Quantum Painlevé equations

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1 Introduction

In this note, we present a quantization of the Painlevé equations with the affine Weyl group actions. Constructing a generalization of Knizhnik - Zamolodchikov equations for $\mathfrak{sl}_2$, we obtain the quantization of the Painlevé equations from the generalization of the Knizhnik-Zamolodchikov equations, in a formal way.

In the ninetieth century, one of the important problems of analysis was to find "a new special functions" defined by nonlinear algebraic differential equations. Under this consideration, P. Painlevé classified the second order nonlinear ordinary differential equations without movable singular points in their solutions, which are now called the Painlevé equations $\mathrm{P_J}$ (J=I,...,VI) [6].

Soon after the discovery of the Painlevé equations, the sixth Painlevé equation was derived from the monodromy preserving deformation by R. Fuchs [1]. Much later, the Painlevé equations were formulated in the general theory of monodromy preserving deformation [3]. As for the quantization of monodromy preserving deformation, it was noticed by N. Reshetikhin [7] that a quantization of the Schlesinger equations which are deformation equations that preserve the monodromy of the rational connection with regular singularities can be viewed as the Knizhnik-Zamolodchikov equations in the conformal field theory. We hope that we would obtain "a quantum sixth Painlevé equation" from the Knizhnik-Zamolodchikov equation in the special case and no one succeeds to obtain it so far.

We attack the problem of quantization of Painlevé equation from another aspect. Since the Painlevé equations are Hamiltonian systems and admit the affine Weyl group actions as Bäcklund transformations, we quantize the Painlevé equations as quantization has the affine Weyl group action. We call them quantum Painlevé equations.

Moreover, generalizing the Knizhnik-Zamolodchikov equations for irregular singular type in the case of $\mathfrak{sl}_2$, we obtain quantum Painlevé equations

QP_{I}, QP_{II}, QP_{III}, QP_{IV} and QP_{V}. This part is a joint work with M. Jimbo and J. Sun.

In the following, we write down all quantum Painlevé equations and explain the relation between the Schlesinger equations and the Knizhnik-Zamolodchikov equations and generalize the Knizhnik-Zamolodchikov equations. Finally we compute QP_{IV} case as example.

2 Quantum Painlevé equations

In this section, we write down quantized Hamiltonians and the affine Weyl group actions.

2.1 The case of QP_{VI}

Let $\mathcal{K}_{VI}$ be a skew field over $\mathbb{C}$ with generators $\hat{q}$, $\hat{p}$, α_1, α_2, α_3, α_4, t and the commutation relations defined by

$$[\hat{p},\hat{q}] = h \quad (h \in \mathbb{C}), \quad [\hat{p},\alpha_i] = [\hat{q},\alpha_i] = [\hat{p},t] = [\hat{q},t] = [t,\alpha_i] = 0 \quad (1 \le i \le 4). \tag{1}$$

Let an element $\widehat{H}_{VI}$ in $\mathcal{K}_{VI}$ be defined by

$$\begin{aligned}\widehat{H}_{VI} =& \frac{1}{6}\left[\hat{q}\hat{p}(\hat{q}-1)\hat{p}(\hat{q}-t) + (\hat{q}-1)\hat{p}(\hat{q}-t)\hat{p}\hat{q} + (\hat{q}-t)\hat{p}\hat{q}\hat{p}(\hat{q}-1) + \right.\\ & \left.(\hat{q}-t)\hat{p}(\hat{q}-1)\hat{p}\hat{q} + (\hat{q}-1)\hat{p}\hat{q}\hat{p}(\hat{q}-t) + \hat{q}\hat{p}(\hat{q}-t)\hat{p}(\hat{q}-1)\right]\\ & -\frac{1}{2}\left[(\alpha_0-1)(\hat{q}\hat{p}(\hat{q}-1)+(\hat{q}-1)\hat{p}\hat{q}) + \alpha_3(\hat{q}\hat{p}(\hat{q}-t)+(\hat{q}-t)\hat{p}\hat{q})\right.\\ & \left.\alpha_4((\hat{q}-1)\hat{p}(\hat{q}-t)+(\hat{q}-t)\hat{p}(\hat{q}-1))\right] + \alpha_2(\alpha_1+\alpha_2)(\hat{q}-t),\end{aligned} \tag{2}$$

where $\alpha_0 = 1-\alpha_1-2\alpha_2-\alpha_3-\alpha_4$. Let a $\mathbb{C}$-derivation ∂_{VI} on $\mathcal{K}_{VI}$ be defined by Heisenberg equation

$$\partial_{VI}(a) = \frac{1}{h}[\widehat{H}_{VI}, a] + t(t-1)\frac{\partial a}{\partial t} \quad (a \in \mathcal{K}), \tag{3}$$

where $\partial/\partial t$ is a $\mathbb{C}$-derivation which takes t to 1 and other generators to 0. We write down Heisenberg equation for $\hat{q}, \hat{p}$.

$$\partial_{VI}(\hat{q}) = \frac{1}{h}[\widehat{H}_{VI}, \hat{q}] = \hat{q}(\hat{q}-1)\hat{p}(\hat{q}-t) + (\hat{q}-t)\hat{p}\hat{q}(\hat{q}-1)$$

$$- \{\alpha_4(\hat{q}-1)(\hat{q}-t) + \alpha_3\hat{q}(\hat{q}-t) + (\alpha_0-1)\hat{q}(\hat{q}-1)\}, \quad (4)$$

$$\partial_{VI}(\hat{p}) = \frac{1}{h}[\widehat{H}_{VI}, \hat{p}] = -\left(\hat{p}\hat{q}\hat{p}\hat{q} + \hat{q}\hat{p}^2\hat{q} + \hat{q}\hat{p}\hat{q}\hat{p}\right) + 2(1+t)\hat{p}\hat{q}\hat{p} - t\hat{p}^2$$
$$+ (\alpha_0 + \alpha_3 + \alpha_4 - 1)(\hat{p}\hat{q} + \hat{q}\hat{p})$$
$$+ \hat{p}\{-\alpha_4(1+t) - \alpha_3 t - \alpha_0 + 1\} - \alpha_2(\alpha_1 + \alpha_2). \quad (5)$$

We define transformations s_i $(0 \le i \le 4)$ for the generators of $\mathcal{K}_{VI}$ as follows:

	α_0	α_1	α_2	α_3	α_4
s_0	$-\alpha_0$	α_1	$\alpha_2 + \alpha_0$	α_3	α_4
s_1	α_0	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	α_4
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$\alpha_4 + \alpha_2$
s_3	α_0	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	α_4
s_4	α_0	α_1	$\alpha_2 + \alpha_4$	α_3	$-\alpha_4$

	$\hat{q}$	$\hat{p}$
s_0	$\hat{q}$	$\hat{p} - \alpha_0(\hat{q}-t)^{-1}$
s_1	$\hat{q}$	$\hat{p}$
s_2	$\hat{q} + \alpha_2 p^{-1}$	$\hat{p}$
s_3	$\hat{q}$	$\hat{p} - \alpha_3(\hat{q}-1)^{-1}$
s_4	$\hat{q}$	$\hat{p} - \alpha_4\hat{q}^{-1}$

Proposition 1 *Transformations s_i preserve the commutation relations, that is, s_i become automorphisms on $\mathcal{K}_{VI}$.*

Theorem 2 *The derivation ∂_{VI} commutes with automorphisms s_i $(0 \le i \le 4)$ and s_i $(0 \le i \le 4)$ give a representation of the affine Weyl group of type $D_4^{(1)}$, namely, s_i satisfy the following relations:*

$$s_i^2 = 1, \quad (s_i s_j)^2 = 1 \quad (i, j \neq 2), \quad (s_i s_2)^3 = 1. \quad (6)$$

2.2 The case of $\mathrm{QP_V}$

Let $\mathcal{K}_V$ be a skew field over $\mathbb{C}$ with generators α_i, $\hat{f}_i$ $(i \in \mathbb{Z}/4\mathbb{Z})$ and the commutation relations

$$[\hat{f}_i, \hat{f}_{i+1}] = h \quad (h \in \mathbb{C}), \quad [\hat{f}_i, \alpha_j] = [\alpha_i, \alpha_j] = 0, \quad (i, j \in \mathbb{Z}/4\mathbb{Z}). \quad (7)$$

Let an element $\widehat{H}_V$ in $\mathcal{K}_V$ be defined by

$$\begin{aligned}\widehat{H}_V =& \hat{f}_0\hat{f}_1\hat{f}_2\hat{f}_3 + h\hat{f}_1\hat{f}_2 + \frac{1}{4}(\alpha_1 + 2\alpha_2 - \alpha_3)\hat{f}_0\hat{f}_1 + \frac{1}{4}(\alpha_1 + 2\alpha_2 + 3\alpha_3)\hat{f}_1\hat{f}_2 \\ &- \frac{1}{4}(3\alpha_1 + 2\alpha_2 + \alpha_3)\hat{f}_2\hat{f}_3 + \frac{1}{4}(\alpha_1 - 2\alpha_2 - \alpha_3)\hat{f}_3\hat{f}_0 + \frac{1}{4}(\alpha_1 + \alpha_3)^2.\end{aligned}$$

Let a $\mathbb{C}$-derivation ∂_V on $\mathcal{K}_V$ be defined by

$$\partial_V \hat{f}_i = \frac{1}{h}[\widehat{H}_V, \hat{f}_i] - (-1)^i \frac{k}{2}\hat{f}_i + \delta_{i,0}kx, \tag{8}$$

$$\partial_V \alpha_i = \frac{1}{h}[\widehat{H}_V, \alpha_i] \quad (0 \le i \le 4), \tag{9}$$

where $k = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ and $x = \hat{f}_0 + \hat{f}_2$. We write down the derivation ∂_V for $\hat{f}_i$.

$$\partial_V \hat{f}_i = \hat{f}_i\hat{f}_{i+1}\hat{f}_{i+2} - \hat{f}_{i+2}\hat{f}_{i+3}\hat{f}_i + \left(\frac{k}{2} - \alpha_i\right)\hat{f}_{i+2} + \alpha_i\hat{f}_{i+2} \quad (i \in \mathbb{Z}/4\mathbb{Z}). \tag{10}$$

If we introduce $\hat{p}, \hat{q}$ by

$$\hat{f}_0 = \frac{x}{\hat{q} - 1}, \quad \hat{f}_1 = (\hat{q} - 1)\frac{\hat{p}}{x}(\hat{q} - 1), \tag{11}$$

then $(\hat{p}, \hat{q})$ is a canonical coordinate, namely it holds $[\hat{p}, \hat{q}] = h$.

We define transformations s_i $(0 \le i \le 3)$ for the generators of $\mathcal{K}_V$ as follows:

	α_0	α_1	α_2	α_3
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	α_2	$\alpha_3 + \alpha_0$
s_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3
s_2	α_0	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$
s_3	$\alpha_0 + \alpha_3$	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$

	$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$	$\hat{f}_3$
s_0	$\hat{f}_0$	$\hat{f}_1 + \alpha_0\hat{f}_0^{-1}$	$\hat{f}_2$	$\hat{f}_3 - \alpha_0\hat{f}_0^{-1}$
s_1	$\hat{f}_0 - \alpha_1\hat{f}_1^{-1}$	$\hat{f}_1$	$\hat{f}_2 + \alpha_1\hat{f}_1^{-1}$	$\hat{f}_3$
s_2	$\hat{f}_0$	$\hat{f}_1 - \alpha_2\hat{f}_2^{-1}$	$\hat{f}_2$	$\hat{f}_3 + \alpha_2\hat{f}_2^{-1}$
s_3	$\hat{f}_0 + \alpha_3\hat{f}_3^{-1}$	$\hat{f}_1$	$\hat{f}_2 - \alpha_3\hat{f}_3^{-1}$	$\hat{f}_3$

Proposition 3 *Transformations s_i preserve the commutation relations, that is, s_i become automorphisms on $\mathcal{K}_V$.*

Theorem 4 *The derivation ∂_V commutes with automorphisms s_i $(0 \le i \le 3)$ and s_i $(0 \le i \le 3)$ give a representation of the affine Weyl group of type $A_3^{(1)}$, namely, s_i satisfy the following relations:*

$$s_i^2 = 1, \quad (s_i s_j)^2 = 1 \quad (i \neq j \pm 1), \quad (s_i s_{i+1})^3 = 1. \tag{12}$$

2.3 The case of $\mathrm{QP}_{\mathrm{IV}}$

Let $\mathcal{K}_{IV}$ be a skew field over $\mathbb{C}$ with generators α_i, $\hat{f}_i$ $(i \in \mathbb{Z}/3\mathbb{Z})$ and the commutation relations

$$[\hat{f}_i, \hat{f}_{i+1}] = h \quad (h \in \mathbb{C}), \quad [\hat{f}_i, \alpha_j] = [\alpha_i, \alpha_j] = 0, \quad (i, j \in \mathbb{Z}/3\mathbb{Z}). \tag{13}$$

Let an element $\widehat{H}_{IV}$ in $\mathcal{K}_{IV}$ be defined by

$$\widehat{H}_{IV} = \hat{f}_0 \hat{f}_1 \hat{f}_2 + h\hat{f}_1 + \frac{1}{3}(\alpha_1 - \alpha_2)\hat{f}_0 + \frac{1}{3}(\alpha_1 + 2\alpha_2)\hat{f}_1 - \frac{1}{3}(2\alpha_1 + \alpha_2)\hat{f}_2. \tag{14}$$

Let a $\mathbb{C}$-derivation ∂_{IV} on $\mathcal{K}_{IV}$ be defined by

$$\partial_{IV} \hat{f}_i = \frac{1}{h}[\widehat{H}_{IV}, \hat{f}_i] + \delta_{i,0} k, \tag{15}$$

$$\partial_{IV} \alpha_i = \frac{1}{h}[\widehat{H}_{IV}, \alpha_i] \quad (0 \le i \le 3), \tag{16}$$

where $k = \alpha_0 + \alpha_1 + \alpha_2$. We write down the derivation ∂_{IV} for $\hat{f}_i$.

$$\partial_{IV} \hat{f}_i = \hat{f}_i \hat{f}_{i+1} - \hat{f}_{i-1} \hat{f}_i + \alpha_i \quad (i \in \mathbb{Z}/4\mathbb{Z}). \tag{17}$$

If we introduce $\hat{p}, \hat{q}$ by

$$\hat{f}_1 = \hat{p}, \quad \hat{f}_2 = \hat{q}, \tag{18}$$

then $(\hat{p}, \hat{q})$ is a canonical coordinate, namely it holds $[\hat{p}, \hat{q}] = h$.

We define transformations s_i $(0 \le i \le 2)$ for the generators of $\mathcal{K}_{IV}$ as follows:

	α_0	α_1	α_2
s_0	$-\alpha_0$	$\alpha_1 + \alpha_0$	$\alpha_2 + \alpha_0$
s_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_2 + \alpha_1$
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$

	$\hat{f}_0$	$\hat{f}_1$	$\hat{f}_2$
s_0	$\hat{f}_0$	$\hat{f}_1 + \alpha_0 \hat{f}_0^{-1}$	$\hat{f}_2 - \alpha_0 \hat{f}_0^{-1}$
s_1	$\hat{f}_0 - \alpha_1 \hat{f}_1^{-1}$	$\hat{f}_1$	$\hat{f}_2 + \alpha_1 \hat{f}_1^{-1}$
s_2	$\hat{f}_0 + \alpha_2 \hat{f}_2^{-1}$	$\hat{f}_1 - \alpha_2 \hat{f}_2^{-1}$	$\hat{f}_2$

Proposition 5 *Transformations s_i preserve the commutation relations, that is, s_i become automorphisms on $\mathcal{K}_{IV}$.*

Theorem 6 *The derivation ∂_{IV} commutes with automorphisms s_i $(0 \le i \le 2)$ and s_i $(0 \le i \le 2)$ give a representation of the affine Weyl group of type $A_2^{(1)}$, namely, s_i satisfy the following relations:*

$$s_i^2 = 1, \quad (s_i s_{i+1})^3 = 1. \tag{19}$$

2.4 The case of $\mathrm{QP_{III}}$

Let $\mathcal{K}_{III}$ be a skew field over $\mathbb{C}$ with generators $\hat{q}$, $\hat{p}$, α_1, α_2, t and the commutation relations defined by

$$[\hat{p}, \hat{q}] = h \quad (h \in \mathbb{C}), \quad [\hat{p}, \alpha_i] = [\hat{q}, \alpha_i] = [\hat{p}, t] = [\hat{q}, t] = [t, \alpha_i] = 0 \quad (1 \le i \le 2). \tag{20}$$

Let an element $\widehat{H}_{III}$ in $\mathcal{K}_{III}$ be defined by

$$\widehat{H}_{III} = \frac{1}{4}\left(\hat{p}\hat{q}(\hat{p}-1)\hat{q} + (\hat{p}-1)\hat{q}\hat{p}\hat{q} + \hat{q}\hat{p}\hat{q}(\hat{p}-1) + \hat{q}(\hat{p}-1)\hat{q}\hat{p}\right) \tag{21}$$

$$+ \frac{1}{2}(\alpha_0 + \alpha_2)(\hat{q}\hat{p} + \hat{p}\hat{q}) - \alpha_0 \hat{q} + t\hat{p} \tag{22}$$

where $\alpha_0 = 1 - 2\alpha_1 - \alpha_2$. Let a $\mathbb{C}$-derivation ∂_{III} on $\mathcal{K}_{III}$ be defined by Heisenberg equation

$$\partial_{III}(a) = \frac{1}{h}[\widehat{H}_{III}, a] + t\frac{\partial a}{\partial t} \quad (a \in \mathcal{K}). \tag{23}$$

We write down Heisenberg equation for $\hat{q}, \hat{p}$.

$$\partial_{III}(\hat{q}) = \frac{1}{h}[\widehat{H}_{III}, \hat{q}] = 2\hat{q}\hat{p}\hat{q} - \hat{q}^2 + (\alpha_0 + \alpha_2)\hat{q} + t, \tag{24}$$

$$\partial_{III}(\hat{p}) = \frac{1}{h}[\widehat{H}_{III}, \hat{p}] = -2\hat{p}\hat{q}\hat{p} + \hat{q}\hat{p} + \hat{p}\hat{q} - (\alpha_0 + \alpha_2)\hat{p} + \alpha_0. \tag{25}$$

We define transformations s_i $(0 \le i \le 2)$ on $\mathcal{K}_{III}$ as follows:

	α_0	α_1	α_2	t	$\hat{q}$	$\hat{p}$
s_0	$-\alpha_0$	$\alpha_1+\alpha_0$	α_2	t	$\hat{q}+\alpha_0\hat{p}^{-1}$	$\hat{p}$
s_1	$\alpha_0+2\alpha_1$	$-\alpha_1$	$\alpha_2+2\alpha_1$	$-t$	$\hat{q}$	$\hat{p}-2\alpha_1\hat{q}^{-1}+t\hat{q}^{-2}$
s_2	α_0	$\alpha_1+\alpha_2$	$-\alpha_2$	t	$\hat{q}+\alpha_2(\hat{p}-1)^{-1}$	$\hat{p}$

Proposition 7 *Transformations s_i preserve the commutation relations, that is, s_i become automorphisms on $\mathcal{K}_{III}$.*

Theorem 8 *The derivation ∂_{III} commutes with automorphisms s_i $(0 \le i \le 2)$ and s_i $(0 \le i \le 2)$ give a representation of the affine Weyl group of type $C_2^{(1)}$, namely, s_i satisfy the following relations:*

$$s_i^2 = 1, \quad (s_0s_1)^4 = 1, \quad (s_1s_2)^4 = 1. \tag{26}$$

2.5 The case of $\mathrm{QP_{II}}$

Let $\mathcal{K}_{II}$ be a skew field over $\mathbb{C}$ with generators α_0, α_1, $\hat{f}_i$ $(0 \le i \le 2)$ and the commutation relations

$$[\hat{f}_1, \hat{f}_0] = 2h\hat{f}_2, \quad [\hat{f}_0, \hat{f}_2] = [\hat{f}_2, \hat{f}_1] = h \quad (h \in \mathbb{C}), \tag{27}$$

$$[\hat{f}_i, \alpha_j] = 0. \tag{28}$$

Let an element $\widehat{H}_{II}$ in $\mathcal{K}_{II}$ be defined by

$$\widehat{H}_{II} = \frac{1}{2}(\hat{f}_0\hat{f}_1 + \hat{f}_1\hat{f}_0) + \alpha_1\hat{f}_2 \tag{29}$$

Let a $\mathbb{C}$-derivation ∂_{II} on $\mathcal{K}_{II}$ be defined by

$$\partial_{II}\hat{f}_i = \frac{1}{h}[\widehat{H}_{II}, \hat{f}_i] + \delta_{i,0}k \quad (i = 0, 1, 2), \tag{30}$$

$$\partial_{II}\alpha_i = \frac{1}{h}[\widehat{H}_{II}, \alpha_i] \quad (i = 0, 1), \tag{31}$$

where $k = \alpha_0 + \alpha_1$. We write down the derivation ∂_{II} for $\hat{f}_i$.

$$\partial_{II}\hat{f}_0 = \hat{f}_0\hat{f}_2 + \hat{f}_2\hat{f}_0 + \alpha_0, \quad \partial_{II}\hat{f}_1 = -\hat{f}_1\hat{f}_2 - \hat{f}_2\hat{f}_1 + \alpha_1, \quad \partial_{II}\hat{f}_2 = \hat{f}_1 - \hat{f}_0, \tag{32}$$

If we introduce $\hat{p}, \hat{q}$ by

$$\hat{f}_1 = \hat{q}, \quad \hat{f}_2 = \hat{p}, \tag{33}$$

then $(\hat{p}, \hat{q})$ is a canonical coordinate, namely it holds $[\hat{p}, \hat{q}] = h$.

We define transformations s_i $(0 \le i \le 2)$ for the generators of $\mathcal{K}_{II}$ as follows:

	α_0	α_1
s_0	$-\alpha_0$	$\alpha_1+2\alpha_0$
s_1	$\alpha_0+2\alpha_1$	$-\alpha_1$

	$\hat{f}_0$	$\hat{f}_1$
s_0	$\hat{f}_0$	$\hat{f}_1-\alpha_0(\hat{f}_2\hat{f}_0^{-1}+\hat{f}_0^{-1}\hat{f}_2)-(\alpha_0\hat{f}_0^{-1})^2$
s_1	$\hat{f}_0+\alpha_1(\hat{f}_2\hat{f}_1^{-1}+\hat{f}_1^{-1}\hat{f}_2)-(\alpha_0\hat{f}_1^{-1})^2$	$\hat{f}_1$

	$\hat{f}_2$
s_0	$\hat{f}_2+\alpha_0\hat{f}_0^{-1}$
s_1	$\hat{f}_2-\alpha_1\hat{f}_1^{-1}$

Proposition 9 *Transformations s_i preserve the commutation relations, that is, s_i become automorphisms on $\mathcal{K}_{II}$.*

Theorem 10 *The derivation ∂_{II} commutes with automorphisms s_0, s_1 and s_0, s_1 give a representation of the affine Weyl group of type $A_1^{(1)}$, namely, s_i satisfy the following relations:*

$$s_i^2=1. \tag{34}$$

3 Schlesinger equation

In this section, we review the Schlesinger equations and their Hamiltonian structure.

The Schlesinger equations are the following:

$$\frac{\partial A_i}{\partial z_j}=\frac{[A_i,A_j]}{z_i-z_j},\quad i\neq j,\quad i,j=1,\ldots,n$$
$$\frac{\partial A_i}{\partial z_i}=-\sum_{j=1,j\neq i}^{n}\frac{[A_i,A_j]}{z_i-z_j},$$

where A_i are $r\times r$ matrices whose entries are functions of z_j $(i,j=1,\ldots,n)$ over $\mathbb{C}$. The Schlesinger equations are derived from isomonodromy deformation for rational connections of regular singular type. Accordingly, the Schlesinger equations are equivalent to the following relations:

$$[\nabla_z,\nabla_i]=0,$$

where

$$\nabla_z = \frac{\partial}{\partial z} - \sum_{i=1}^{n} \frac{A_i}{z - z_i}, \quad \nabla_i = \frac{\partial}{\partial z_i} + \frac{A_i}{z - z_i}.$$

We define Hamiltonians H_i by

$$H_i = \sum_{j=1, j \neq i}^{n} \frac{\operatorname{tr}(A_i A_j)}{z_j - z_i},$$

and Poisson bracket by

$$\{(A_i)_{ab}, (A_j)_{cd}\} = \delta_{ij} \left(\delta_{bc}(A_i)_{ad} - \delta_{da}(A_i)_{cb}\right).$$

Then, it holds

$$\frac{\partial A_i}{\partial z_j} = \{H_j, A_i\}.$$

Remark 11 *The Poisson bracket above is induced from the corresponding Lie algebra $\mathfrak{gl}_r$. We can induce the Poisson bracket to the dual of a Lie algebra from the Lie bracket of the Lie algebra.*

Remark 12 *In order to derive the sixth Painlevé equation from the Schlesinger equation, we consider the case of four singular points and $r = 2$. At this moment, we have a nonlinear third order differential equation. Moreover, we take a reduction by $SL(2)$ action to introduce a Poisson bracket appropriately. Then we obtain a nonlinear second order differential equation, and that equation is the sixth Painlevé equation.*

4 Knizhnik-Zamolodchikov equation

In this section, we construct the Knizhnik-Zamolodchikov equations by Lie algebra and we see that the Knizhnik-Zamolodchikov equations are quantization of the Schlesinger equations.

Let (ρ_i, V_i) be representations of a simple Lie algebra $\mathfrak{g}$ and let

$$V = V_1 \otimes V_2 \otimes \cdots \otimes V_n, \quad \Omega_{ij} = \sum_{a=1}^{d} \rho_i(J^a)\rho_j(J_a),$$

where $\{J^a\}$ is a basis of $\mathfrak{g}$ and $\{J_a\}$ is the dual basis.

The Knizhnik-Zamolodchikov equations are the following:

$$\kappa\frac{\partial}{\partial z_i}\psi = \left(\sum_{j=1, j\neq i}^{n}\frac{\Omega_{ij}}{z_i - z_j}\right)\psi, \quad i = 1, \ldots, n,$$
$$\psi(z_1, \ldots, z_n) \in V.$$

To obtain quantum Schlesinger equations, we move from Schlesinger picture to Heisenberg picture. As usual, introducing the invertible element U satisfying

$$\kappa\frac{\partial U}{\partial z_i} = \left(\sum_{j=1, j\neq i}^{n}\frac{\Omega_{ij}}{z_i - z_j}\right)U, \quad (i = 1, \ldots, n)$$

and $\widetilde{U} \in V_0 \otimes V_1 \otimes \cdots \otimes V_n$ satisfying

$$\kappa\frac{\partial \widetilde{U}}{\partial z} = \left(\sum_{j=1}^{n}\frac{\Omega_{0j}}{z - z_j}\right)\widetilde{U},$$

we have for $Y = U^{-1}\widetilde{U}$,

$$\nabla_z Y = 0, \quad \nabla_i Y = 0, \tag{35}$$

where

$$\nabla_z = \frac{\partial}{\partial z} - \sum_{i=1}^{n}\frac{\widehat{A}_i}{z - z_i}, \quad \nabla_i = \frac{\partial}{\partial z_i} + \frac{\widehat{A}_i}{z - z_i}, \quad \widehat{A_i} = U^{-1}\Omega_0 U.$$

If we take V_0 as a matrix representation, $\widehat{A}_i$ is a matrix whose entries are elements in $U(\mathfrak{g})$ and the compatibility condition of (35) can be regarded as a quantization of the Schlesinger equations.

5 Generalized Knizhnik-Zamolodchikov equations

In this section, we give generalized Knizhnik-Zamolodchikov equations. This is a joint work with M. Jimbo and J. Sun.

Let $\mathfrak{g}$ be $\mathfrak{sl}_2 = \langle e, f, h\rangle$ and M_i $(i = 1, \dots, n)$ Verma modules of $\mathfrak{g}$ with highest weights m_i and with highest weight vectors v_i, and M_∞ a module $U(\lambda\mathfrak{g}[\lambda]/\lambda^{r+1}\mathfrak{g}[\lambda] \oplus d_\infty)v_\infty$ such that

$$(e[p])v_\infty = 0, \quad (h[p])v_\infty = \gamma_p v_\infty, \quad p = 1, \dots, r, \quad x[p] = x \otimes \lambda^p,$$

$$d_\infty v_\infty = 0, \quad [d_\infty, h[p]] = 0, \quad [d_\infty, e[p]] = e[p], \quad [d_\infty, f[p]] = -f[p].$$

We define differential operators for $u(z_i, \gamma_p) \in M_1 \otimes \cdots \otimes M_n \otimes M_\infty$:

$$D_l = \sum_{p=1}^{r-l} p\gamma_{p+l}\frac{\partial}{\partial \gamma_p} \quad (0 \le l \le r-1),$$

$$\nabla_i = \kappa\frac{\partial}{\partial z_i} - \left(\sum_{j=1, j\neq i}^{n} \frac{\Omega_{ij}}{z_i - z_j} - \sum_{p=1}^{r} z_i^{p-1}\Omega_{i\infty}[p]\right) \quad (1 \le i \le n),$$

$$\nabla_\infty^{(l)} = \kappa D_l - \left(\sum_{j=1}^{n}\sum_{p=0}^{r-l} z_j^p \Omega_{j\infty}[p+l] - \Omega_\infty[l]\right),$$

where

$$\Omega_{ij} = \frac{1}{2}h_i h_j + e_i f_j + e_j f_i, \quad \Omega_{i\infty}[p] = \frac{1}{2}h_i h[p] + e_i f[p] + f_i e[p],$$

$$\Omega_\infty[l] = \frac{1}{2}\sum_{\mu+\nu=l}\left(\frac{1}{2}h[\mu]h[\nu] + e[\mu]f[\nu] + f[\mu]e[\nu]\right).$$

Proposition 13 *we have the following for* $1 \le i, j \le n$ *and* $0 \le l, m \le r-1$.

$$[\nabla_i, \nabla_j] = 0,$$
$$[\nabla_i, \nabla_\infty^{(l)}] = 0,$$
$$[\nabla_\infty^{(l)}, \nabla_\infty^{(m)}] = \kappa(l-m)\nabla_\infty^{(l+m)}.$$

Generalized KZ equations are the following:

$$\nabla_i u = 0 \quad (1 \le i \le n),$$

$$\nabla_\infty^{(l)} u = \left(\frac{1}{4}\sum_{\mu+\nu=l}\gamma_\mu\gamma_\nu + \left(\frac{l-1}{2} - d\right)\gamma_l\right) u,$$

$$(d = \frac{1}{2}\sum_{i=1}^{n} h_i + d_\infty, \quad 1 \le l \le r-1),$$

$$\kappa\left(\sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} - D_0\right) u =$$

$$\left(d(d+1) - \sum_{i=1}^{n} \frac{m_i}{2}\left(\frac{m_i}{2}+1\right) + \frac{1}{2}\sum_{p=1}^{r}\gamma_p \sum_{i=1}^{n} m_i z_i^{p-1}\right) u.$$

Solution to generalized Knizhnik-Zamolodchikov equations are given by a straightforward generalization of the known integral formula for the KZ equation.

We set

$$u = \int \Phi \omega_k v, \quad v = v_1 \otimes \cdots \otimes v_n \otimes v_\infty$$

$$\Phi = \prod_{j=1}^{n} \exp\left[-\frac{m_j}{2\kappa}\phi^{(r)}(z_j)\right] \prod_{1 \le j < l \le n} (z_j - z_l)^{\frac{m_j m_l}{2\kappa}}$$

$$\times \prod_{a=1}^{k}\left\{\exp\left[\frac{1}{\kappa}\phi^{(r)}(t_a)\right]\prod_{j=1}^{n}(z_j - t_a)^{-\frac{m_j}{\kappa}}\right\} \prod_{1 \le a < b \le k} (t_a - t_b)^{\frac{2}{\kappa}},$$

$$\omega_k = \omega_1(t_1) \wedge \cdots \wedge \omega_1(t_k), \quad \omega_1(t) = \left(\sum_{i=1}^{n}\frac{f_i}{z_i - t} + \sum_{p=1}^{r} f[p]t^{p-1}\right) dt,$$

$$\phi^{(r)}(t) = \sum_{p=1}^{r}\gamma_p \frac{t^p}{p}.$$

Proposition 14 *The element u is a solution to the generalized KZ equations.*

Now, we compute an example corresponding to $\mathrm{QP}_{\mathrm{IV}}$ case. For $n = 2$, $z_1 = z$, $z_2 = 0$, $r = 2$ and $M = M_a \otimes M_0 \otimes M_\infty$, we have

$$\kappa \frac{\partial}{\partial z}\widetilde{U} = \left(\frac{\Omega_{a0}}{z} - \Omega_{a,\infty}[2]z - \Omega_{a,\infty}[1]\right)\widetilde{U},$$

$$\kappa\gamma_2 \frac{\partial}{\partial \gamma_1}\widetilde{U} = \left(-\Omega_{a,\infty}[2]z - \Omega_{a,\infty}[1] - \Omega_{0\infty}[1] + \gamma_1\left(\frac{1}{2}h_a + \frac{1}{2}h_0 + d_\infty\right)\right)\widetilde{U},$$

and for $n = 1$, $z_1 = 0$, $r = 2$ and $M' = M_0 \otimes M_\infty$, we have

$$\kappa\gamma_2 \frac{\partial}{\partial\gamma_1} U = (-\Omega_{0\infty}[1] + \gamma_1 d)\, U$$

Choosing parameters,

$$\begin{aligned} &\gamma_1 = -2\kappa t, \quad \gamma_2 = -2\kappa, \quad \hbar = -\frac{1}{\kappa}, \\ &\bar{x}[p] = \hbar\, U^{-1} x[p] U \quad (p = 1, 2), \quad \bar{x}_0 = \hbar\, U^{-1} x_0 U, \end{aligned}$$

and setting

$$h[1] = \gamma_1, \quad e[2] = f[2] = 0, \quad h[2] = \gamma_2,$$

we have Lax equations for $Y = U^{-1}\widetilde{U}$:

$$\begin{aligned} &\frac{\partial}{\partial z} Y = \left(Az + B - Cz^{-1}\right) Y, \quad \frac{\partial}{\partial t} Y = (Az + B')\, Y, \\ &A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = B' + tA, B' = \begin{pmatrix} 0 & \bar{f}[1] \\ \bar{e}[1] & 0 \end{pmatrix}, C = \begin{pmatrix} \bar{h}_0/2 & \bar{f}_0 \\ \bar{e}_0 & -\bar{h}_0/2 \end{pmatrix}. \end{aligned}$$

From the Lax equations, we obtain the deformation equations

$$\begin{aligned} &\frac{\partial}{\partial t}\bar{f}[1] = -2t\bar{f}[1] - 2\bar{f}_0, \quad \frac{\partial}{\partial t}\bar{e}[1] = 2t\bar{e}[1] + 2\bar{e}_0, \\ &\frac{\partial}{\partial t}\bar{h}_0 = 2(\bar{f}[1]\bar{e}_0 - \bar{f}_0\bar{e}[1]), \\ &\frac{\partial}{\partial t}\bar{f}_0 = -\bar{f}[1]\bar{h}_0, \quad \frac{\partial}{\partial t}\bar{e}_0 = \bar{e}[1]\bar{h}_0. \end{aligned}$$

Since first integrals are the following

$$C_0 = \bar{f}_0\bar{e}_0 + \frac{\bar{h}_0}{2}\left(\frac{\bar{h}_0}{2} + \hbar\right), \quad C_1 = \bar{f}[1]\bar{e}[1] - \bar{h}_0,$$

we set

$$C_0 = (\theta_0 - \frac{\hbar}{2})(\theta_0 + \frac{\hbar}{2}), \quad C_1 = -2\theta_\infty - \hbar.$$

In order to obtain a non-commutative Painlevé equation, we introduce new variables u, ζ, η by

$$\bar{f}[1] = u, \quad \bar{f}[1]\bar{e}[1] = 2(\zeta - \theta_0 - \theta_\infty - \frac{\hbar}{2}),$$

$$\bar{f}_0 = \frac{1}{2}u\eta, \qquad \bar{e}_0 = -2(u\eta)^{-1}\left(\zeta + \frac{\hbar}{2}\right)\left(\zeta + \frac{\hbar}{2} - 2\theta_0\right),$$
$$\bar{h}_0 = 2(\zeta - \theta_0).$$

Commutation relations for $u, \zeta, \eta, \theta_0, \theta_\infty$ are the following:

$$[\theta_\infty, \eta] = 0, \ [\theta_\infty, \zeta] = 0, \ [\theta_\infty, u] = -\hbar\, u,$$
$$[u, \eta] = 0, \ [u, \zeta] = 0, \ [\eta, \zeta] = \hbar\, \eta, \ \theta_0 \text{ is central.}$$

Then we have the following non commutative differential equations

$$\frac{\partial}{\partial t}\eta = -4(\zeta - \theta_0) + (\eta + 2t)\eta,$$
$$\frac{\partial}{\partial t}\zeta = -\frac{2}{\eta}\left(\zeta + \frac{\hbar}{2}\right)\left(\zeta + \frac{\hbar}{2} - 2\theta_0\right) - \eta(\zeta - \theta_0 - \theta_\infty - \frac{\hbar}{2})$$
$$\frac{\partial}{\partial t}u = -u(\eta + 2t).$$

Taking

$$\eta = -\hat{f}_2, \quad \zeta = -\frac{1}{4}(\hat{f}_1\hat{f}_2 + \hat{f}_2\hat{f}_1), \quad \hat{f}_0 + \hat{f}_1 + \hat{f}_2 = 2t,$$
$$\theta_0 = -\frac{\alpha_2}{2}, \quad \theta_\infty = \frac{2\alpha_1 + \alpha_2}{4}, \quad \hbar = \frac{h}{2}, \quad \frac{\partial}{\partial t} = \partial_{IV},$$

we obtain $\mathrm{QP_{IV}}$

$$\partial_{IV}(\hat{f}_i) = \hat{f}_i\hat{f}_{i+1} - \hat{f}_{i-1}\hat{f}_i + \alpha_i, \quad [\hat{f}_i, \hat{f}_{i+1}] = h, \quad (i \in \mathbb{Z}/3\mathbb{Z}).$$

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