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CMC-trinoids with embedded ends: a closer look

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Contents

1 Introduction

Among the surfaces of constant mean curvature $H \neq 0$, CMC-surfaces for short, only a few subclasses have been classified. The first ones have been the surfaces of rotation among the CMC-surfaces, the Delaunay surfaces. They were found almost 200 years ago [3], and are still of interest, since every embedded CMC-end is asymptotically a Delaunay surface [15].

More generally, CMC-immersions of round cylinders into $\mathbb{R}^3$ are fairly well understood. The class of CMC-tori has been investigated extensively using different mathematical techniques. While perhaps not explicitly classifiable, the class of CMC-tori is clearly so far the best investigated one among all CMC-immersions.

All the surface classes mentioned so far have an abelian fundamental group. The simplest non-abelian groups are perhaps those which are free and have only two generators. Thus it seems to be particularly important to understand the CMC-immersions of the *trinoid* $\mathcal{T}_3 = S^2 \setminus \{0, 1, \infty\}$.

Among the CMC-immersions from $\mathcal{T}_3$ to $\mathbb{R}^3$ clearly the embeddings are of particular interest. It seems to be diffcult to classify this class of CMC-immersions. However, in a beautiful piece of work, Große-Brauckmann, Kusner and Sullivan have classified the Alexandrov embedded CMC-surfaces from $\mathcal{T}_3$ to $\mathbb{R}^3$ [12].

In [16] it was shown, however, that there are CMC-immersions from $\mathcal{T}_3$ to $\mathbb{R}^3$ which have embedded ends, but are not Alexandrov embedded. Related to this is the question whether one can add "bubbles" to all trinoids and where on the surface such bubbles can be addded, if at all. It is therefore of great importance to the understanding of all CMC-immersions to understand in detail CMC-trinoids with embedded ends.

Examples of such surfaces have been given in [11] using the generalized Weierstraß representation [10]. In [4] it is shown that all CMC-immersions from $\mathcal{T}_3$ to $\mathbb{R}^3$ with embedded ends can be obtained this way. However, an answer to the questions raised above requires a much more detailed study of the monodromy matrices associated with the ends than given in [4] or [16].

In this note we announce recent work characterizing the monodromy matrices of CMC-trinoids with embedded ends. In particular, we investigate the monodromy matrices of such CMC-surfaces which are invariant under the rotation by 120° about some axis.

Our approach uses the loop group method [10] and more specifically the potentials presented in [11]. It is well known that in this setting everything can be expressed in terms of hypergeometric functions. In particular, under certain normalizations, an explicit expression can be found for the monodromy matrices (section 3). Finally, in the last section we investigate trinoids which are invariant under a 120° rotation about some axis. In [9] we prove that after some $\lambda-$dependent rotation we can assume that all surfaces of the associated family have the same axis of rotation. Moreover, all possible monodromy matrices can be determined explicitly.

2 Outline of the loop group method

We begin by giving a brief review of the "loop group method" for the construction of constant mean curvature surfaces from holomorphic potentials as presented in [10]. (This method is often also referred to as the "DPW-method".) This review will involve introducing the basic concepts (loop groups, Iwasawa decomposition, holomorphic potentials) as well as giving an outline of the loop group method itself, illustrated by the simple example of the so called "Delaunay surfaces".

2.1 Loop Groups

Let $SL(2,\mathbb{C})$ denote the special linear group of complex 2×2 matrices and $\sigma : SL(2,\mathbb{C}) \rightarrow SL(2,\mathbb{C})$ denote the conjugation by the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For each $r \in (0,1]$, we define by

$$\Lambda_r SL(2,\mathbb{C})_\sigma = \{g : C^{(r)} \rightarrow SL(2,\mathbb{C}) \text{ smooth } ; \ g(-\lambda) = \sigma(g(\lambda))\} \qquad (2.1.1)$$

the *(twisted) loop group* of smooth maps from the r-cirle $C^{(r)} = \{\lambda \in \mathbb{C}; |\lambda| = r\}$ into $SL(2,\mathbb{C})$. The condition $g(-\lambda) = \sigma(g(\lambda))$ will be referred to as the "twisting condition" for g. The Lie algebra $\Lambda_r sl(2,\mathbb{C})_\sigma$ of $\Lambda_r SL(2,\mathbb{C})_\sigma$ is given by

$$\Lambda_r sl(2,\mathbb{C})_\sigma = \{x : C^r \to sl(2,\mathbb{C}) \text{ smooth } ; x(-\lambda) = \sigma_3 x(\lambda)\sigma_3\}, \tag{2.1.2}$$

where $sl(2,\mathbb{C})$ denotes the Lie algebra of $SL(2,\mathbb{C})$.

Furthermore, we denote by $\Lambda_r^+ SL(2,\mathbb{C})_\sigma$ the subgroup of maps $g \in \Lambda_r SL(2,\mathbb{C})_\sigma$ that extend holomorphically to the open disc $I^{(r)} = \{\lambda \in \mathbb{C}; |\lambda| < r\}$, and - by abuse of notation - by $\Lambda_r SU(2)_\sigma$ the subgroup of maps $g \in \Lambda_r SL(2,\mathbb{C})_\sigma$ that extend holomorphically to the open annulus $A^{(r)} = \{\lambda \in \mathbb{C}; r < |\lambda| < \frac{1}{r}\}$ and take values in the special unitary group $SU(2)$ on the unit circle $S^1 = C^{(1)}$.

In the case $r = 1$, we will omit any subscripts "r", simply denoting the groups $\Lambda_1 SL(2,\mathbb{C})_\sigma$, $\Lambda_1^+ SL(2,\mathbb{C})_\sigma$, $\Lambda_1 SU(2)_\sigma$ by $\Lambda SL(2,\mathbb{C})_\sigma$, $\Lambda^+ SL(2,\mathbb{C})_\sigma$, $\Lambda SU(2)_\sigma$, respectively. We deal analogously with the corresponding Lie algebras.

The topology introduced above for the loop groups and Lie algebras is a Frechet topology. Sometimes it is preferable to work with Banach structures instead of with Frechet structures. In this case one could require, e.g., that all matrix coefficients are contained in the Wiener algebra on the unit circle. For the purposes of this article the topology of the groups will play a minor role.

2.2 Iwasawa decomposition

It is known from [17] that the multiplication map $\Lambda_r SU(2)_\sigma \times \Lambda_r^+ SL(2,\mathbb{C})_\sigma \to \Lambda_r SL(2,\mathbb{C})_\sigma$ is surjective, that is, any $g \in \Lambda_r SL(2,\mathbb{C})_\sigma$ may be written as

$$g = g_u g_+, \tag{2.2.1}$$

where $g_u \in \Lambda_r SU(2)_\sigma$ and $g_+ \in \Lambda_r^+ SL(2,\mathbb{C})_\sigma$. The splitting (2.2.1) is called an *r-Iwasawa decomposition* of $g \in \Lambda_r SL(2,\mathbb{C})_\sigma$, or, if $r = 1$, just *Iwasawa decomposition* of g. By additionally requiring that $g_+(0)$ is diagonal with positive real entries, the factors of the splitting (2.2.1) are uniquely determined. In this case the multiplication map is a real-analytic diffeomorphism, therefore we will speak of the *unique r-Iwasawa decomposition* (resp. *unique Iwasawa decomposition*) of g.

2.3 Holomorphic potentials

Next we will outline how one obtains from an immersion $\psi : \tilde{M} \to \mathbb{R}^3$ of constant mean curvature $H \neq 0$ on a simply connected domain $\tilde{M} \subseteq \mathbb{C}$ an $sl(2,\mathbb{C})$-valued holomorphic differential one-form on $\tilde{M}$ involving a loop parameter $\lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, the so called "holomorphic potential" η. As we are interested in constructing CMC-immersions $\phi : M \to \mathbb{R}^3$ on a not necessarily simply connected Riemann surface M, we think of $\tilde{M}$ as of the universal cover of M, that is $M = \tilde{M}/\Gamma$, where Γ denotes the fundamental group of M (acting freely and discontinously on $\tilde{M}$). While the loop group method will allow to recover $\psi : \tilde{M} \to \mathbb{R}^3$ from the holomorphic potential η (see section 2.4), we will need to turn to the question under which circumstances ψ descends to an CMC-immersion $\phi : M \to \mathbb{R}^3$, at least for a special choice of the loop parameter λ (see section 2.5).

An immersion $\psi : \tilde{M} \to \mathbb{R}^3$ of constant mean curvature $H \neq 0$ may be interpreted as a member of an "associated family" $\psi_\lambda : \tilde{M} \to \mathbb{R}^3$, $\lambda \in S^1$, of immersions of the same constant mean curvature H. Say, $\psi = \psi_{\lambda=1}$. We may obtain the associated family ψ_λ from ψ in the following way: First, consider the "extended frame" $F : \tilde{M} \to \Lambda SU(2)_\sigma$ corresponding to ψ as defined in [10]. Note that, by construction, F involves the loop parameter $\lambda = e^{i\theta} \in S^1$. Then compute ψ_λ from F by evaluating the Sym-Bobenko formula

$$\psi_\lambda = -\frac{1}{2H}\left(\frac{\partial}{\partial\theta}F \cdot F^{-1} + \frac{i}{2}F\sigma_3 F^{-1}\right) \tag{2.3.1}$$

Remark 2.1. The extended frame F associated with ψ is not determined uniquely, but only up to the choice of some initial value $F(z_*, \lambda) \in \Lambda SU(2)_\sigma$ for some $z_* \in \tilde{M}$. It is in particular always possible to achieve $F(z_*, \lambda) = I$ for a chosen base point $z_* \in \tilde{M}$, meaning that the surface is moved by a rigid motion into the "right position".

Remark 2.2. Note that ψ_λ generated by the Sym-Bobenko formula (2.3.1) actually takes values in $su_2 = \{\frac{-i}{2}\begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix};\ x,y,z \in \mathbb{R}^3\}$. We interpret ψ_λ, $\lambda \in S^1$, as a family of CMC-immersions $\tilde{M} \to \mathbb{R}^3$ by identifying su_2 with $\mathbb{R}^3$ via the isomorphism

$$J : \mathbb{R}^3 \to su_2, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \frac{-i}{2}\begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix}. \tag{2.3.2}$$

According to [10], there exists $B_+ \in \Lambda^+ SL(2,\mathbb{C})_\sigma$ such that

$$H = FB_+ \text{ is holomorphic in both } z \in \tilde{M} \text{ and } \lambda \in \mathbb{C}^* \tag{2.3.3}$$

H is called an *holomorphic frame* associated with ψ. The corresponding *Maurer-Cartan form* $\eta = H^{-1}\mathrm{d}H$ is holomorphic in both $z \in \tilde{M}$ and $\lambda \in \mathbb{C}^*$ as well and is called the *holomorphic potential* associated with the immersion ψ.

2.4 The loop group method

As indicated above, we can construct immersions of constant mean curvature $H \neq 0$ defined on the universal cover $\tilde{M}$ of a Riemann surface $M = \tilde{M}/\Gamma$ from holomorphic potentials introduced in section 2.3 by applying the "loop group method" presented in [10]. Carrying out this procedure involves the following three steps:

1. Given a holomorphic potential η, solve the differential equation
$$\mathrm{d}H = H\eta. \tag{2.4.1}$$

2. Perform (for each $z \in \tilde{M}$) an r-Iwasawa decomposition
$$H = FB_+. \tag{2.4.2}$$

3. Interpreting F as the extended frame of a CMC-immersion ψ, compute the associated family of ψ by (2.3.1) to obtain for every $\lambda_0 \in S^1$ a CMC-immersion $\psi_{\lambda=\lambda_0}$ defined on $\tilde{M}$.

Remark 2.3. For our purposes, that is for the construction of CMC-trinoids from holomorphic potentials (cf. section 3), we can think of the starting potential η on $\tilde{M}$ as the pullback of some potential defined on $M = \tilde{M}/\Gamma$ [5]. Thus we ensure that η is invariant under the fundamental group Γ (viewed as a subgroup of the automorphism group of $\tilde{M}$). We will return to this in section 2.5.

By the theory of ordinary differential equations the solution to (2.4.1) is uniquely determined, once we prescribe an "initial value condition"

$$H(z_*) = H_0 \tag{2.4.3}$$

for an arbitrary base point $z_* \in \tilde{M}$ and some $H_0 \in \Lambda_r SL(2,\mathbb{C})_\sigma$ for some $r \in (0,1]$. Given a solution H to (2.4.1) with some initial value $H(z_*)$, it is easy to verify that $\hat{H}(z) = H_0 H(z_*)^{-1} H(z)$ will also satisfy (2.4.1) and, moreover, will meet the initial value condition (2.4.3).

In general, we will refer to the action of replacing H by

$$\hat{H} = TH, \tag{2.4.4}$$

where T denotes some z-independent loop in $\Lambda_r SL(2,\mathbb{C})_\sigma$, as *r-dressing* or simply *dressing* H by T. By dressing a solution H of (2.4.1), we will obtain a new solution $\hat{H}$ of (2.4.1), "only" changing the initial condition. Such a change, however, will have profound consequences in step two of the loop group method, as there is no trivial relation between the frames F and $\hat{F}$ involved in the Iwasawa decompositions of H and $\hat{H}$, respectively. This means, that dressing a solution H of (2.4.1) will (in general) give rise to significant changes in the CMC-immersion $\psi = \psi_{\lambda=1}$ generated by step three of the loop group method. In fact, the manipulation of the initial value H_0 given by (2.4.4) turns out to be crucial for our purposes, as it plays the decisive role when it comes to deciding whether ψ will descend to a CMC-immersion ϕ on M or not. This issue will be discussed further in the following section.

2.5 Monodromy

Next we investigate under which circumstances a given immersion ψ on the universal cover $\tilde{M}$ of a Riemann surface M will descend to an immersion ϕ defined on $M = \tilde{M}/\Gamma$. The answer to this question is closely linked to the behaviour of the holomorphic frame H associated with ψ (cf. section 2.3) under the deck transformations γ corresponding to the elements of the fundamental group Γ. This transformation behaviour of H is expressed by a z-independent matrix, the "monodromy matrix" $M = M(\gamma, \lambda)$. We will briefly state the results pertinent to this article, for more details see section 2.4 of [11].

Lemma 2.4. *Given a holomorphic potential η on $\tilde{M}$ which is invariant under Γ in the sense of remark 2.3, any solution H of (2.4.1) will transform under $\gamma \in \Gamma$ according to*

$$H(\gamma(z), \lambda) = M(\gamma, \lambda) H(z, \lambda), \tag{2.5.1}$$

where $M(\gamma,\lambda)$ denotes some $\Lambda SL(2,\mathbb{C})_\sigma$ matrix depending on γ, but independent of z. For each $\gamma \in \Gamma$, $M(\gamma,\lambda)$ is called the monodromy matrix *of H with respect to γ.*

The basic theorem for all considerations in this note is obtained from theorem 2.7 of [7]:

Theorem 2.5. *Let η, Γ and H be as in the above lemma. Then, ψ descends to a CMC-immersion ϕ on $M = \tilde{M}/\Gamma$ if and only if*

1. *$M(\gamma, \lambda)$ is unitary for all $\gamma \in \Gamma$, $\lambda \in S^1$* and
2. *$M(\gamma, \lambda = 1) = \pm\mathrm{I}$ for all $\gamma \in \Gamma$* and
3. *$\partial_\lambda M(\gamma, \lambda)|_{\lambda=1} = 0$ for all $\gamma \in \Gamma$.*

Theorem 2.5 provides the key for "tuning" the loop group method such that it will generate a CMC-immersion ψ on $\tilde{M}$ that descends to an immersion ϕ on M for, say, $\lambda = 1$: Assuming the requirements of theorem 2.5 are met, it is easy to verify that dressing a solution H of (2.4.1) by $T = T(\lambda) \in \Lambda_r SL(2,\mathbb{C})_\sigma$, as defined in section 2.4, will produce a solution $\hat{H} = TH$ with monodromy matrices $\hat{M}(\gamma, \lambda) = T(\lambda)M(\gamma, \lambda)(T(\lambda))^{-1}$ for each $\gamma \in \Gamma$, where $M(\gamma, \lambda)$, $\gamma \in \Gamma$, denote the corresponding monodromy matrices of H. Thus, to obtain a CMC-immersion $\phi : M \to \mathbb{R}^3$ from a given potential η, the strategy will be to find an appropriate dressing matrix T that will modify a given solution H with monodromy matrices $M(\gamma, \lambda)$,$\gamma \in \Gamma$, such that the monodromy matrices $\hat{M}(\gamma, \lambda)$ of $\hat{H} = TH$ will meet the conditions given in theorem 2.5.

2.6 Delaunay surfaces

As pointed out in the introduction, for the study of CMC-immersions with embedded ends "Delaunay surfaces" are of particular importance. These are CMC-surfaces of revolution around an axis x_1 in $\mathbb{R}^3$. For a detailed discussion of Delaunay surfaces, we refer to [8]. Here, we will only summarize some basic results, which we will use in this article.

By section 3.2.1 of [8] all Delaunay surfaces (up to rigid motions) can be constructed from holomorphic potentials of the form

$$\eta = \frac{1}{z} A\mathrm{d}z = \frac{1}{z}\begin{pmatrix} 0 & X \\ \overline{X} & 0 \end{pmatrix} \mathrm{d}z, \tag{2.6.1}$$

where $X(\lambda) = s\lambda^{-1} + t\lambda$, $\overline{X}(\lambda) = s\lambda + t\lambda^{-1}$ and $s, t \in \mathbb{R}$ with $(s+t)^2 = \frac{1}{4}$. The matrix A is called a *Delaunay matrix.*

The choice of s and t will determine the special shape of the produced Delaunay surface (see [8] for details). We will be especially interested in *embedded* Delaunay surfaces, which are also referred to as *unduloids.* These correspond to s and t such that $st > 0$, w.l.o.g. $s > 0$ and $t > 0$.

Given a potential η of the form (2.6.1), it is easy to verify that

$$H = \exp(\ln(z)A) \tag{2.6.2}$$

solves the differential equation (2.4.1). Around the singularity at $z = 0$ of η, H picks up the *Delaunay monodromy matrix* $M(\lambda)$:

$$H(\gamma(z), \lambda) = M(\gamma, \lambda)H(z, \lambda), \tag{2.6.3}$$

where γ denotes the deck transformation on $\tilde{M}$ corresponding to the simply closed curve in $\mathbb{C}$, which encloses the singularity $z = 0$, w.l.o.g. defined by $(-\pi, \pi) \to \mathbb{C}$, $t \mapsto e^{it}$. Note that γ already generates the fundamental group Γ of $M = \mathbb{C}\backslash\{0\}$. A simple computation yields

$$M(\gamma, \lambda) = exp(2\pi i A), \tag{2.6.4}$$

where $\mu = \sqrt{X\overline{X}}$. By use of lemma 2.5, we conclude that H gives rise to a CMC-immersion $\psi = \psi_{\lambda=1}$ which under our conditions descends to M, thus defining a CMC-surface ϕ on M.

3 Trinoids

In this note we are concerned with constant mean curvature surfaces parameterized by the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with three points removed, which are singularities of the parameterizing immersion ϕ and thus induce three "ends" of the surface. By applying an appropriate coordinate transformation on $\hat{\mathbb{C}}$ we may assume w.l.o.g. that the singularities are located at $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$. While we allow arbitrary self-intersections of the surface away from its singularities, we require the ends to be "embedded". That is, we require that for sufficiently small pointed discs $\mathbb{D}^*_\epsilon(z_j) = \{z \in \hat{\mathbb{C}}\backslash\{0, 1, \infty\}; |z - z_j| < \epsilon\}$ the immersion ϕ is a CMC-embedding. Therefore, according to [15], the ends asymptotically show the behaviour of (unduloidal) Delaunay surfaces (cf. section 2.6). Constant mean curvature surfaces with three asymptotic Delaunay ends will be called "CMC-trinoids with embedded ends" or "trinoids" for short.

It is well known that the universal cover of $M = \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$ is given by the upper half plane $\tilde{M} = \mathbb{H} = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$. The fundamental group $\Gamma = \pi_1(M)$ is generated by three simply closed curves γ_j in M, which start at a base point $z_* \in M$ and surround only one z_j. The corresponding monodromy matrices associated with a solution H of $\mathrm{d}H = H\eta$, where η is a potential defined on $\tilde{M}$, will be denoted by $M_j(\lambda) = M(\gamma_j, \lambda)$. We note (see section 2.5 of [11]) that

$$M_0 M_1 M_\infty = \pm I.$$

3.1 The trinoid potential

For the construction of trinoids via the loop group method we start with a potential η with three singularities at $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$. These singularities will carry over to the solution of the differential equation $\mathrm{d}H = H\eta$ as well as to the induced immersion ϕ parameterizing the surface and thus will generate the three trinoid ends. As explicated in section 3.1 of [11], we may restrict without loss of much generality to the case where η is off-diagonal, that is of the form

$$\eta = \begin{pmatrix} 0 & \nu(z,\lambda) \\ \tau(z,\lambda) & 0 \end{pmatrix} \mathrm{d}z, \tag{3.1.1}$$

where, for now, ν and τ denote some holomorphic functions in $z \in \hat{\mathbb{C}}\backslash\{z_0, z_1, z_\infty\}$ which also depend on $\lambda \in \mathbb{C}^*$.

Since we would like to construct CMC-trinoids with embedded (and thus, as pointed out above, asymptotically Delaunay) ends, we further assume that the potential η near each end z_j adopts some of the properties of the corresponding (unduloidal) Delaunay potential $\frac{1}{z-z_j}D_j\mathrm{d}z$ involving the off-diagonal Delaunay matrix

$$D_j = \begin{pmatrix} 0 & X_j \\ \overline{X_j} & 0 \end{pmatrix}, \tag{3.1.2}$$

where

$$X_j = s_j\lambda^{-1} + t_j\lambda, \ \ \overline{X_j} = s_j\lambda + t_j\lambda^{-1}, \tag{3.1.3}$$

$$s_j, t_j \in (0, \frac{1}{2}), \ \ s_j + t_j = \frac{1}{2}. \tag{3.1.4}$$

First of all, as the singularities of Delaunay potentials are regular, we require all singularities z_j of η to be regular singularities. More precisely, they will be regular singular points of a second order scalar ODE corresponding to the differential equation

$$\mathrm{d}H = H\eta \tag{3.1.5}$$

in the sense of the following straightforward lemma.

Lemma 3.1. *Every solution H of the differential equation* (3.1.5) *can be written in the form*

$$H = \begin{pmatrix} \frac{y_1'}{\nu} & y_1 \\ \frac{y_2'}{\nu} & y_2 \end{pmatrix}, \tag{3.1.6}$$

where y_1, y_2 is a fundamental system of the differential equation

$$y'' - \frac{\nu'}{\nu}y' - \nu\tau y = 0. \tag{3.1.7}$$

We require that z_0, z_1, z_∞ are regular singular points of (3.1.7), i.e. we require that equation (3.1.7) is of "Fuchsian type" (cf., e.g., chapter 7 of [1]) with three singular points. In general, a Fuchsian equation can have a singularity which, however, does not show up in the solutions. Such a singularity is called "apparent singularity". In our case we do not want any apparent singularities, since otherwise we would have fewer than three embedded ends. From [1] and sections 3.3 and 3.5 of [11] we obtain that the three ends at $0, 1$ and ∞ are non-apparent regular singular points if and only if:

$$\nu(z,\lambda) = \lambda^{-1}z^{-a_0}(z-1)^{-a_1}, \tag{3.1.8}$$

$$\tau(z,\lambda) = -\lambda z^{a_0}(z-1)^{a_1}\left[\frac{b_0(\lambda)}{z^2} + \frac{b_1(\lambda)}{(z-1)^2} + \frac{c_0(\lambda)}{z} + \frac{c_1(\lambda)}{z-1}\right], \tag{3.1.9}$$

for some integers a_0, a_1, a_∞ and some even functions b_0, b_1, b_∞, c_0, c_1 in $\lambda \in \mathbb{C}^*$ satisfying

$$a_0 + a_1 + a_\infty = 2, \tag{3.1.10}$$

$$b_0(\lambda) + b_1(\lambda) + 0 \cdot c_0(\lambda) + 1 \cdot c_1(\lambda) = b_\infty(\lambda), \tag{3.1.11}$$

$$c_0(\lambda) + c_1(\lambda) = 0. \tag{3.1.12}$$

The potential above is defined on M. Therefore, its pullback to the universal cover $\tilde{M}$ is invariant under the deck transformations which correspond to surrounding an end. Thus, given a solution H of (3.1.5), after surrounding an end we obtain a monodromy matrix (cf. section 2.5). One should expect that the monodromy matrix M_j corresponding to the end at z_j is somehow related to the monodromy matrix of the Delaunay surface which is the asymptotic shape of the end at z_j. It is therefore natural to assume that M_j is conjugate to the monodromy matrix of the corresponding Delaunay surface. Actually, one can prove that this is necessarily the case [4]. Referring to section 3.6 of [11], this is equivalent to requiring for each $j \in \{0, 1, \infty\}$

$$b_j(\lambda) = \frac{1}{4}(1 - a_j)^2 - \mu_j^2, \tag{3.1.13}$$

where

$$\mu_j = \sqrt{X_j \overline{X_j}} \tag{3.1.14}$$

and $\pm\mu_j$ are the eigenvalues of D_j.

By the choice of D_0, D_1, D_∞ and some integers a_0, a_1, a_∞ satisfying (3.1.10) the functions b_j and c_j are given by equations (3.1.13), (3.1.11) and (3.1.12) explicitly, whereby η is determined completely. While the choice of the a_j seems to be arbitrary (in fact, it is unknown, if there is any geometric meaning in the choice of the a_j at all), the choice of the D_j will determine whether the associated potential η will give rise to a "descending" CMC-immersion ψ in the sense of theorem 2.5. In order to ensure this, we further need to require for $\lambda \in S^1$

$$0 \leq \frac{cos(\pi(\mu_0 - \mu_1 - \mu_\infty))cos(\pi(\mu_0 - \mu_1 + \mu_\infty))}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \leq 1. \tag{3.1.15}$$

Equation (3.1.15) will be referred to as the "unitarizability condition", as it is equivalent with the existence of a solution H of (3.1.5), η corresponding to D_0, D_1, D_∞ which has *unitary* monodromy matrices at the singularities at z_0, z_1, z_∞. In other words, (3.1.15) holds if and only if any solution of (3.1.5) can be "dressed" (cf. section 2.4) into a new solution with unitary monodromy matrices.

Altogether, by [11], theorem 5.4.1 and corollary 5.4.2, we have in fact

Theorem 3.2. *Let D_0, D_1, D_∞ be Delaunay matrices satisfying* (3.1.15) *for all $\lambda \in S^1$. Let η be of the form* (3.1.1) *associated with the given Delaunay matrices. Assume that η satisfies equations* (3.1.8) *to* (3.1.13). *Then, η yields a CMC-trinoid with embedded ends after some appropriate r-dressing.*

Any potential η meeting the requirements of the theorem above will be called a "trinoid potential". Note that such potentials are holomorphic in $z \in \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$ and in $\lambda \in \mathbb{C}^*$, as well as meromorphic in $z \in \hat{\mathbb{C}}$ with singularities at $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$.

Remark 3.3. It is claimed in [4] that all CMC-trinoids with embedded ends can be constructed via the loop group method from potentials of the form above.

Starting with a trinoid potential η, any solution H of (3.1.5), by theorem 3.2, can be dressed by some appropriately chosen matrix $T = T(\lambda)$ into a solution $\hat{H} = TH$, which in turn will produce a CMC-trinoid with embedded ends, defined on M. If H

picks up the monodromy matrix $M_j(\lambda)$ around the end z_j, $\hat{H}$ has the monodromy matrix $\hat{M}_j(\lambda) = T(\lambda)M_j(\lambda)(T(\lambda))^{-1}$ at z_j (cf. section 2.5). These monodromy matrices are, by theorem 2.5, unitary on S^1. Thus, in order to find a solution $\hat{H}$ yielding a CMC-trinoid on M in the sense of theorem 2.5, we will perform the following two steps:

1. We compute a solution H of (3.1.5) with monodromy matrix M_j at z_j (see sections 3.2 and 3.3).
2. We determine the dressing matrix T explicitly such that the "dressed monodromy matrices" $\hat{M}_j$ are unitary on S^1 (see section 3.4).

The "right" solution $\hat{H}$ is then given by $\hat{H} = TH$.

Remark 3.4. Note that, in fact, we need to make sure that the dressed monodromy matrices $\hat{M}_j$ meet all three conditions of theorem 2.5. It turns out, however, that finding a T which simultaneously unitarizes all the monodromy matrices M_j poses the main difficulty. Furthermore keep in mind that verifying the three conditions of theorem 2.5 for the three monodromy matrices $\hat{M}_j$ at the ends z_j will imply that these conditions hold for any other monodromy matrix $\hat{M}(\gamma, \lambda)$, $\gamma \in \Gamma$, as well (since the curves γ_j corresponding to $\hat{M}_j$, $j = 0, 1, \infty$, generate the fundamental group Γ). In view of $M_0 M_1 M_\infty = \pm I$, it will even suffice to check the conditions for two of the three monodromy matrices, w.l.o.g. for $\hat{M}_0$ and $\hat{M}_1$.

3.2 The Fuchsian ODE

In order to solve (3.1.5) we take a closer look at the Fuchsian differential equation (3.1.7), which reads more explicitly as

$$y'' + \left(\frac{a_0}{z} + \frac{a_1}{z-1}\right) y' + \left(\frac{b_0}{z^2} + \frac{b_1}{(z-1)^2} + \frac{c_0}{z} + \frac{c_1}{z-1}\right) y = 0. \tag{3.2.1}$$

The corresponding indicial equations around the singularities $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$ are given by $w(w-1) + a_j w + b_j$ for $j = 0, 1, \infty$, respectively (cf. section 7.2 of [1]), and possess the roots

$$r_{j,\pm} = \frac{1}{2}\left(1 - a_j \pm \sqrt{(1-a_j)^2 - 4b_j}\right). \tag{3.2.2}$$

Defining $r_j := r_{j,+}$ and substituting

$$y = z^{r_0}(z-1)^{r_1} w, \tag{3.2.3}$$

equation (3.2.1) translates into the hypergeometric differential equation

$$w'' + \frac{-\gamma + (1 + \alpha + \beta)z}{z(z-1)} w' + \frac{\alpha\beta}{z(z-1)} w = 0, \tag{3.2.4}$$

where

$$\alpha = r_{0,+} + r_{1,+} + r_{\infty,+}, \tag{3.2.5}$$
$$\beta = r_{0,+} + r_{1,+} + r_{\infty,-}, \tag{3.2.6}$$
$$\gamma = 1 + r_{0,+} - r_{0,-}. \tag{3.2.7}$$

Assuming $\gamma \notin \mathbb{Z}$ and $\alpha + \beta - \gamma \notin \mathbb{Z}$, which is the case for all $\lambda \in \mathbb{C}^*\backslash S$, where S is a discrete subset of $\mathbb{C}^*$, there are natural fundamental systems w_{j1}, w_{j2} of (3.2.4) at z_j, $j = 0, 1$ (cf. chapter 8 of [1]):

$$w_{01} = F(\alpha, \beta, \gamma; z), \tag{3.2.8}$$
$$w_{02} = z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z), \tag{3.2.9}$$

$$w_{11} = F(\alpha, \beta, \alpha + \beta - \gamma + 1; 1 - z), \tag{3.2.10}$$
$$w_{12} = (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1; 1 - z). \tag{3.2.11}$$

where F denotes the hypergeometric series

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \frac{z^n}{n!}. \tag{3.2.12}$$

Remark 3.5. In view of remark 3.4 it suffices to unitarize the monodromy matrices of two of the three ends, since they generate the monodromy group. Therefore, our discussion of a solution H to the equation (3.1.5) can be restricted to the analysis of its behaviour "near" the two singularities at $z_0 = 0$ and $z_1 = 1$. More precisely, as a typical set of definition we will consider an open set, containing 0 and 1, from which we cut out all points contained in the half-lines extending on the real axis from 0 to $-\infty$ and from 1 to ∞. It therefore suffices to consider in this article only fundamental systems to the equation (3.2.4) defined near z_0 and z_1.

According to [1], p. 235, on the simply connected set $\mathcal{D} = \mathbb{C}\backslash\{x \in \mathbb{R}; x < 0 \text{ or } x > 1\}$, where all w_{jk} are defined, the following relations hold:

$$w_{01} = \kappa_{11}^{01} w_{11} + \kappa_{12}^{01} w_{12}, \tag{3.2.13}$$
$$w_{02} = \kappa_{11}^{02} w_{11} + \kappa_{12}^{02} w_{12}, \tag{3.2.14}$$

where

$$\kappa_{11}^{01} = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \tag{3.2.15}$$
$$\kappa_{12}^{01} = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \tag{3.2.16}$$
$$\kappa_{11}^{02} = \frac{\Gamma(\gamma - \alpha - \beta)\Gamma(2 - \gamma)}{\Gamma(1 - \alpha)\Gamma(1 - \beta)}, \tag{3.2.17}$$
$$\kappa_{12}^{02} = \frac{\Gamma(\alpha + \beta - \gamma)\Gamma(2 - \gamma)}{\Gamma(\alpha - \gamma + 1)\Gamma(\beta - \gamma + 1)}, \tag{3.2.18}$$

and Γ denotes the Gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

From (3.2.8) to (3.2.11) together with (3.2.3) we obtain fundamental systems y_{j1}, y_{j2} around z_j solving the Fuchsian equation (3.2.1):

$$y_{01} = z^{r_0} (1 - z)^{r_1} F(\alpha, \beta, \gamma; z), \tag{3.2.19}$$
$$y_{02} = z^{r_0 + 1 - \gamma} (1 - z)^{r_1} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; z), \tag{3.2.20}$$

$$y_{11} = z^{r_0}(1-z)^{r_1}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-z), \tag{3.2.21}$$

$$y_{12} = z^{r_0}(1-z)^{r_1+\gamma-\alpha-\beta}F(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;1-z). \tag{3.2.22}$$

Note that relations (3.2.13) and (3.2.14) relating the fundamental system w_{01}, w_{02} to w_{11}, w_{12} are equivalent to the relations

$$y_{01} = \kappa_{11}^{01}y_{11} + \kappa_{12}^{01}y_{12}, \tag{3.2.23}$$

$$y_{02} = \kappa_{11}^{02}y_{11} + \kappa_{12}^{02}y_{12}, \tag{3.2.24}$$

for the fundamental systems y_{01}, y_{02} and y_{11}, y_{12}.

Applying lemma 3.1 we infer that any solution to (3.1.5) can be expressed locally near z_j, $j = 0, 1$ in terms of the fundamental system y_{j1}, y_{j2}. More precisely, this is possible on a cut disc around z_j. We will keep this in mind for later use in the next section.

3.3 Solving $\mathrm{d}H = H\eta$

We want to understand the solutions to the differential equation (3.1.5) near the ends. For this it will turn out to be helpful to use special forms of the potential and the solutions. Let's consider, for $j = 0, 1$, the "gauged" potential

$$\tilde{\eta}_j = V_{+,j}^{-1}\eta V_{+,j} + V_{+,j}^{-1}\mathrm{d}V_{+,j}, \tag{3.3.1}$$

where

$$V_{+,j} = \begin{pmatrix} \sqrt{\lambda(z-z_j)\nu}\cdot\sqrt{\lambda X_j}^{-1} & 0 \\ \frac{\lambda}{2}(a_j-1)\sqrt{\lambda(z-z_j)\nu}^{-1}\cdot\sqrt{\lambda X_j}^{-1} & \sqrt{\lambda(z-z_j)\nu}^{-1}\cdot\sqrt{\lambda X_j} \end{pmatrix}. \tag{3.3.2}$$

Remark 3.6. We note that $V_{+,j}$ takes values in $\Lambda_r^+SL(2,\mathbb{C})_\sigma$ for some $r \in (0,1]$ (see section 5.2 of [11] for details). The transformation of η by $V_{+,j} \in \Lambda_r^+SL(2,\mathbb{C})_\sigma$, as described in (3.3.1), will be referred to as *gauging* η by $V_{+,j}$.

By section 4.2 of [11] there exists an *EDP-representation*[1] $\tilde{H}_j$, defined around z_j, of $\mathrm{d}\tilde{H}_j = \tilde{H}_j\tilde{\eta}_j$. We will call $\tilde{H}_j$ an *EDP-solution* for short.

$$\tilde{H}_j = e^{\ln(z-z_j)D_j}\cdot P_j, \tag{3.3.3}$$

where $P_j = I + (z-z_j)P_{j,1} + (z-z_j)^2P_{j,2} + \ldots$ is holomorphic at $z = z_j$, $P(z = z_j) = I$ and P_j is uniquely determined by these properties [2].

Thus, for $j = 0, 1$, we obtain a local solution H_j of the original differential equation (3.1.5) around z_j of the form

$$H_j = \tilde{H}_jV_{+,j}^{-1} = e^{\ln(z-z_j)D_j}P_jV_{+,j}^{-1}. \tag{3.3.4}$$

By lemma 3.1, H_j may be described in terms of an appropriate fundamental system solving (3.1.7) around z_j, which itself may be expressed in terms of the fundamental system y_{j1}, y_{j2} given in equations (3.2.19) to (3.2.22). That is, we may write for $j = 0, 1$

$$H_j = \begin{pmatrix} \frac{\alpha_jy_{j1}'+\beta_jy_{j2}'}{\nu} & \alpha_jy_{j1}+\beta_jy_{j2} \\ \frac{\delta_jy_{j1}'+\epsilon_jy_{j2}'}{\nu} & \delta_jy_{j1}+\epsilon_jy_{j2} \end{pmatrix}, \tag{3.3.5}$$

[1]The expression EDP-representation is an abbreviation for exponential-Delaunay-powerseries-representation.

where α_j, β_j, δ_j, ϵ_j denote z-independent functions in λ. Since y_{01}, y_{02}, y_{11}, y_{12} may be extended holomorphically to the complex plane excluding two "cuts" from 0 to $-\infty$ and from 1 to ∞, this also holds for H_0 and H_1. Denoting the extensions again by H_0 and H_1, respectively, we obtain two solutions of (2.4.1) - now defined globally for z from the simply connected, "double-cut" complex plane - which will only differ by a z-independent matrix $A = A(\lambda)$. That is, we have

$$H_0 = A(\lambda)H_1. \tag{3.3.6}$$

It turns out that, by evaluating in (3.3.4) the properties of both P_j and the fundamental system y_{j1}, y_{j2} at z_j, especially the holomorphy on a cut disc around z_j, the connection coefficients α_j, β_j, δ_j, ϵ_j can be computed explicitly:

$$\alpha_j = -\beta_j = \frac{1}{2}\frac{\sqrt{(-1)^{j-a_1}}}{\sqrt{\lambda X_j}}\frac{X_j}{\mu_j}, \tag{3.3.7}$$

$$\delta_j = \epsilon_j = \frac{1}{2}\frac{\sqrt{(-1)^{j-a_1}}}{\sqrt{\lambda X_j}}. \tag{3.3.8}$$

Thus, by (3.3.5) together with (3.3.7) and (3.3.8), we have explicitly determined two solutions H_0, H_1 of the differential equation (3.1.5). Furthermore, these solutions are linked by (3.3.6), and with regard to (3.2.23) and (3.2.24) we obtain

$$A = i\sqrt{\frac{\mu_1}{\mu_0}}R_0 S \begin{pmatrix} \kappa_{11}^{01} & \kappa_{12}^{01} \\ \kappa_{11}^{02} & \kappa_{12}^{02} \end{pmatrix} S^{-1}R_1^{-1}, \tag{3.3.9}$$

where $R_j = \begin{pmatrix} \sqrt{\frac{X_j}{\mu_j}} & 0 \\ 0 & \sqrt{\frac{X_j}{\mu_j}} \end{pmatrix}$ and $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Moreover, using equation (3.3.4), it is not difficult to compute the monodromy matrix of H_j around the end z_j: Choosing the curves $\gamma_0, \gamma_1 \in \Gamma$ for z close enough to z_j, such that γ_j only encloses the singularity z_j, w.l.o.g. by

$$\gamma_j(t) = z_j + \epsilon e^{it}, \quad -\pi < t < \pi, \tag{3.3.10}$$

for a small enough $\epsilon > 0$, we obtain by (3.3.4)

$$H_j(\gamma_j(z), \lambda) = \pm e^{2\pi i D_j} H_j(z, \lambda), \tag{3.3.11}$$

where γ_j now denotes the deck transformation on $\tilde{M}$ corresponding to the curve γ_j defined in (3.3.10).

Remark 3.7. Note that, by construction, P_j is holomorphic around z_j and therefore doesn't possess any monodromy around z_j. Moreover, it is easy to verify, that $V_{+,j}^{-1}$ picks up the factor $\sqrt{e^{\pi i(1-a_j)}}$, hence at most a sign, under γ_j. Therefore the monodromy matrix of H_j around z_j is up to a sign given by the Delaunay monodromy matrix $e^{2\pi i D_j}$.

Taking into account (3.3.6), we are now able to explicitly compute the monodromy matrices of H_0 at z_0 and z_1, as well as the monodromy matrices of H_1 at z_0 and z_1. It suffices to consider the solution

$$H = H_0, \tag{3.3.12}$$

which by (3.3.11) and (3.3.6) satisfies

$$H(\gamma_j(z),\lambda) = M_j(\lambda)H(z,\lambda), \tag{3.3.13}$$

where

$$M_0(\lambda) = e^{2\pi i D_0} = \cos(2\pi\mu_0)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_0)\begin{pmatrix}0 & \frac{\chi_0}{\mu_0}\\ \frac{\overline{\chi_0}}{\mu_0} & 0\end{pmatrix}, \tag{3.3.14}$$

$$M_1(\lambda) = Ae^{2\pi i D_1}A^{-1} = \cos(2\pi\mu_1)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_1)A\begin{pmatrix}0 & \frac{\chi_1}{\mu_1}\\ \frac{\overline{\chi_1}}{\mu_1} & 0\end{pmatrix}A^{-1}. \tag{3.3.15}$$

3.4 Simultaneous unitarization of the monodromy matrices

In the previous section, we have determined a solution H of (3.1.5), given explicitly by

$$H = H_0 = e^{\ln(z)D_0}P_0V_{+,0}^{-1} = \begin{pmatrix}\frac{\alpha_0 y_{01}'+\beta_0 y_{02}'}{\nu} & \alpha_0 y_{01}+\beta_0 y_{02}\\ \frac{\delta_0 y_{01}'+\epsilon_0 y_{02}'}{\nu} & \delta_0 y_{01}+\epsilon_0 y_{02}\end{pmatrix}, \tag{3.4.1}$$

where the fundamental system y_{01}, y_{02} is given by (3.2.19) and (3.2.20) and the connection coefficients $\alpha_0, \beta_0, \delta_0, \epsilon_0$ are defined in (3.3.7) and (3.3.8). We also know the monodromy matrices $M_0(\lambda)$ and $M_1(\lambda)$ of H around z_0 and z_1, respectively, by (3.3.14) and (3.3.15). By an easy calculation, we observe that, while M_0 is unitary for λ on S^1, this is not the case for M_1. Thus, returning to theorems 2.5 and 3.2, we still need to modify H by "an appropriate dressing (matrix)" T so that altogether the dressed solution $\hat{H} = TH$ yields a CMC-trinoid with embedded ends. Our next goal will be to compute explicitly a dressing matrix $T = T(\lambda)$ such that $\hat{H}$ has *unitary* monodromy matrices $\hat{M}_j = TM_jT^{-1}$ for all ends z_j, $j = 0, 1, \infty$, and for all $\lambda \in S^1$. By remark 3.4, it will suffice to require unitarity (on S^1) for $\hat{M}_0$ and $\hat{M}_1$. That is, we would like to have

$$\left.\begin{matrix}\hat{M}_0 = TM_0T^{-1}\\ \hat{M}_1 = TM_1T^{-1}\end{matrix}\right\}\text{ unitary on } S^1. \tag{3.4.2}$$

In view of (3.3.14) and (3.3.15) we observe that any conjugate of $\begin{pmatrix}0 & \frac{\chi_j}{\mu_j}\\ \frac{\overline{\chi_j}}{\mu_j} & 0\end{pmatrix}$ will be of the form $\begin{pmatrix}p_j & r_j\\ q_j & -p_j\end{pmatrix}$ for some functions $p_j(\lambda)$, $q_j(\lambda)$, $r_j(\lambda)$. Hence the monodromy matrices $\hat{M}_j$ are of the general form

$$\hat{M}_j = TM_jT^{-1} = \cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & r_j\\ q_j & -p_j\end{pmatrix}, \tag{3.4.3}$$

where $p_j^2 + q_jr_j = 1$, since $\det M_j = 1$ and conjugating a matrix does not affect its determinant.

In order to express (3.4.2) in a different way, we note that a matrix U is in $SU(2)$ iff U is of the form $U = \begin{pmatrix}u & v\\ -\bar{v} & \bar{u}\end{pmatrix}$. Applying this to (3.4.3), we see that $\hat{M}_j$ is unitary on S^1 iff it is of the form

$$\hat{M}_j = \cos(2\pi\mu_j)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin(2\pi\mu_j)\begin{pmatrix}p_j & \overline{q_j}\\ q_j & -p_j\end{pmatrix} \tag{3.4.4}$$

with

$$p_j = \overline{p_j} \quad \text{and} \quad p_j^2 + q_j\overline{q_j} = 1, \tag{3.4.5}$$

where

$$\begin{pmatrix} p_0 & \overline{q_0} \\ q_0 & -p_0 \end{pmatrix} = T \begin{pmatrix} 0 & \frac{X_0}{\mu_0} \\ \frac{X_0}{\mu_0} & 0 \end{pmatrix} T^{-1}, \tag{3.4.6}$$

$$\begin{pmatrix} p_1 & \overline{q_1} \\ q_1 & -p_1 \end{pmatrix} = TA \begin{pmatrix} 0 & \frac{X_1}{\mu_1} \\ \frac{X_1}{\mu_1} & 0 \end{pmatrix} A^{-1}T^{-1}. \tag{3.4.7}$$

Remark 3.8. We can assume that $q_0, q_1 \neq 0$: If, by accident, some T should give $q_j = 0$ for some j, it is easy to show that we can modify T by multiplying from the left by a *constant* unitary matrix U, such that $\hat{T} = UT$ will yield $q_0, q_1 \neq 0$, while unitarity on S^1 will be maintained. That is, if simultaneously unitarizing M_0, M_1 is possible at all, it is also possible in a way such that $q_0, q_1 \neq 0$.

Equations (3.4.4) to (3.4.7) give necessary and sufficient conditions for T to unitarize both M_0 and M_1. More precisely, T will render M_0 and M_1 unitary iff there exist functions p_0, q_0, p_1, q_1 depending on λ such that equations (3.4.5) to (3.4.7) hold. In this case the unitarized monodromy matrices $\hat{M}_j$ are given by (3.4.4).

Based upon this, one can proof the following theorem:

Theorem 3.9. *Let $M_0(\lambda)$, $M_1(\lambda)$ be given by* (3.3.14), (3.3.15), *respectively. Then we obtain a matrix function $T = T(\lambda)$ simultaneously unitarizing $M_0(\lambda)$ and $M_1(\lambda)$ by the following steps:*

1. *Solve*

$$p_0p_1 + \frac{q_0\overline{q_1} + \overline{q_0}q_1}{2} = \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \tag{3.4.8}$$

for functions p_0, q_0, p_1, q_1 satisfying (3.4.5).

2. *Compute ω_0 from*

$$\omega_0 = \frac{1}{2i\sqrt{\kappa_{11}^{01}\kappa_{12}^{01}}\sqrt{q_0q_1}}\sqrt{\frac{\mu_0}{\mu_1}}\sqrt{(q_0 + q_1 - p_0q_1 + p_1q_0)(-q_0 + q_1 - p_0q_1 + p_1q_0)}. \tag{3.4.9}$$

3. *Compute T from*

$$T = \frac{1}{2\sqrt{q_0}} \begin{pmatrix} \sqrt{\frac{\mu_0}{X_0}}\left[(\omega_0 + \omega_0^{-1}) + p_0(\omega_0 - \omega_0^{-1})\right] & \sqrt{\frac{X_0}{\mu_0}}\left[(\omega_0 - \omega_0^{-1}) + p_0(\omega_0 + \omega_0^{-1})\right] \\ \sqrt{\frac{\mu_0}{X_0}}q_0(\omega_0 - \omega_0^{-1}) & \sqrt{\frac{X_0}{\mu_0}}q_0(\omega_0 + \omega_0^{-1}) \end{pmatrix}. \tag{3.4.10}$$

Remark 3.10. Note that theorem 3.2 guarantees the existence of the matrix T and therefore also the solvability of equation (3.4.8) for functions p_0, q_0, p_1, q_1 satisfying (3.4.5) as long as the eigenvalues μ_j of the Delaunay matrices D_j inducing the potential η as explicated in section 3.1 meet the unitarizability condition (3.1.15). In fact, the necessity of (3.1.15) can be derived from equations (3.4.8) and (3.4.5) itself: It is easy to see that for any functions p_j, q_j satisfying (3.4.5) the left hand side of (3.4.8) will only take values in $[-1, 1]$. This needs to hold for the right hand side as well, that is

$$-1 \leq \frac{\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)}{\sin(2\pi\mu_0)\sin(2\pi\mu_1)} \leq 1. \tag{3.4.11}$$

Since on S^1 we have $0 \leq \mu_j \leq \frac{1}{2}$ and therefore $\sin(2\pi\mu_0)\sin(2\pi\mu_1) \geq 0$, this is equivalent to

$$|\cos(2\pi\mu_0)\cos(2\pi\mu_1) + \cos(2\pi\mu_\infty)| \leq \sin(2\pi\mu_0)\sin(2\pi\mu_1), \tag{3.4.12}$$

which in turn - as carried out in remark 3.7.4 of [11] - is equivalent to (3.1.15).

4 Rotationally symmetric trinoids

4.1 Introductory conventions

In this section we discuss a special class of trinoids, namely CMC-trinoids which are invariant under the rotation R by the angle $\frac{2\pi}{3}$ around an axis $v = (v_1, v_2, v_3)^T \in \mathbb{R}^3$:

$$(R \circ \psi)(\tilde{M}) = \psi(\tilde{M}). \tag{4.1.1}$$

Here $\psi = \psi_{\lambda=1}$ denotes the CMC-immersion, defined on $\tilde{M} = \mathbb{H}$ and generated by a trinoid potential η via the loop group method as described in section 2.4. A CMC-trinoid satisfying equation (4.1.1) will be called a *rotationally symmetric trinoid.*

Remark 4.1. In section 3 we have introduced CMC-trinoids as immersions defined on $M = \hat{\mathbb{C}}\backslash\{0, 1, \infty\}$. When we talk about CMC-trinoids ψ on $\tilde{M}$, we mean CMC-immersions defined on $\tilde{M}$ which descend to M (in the sense of section 3) to yield a CMC-trinoid ϕ with embedded ends on M. Note that, since we have by construction $\psi(\tilde{M}) = \phi(M)$, equation (4.1.1) is obviously equivalent to $(R \circ \phi)(M) = \phi(M)$.

Referring to remark 2.2, we recall that the CMC-immersion $\psi = \psi_{\lambda=1}$ obtained from a holomorphic potential η by the loop group method actually takes values in $su_2 = \{\frac{-i}{2}\begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix};\ x, y, z \in \mathbb{R}^3\}$. To carry out explicit computations, we therefore need to translate the rotation R on $\mathbb{R}^3$ into the "su_2-modell" by use of the isomorphism J defined in (2.3.2). A straightforward computation shows that the corresponding rotation $J \circ R \circ J^{-1}$ on su_2 is given by conjugation with the matrix

$$D = \pm\frac{1}{2}(I - i\sqrt{3}\begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}) \in SU(2). \tag{4.1.2}$$

From now on, we will no longer distinguish between the rotations R (on $\mathbb{R}^3$) and $J \circ R \circ J^{-1}$ (on su_2). Thus, we denote the latter again by R and obtain

$$R \circ \psi = D\psi D^{-1}. \tag{4.1.3}$$

4.2 Implications of rotational symmetry

Next, we collect some useful implications of equation (4.1.1).

First we note that (4.1.1) implies that the embedded ends of a rotationally symmetric trinoid are rotated by R into each other. This means that the asymptotic Delaunay surfaces associated with the ends are rotated into each other as well. Hence, these Delaunay surfaces only differ by a rigid motion on $\mathbb{R}^3$. This implies in particular that the corresponding Delaunay matrices D_j, $j = 0, 1, \infty$, (see section 3.1 for more details) all possess the same eigenvalues. Therefore, in the case of a rotationally symmetric trinoid, we necessarily have

$$\mu_0 = \mu_1 = \mu_\infty. \tag{4.2.1}$$

By equipping $\tilde{M}$ with the pullback metric induced by ψ, $\tilde{M}$ becomes a complete Riemannian manifold. Thus, we can make use of the following lemma. While the first part of the lemma is a consequence of proposition I.11.3 of [13], the second part is obtained by a simple argument.

Lemma 4.2. *Let ψ be an immersion on $\tilde{M}$ and R an element of $\mathrm{Aut}(\psi(\tilde{M}))$, the automorphism group of $\psi(\tilde{M})$. Then, if $\tilde{M}$ is complete, there exists an automorphism $\tilde{\sigma} \in \mathrm{Aut}(\tilde{M})$ such that the following holds:*

1.
$$R \circ \psi = \psi \circ \tilde{\sigma}. \tag{4.2.2}$$

2. *There exists $z_* \in \tilde{M}$ such that*
$$\tilde{\sigma}(z_*) = z_*. \tag{4.2.3}$$

Let $\tilde{\sigma}$ be an automorphism of $\tilde{M}$ with fixed point z_*, associated with the rotation R given by (4.1.3) in the sense of the above lemma. By theorem 2.4.1 of [11], the extended frame $F : \tilde{M} \longrightarrow \Lambda SU(2)_\sigma$ corresponding to ψ (cf. section 2.3) transforms under $\tilde{\sigma}$ as follows:

$$F(\tilde{\sigma}(z), \lambda) = \chi(\lambda) F(z, \lambda) k(z), \tag{4.2.4}$$

where $\chi \in \Lambda SU(2)_\sigma$ is independent of z and k denotes a function of z, which is independent of λ. Furthermore, we have

$$\chi(1) = D. \tag{4.2.5}$$

By remark 2.1 it is possible to assume w.l.o.g. $F(z_*, \lambda) = I$. Thus, (4.2.4) implies

$$\chi(\lambda) = (k(z_*))^{-1}, \tag{4.2.6}$$

which means that χ is actually independent of λ. Equation (4.2.5) then implies $\chi(\lambda) = D$ for all $\lambda \in S^1$, and we obtain for all $\lambda \in S^1$

$$F(\tilde{\sigma}(z), \lambda) = D F(z, \lambda) k(z). \tag{4.2.7}$$

This carries over to the holomorphic frame H (see section 2.4 of [11]):

$$H(\tilde{\sigma}(z), \lambda) = D H(z, \lambda). \tag{4.2.8}$$

In this context, one can prove the following theorem:

Theorem 4.3. *Let $\psi : \tilde{M} = \mathbb{H} \to su_2$ be a rotationally symmetric trinoid with respect to the rotation R, which is carried out on su_2 by conjugation with the matrix D. Assume R rotates the trinoid end at z_j into the end at z_k. Furthermore, let $\tilde{\sigma}$ be an automorphism of $\tilde{M}$ given by lemma 4.2 with fixed point $z_* \in \tilde{M}$.*

Then there exists a holomorphic frame H corresponding to ψ which transforms under $\tilde{\sigma}$ as in (4.2.8). Moreover, the monodromy matrices $M_j(\lambda)$ associated with H by surrounding the singularities z_j, $j = 0, 1, \infty$, are related by

$$M_k(\lambda) = DM_j(\lambda)D^{-1}. \tag{4.2.9}$$

4.3 Simultaneous unitarization of the monodromy matrices associated with rotationally symmetric trinoids

Let η be a trinoid potential generating a rotationally symmetric CMC-trinoid ψ with regard to the rotation R. In particular, by the previous section, the three Delaunay matrices D_0, D_1, D_∞ associated with η satisfy $\mu_0 = \mu_1 = \mu_\infty$. We will now explicitly carry out the procedure given in section 3.4 to obtain the "right" solution $\hat{H}$ of the differential equation $\mathrm{d}H = H\eta$, which produces ψ. This procedure will simplify, as we are going to implement the implications of the rotational symmetry of the trinoid, which have been explicated in section 4.2.

As the image $\tilde{\psi}(\tilde{M})$ of a rotational symmetric trinoid $\tilde{\psi}$ with arbirary axis of rotation v differs from the image $\psi(\tilde{M})$ of a rotational symmetric trinoid ψ with axis of rotation $v_0 = (0,0,1)^T$ only by a rigid motion on $\mathbb{R}^3$, we will restrict our discussion of rotational symmetric trinoids from now on without loss of generality to the case where R denotes the rotation by the angle $\frac{2\pi}{3}$ around the z-axis in $\mathbb{R}^3$, that is

$$v = v_0 = (0,0,1)^T. \tag{4.3.1}$$

In this case, the matrix D corresponding to the rotation R by (4.1.3) is given by

$$D = \pm \begin{pmatrix} e^{\frac{-\pi i}{3}} & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{pmatrix}. \tag{4.3.2}$$

Furthermore, we assume w.l.o.g. that the ends of the trinoid are permuted under R as $z_0 \mapsto z_1 \mapsto z_\infty \mapsto z_0$. (The second possibility of permuting the ends just correponds to a rotation in the opposite direction.) The Moebius transformation on $\hat{\mathbb{C}}$ satisfying $0 \mapsto 1 \mapsto \infty \mapsto 0$ as well as the covering function $\pi : \tilde{M} = \mathbb{H} \to M = \hat{C}\backslash\{0,1,\infty\}$ are well known (cf., e.g., [14], p. 52 for the latter). Therefore, an automorphism $\tilde{\sigma}$ corresponding to R in the sense of lemma 4.2 can be explicitly computed:

$$\tilde{\sigma} : \tilde{M} \to \tilde{M}, \quad z \mapsto \frac{-1}{1+z}. \tag{4.3.3}$$

In particular, $\tilde{\sigma}$ has fixed points at $\frac{-1 \pm i\sqrt{3}}{2}$.

Recall that we obtain the "right" solution $\hat{H}$ by dressing the solution H given in (3.4.1) with the λ dependent matrix T from (3.4.10) determined by functions $p_0, q_0, p_1, q_1, \omega_0$ satisfying equations (3.4.5), (3.4.9) and (3.4.8). In particular, $\hat{H} = TH$ will have unitary monodromy matrices $\hat{M}_j = TM_jT^{-1}$ given by (3.4.4). By applying theorem 4.3 to $\hat{H}$, we furthermore need to require

$$\hat{M}_1 = D\hat{M}_0D^{-1}. \tag{4.3.4}$$

By (3.4.4), this immediately implies

$$p_1 = p_0, \quad q_1 = e^{\frac{2\pi i}{3}} q_0. \tag{4.3.5}$$

Thus, in the case of a rotationally symmetric trinoid, we obtain the following equivalent reformulations of (3.4.5) and (3.4.8):

$$p_0 = \overline{p_0} \quad \text{and} \quad p_0^2 + q_0\overline{q_0} = 1, \tag{4.3.6}$$

$$p_0^2 - \frac{q_0\overline{q_0}}{2} = \frac{\cos^2(2\pi\mu) + \cos(2\pi\mu)}{\sin^2(2\pi\mu)}, \tag{4.3.7}$$

where $\mu := \mu_0 = \mu_1 = \mu_\infty$.

An easy calculation shows that *all* solutions p_0, q_0 to this system of equations are given by

$$p_0 = \pm\frac{\cos(\pi\mu)}{\sqrt{3}\sin(\pi\mu)}, \tag{4.3.8}$$

$$q_0 = \frac{\zeta_0}{\sqrt{3}\sin(\pi\mu)}, \tag{4.3.9}$$

where ζ_0 is obtained by solving

$$\zeta_0\overline{\zeta_0} = 4\sin^2(\pi\mu) - 1. \tag{4.3.10}$$

Remark 4.4. Since the expression $4\sin^2(\pi\mu) - 1$ is holomorphic on $\mathbb{C}^*$, it can be written in a "Weierstraß product representation" as a product of factors of the form $(\lambda - \lambda_j)$ times $e^{f(\lambda)}$, where λ_j denote the roots of $4\sin^2(\pi\mu) - 1$ and f is a function of λ. Note that the roots appear in pairs λ_j, $\frac{1}{\lambda_j}$. It turns out that this ensures the solvability of (4.3.10). As $4\sin^2(\pi\mu) - 1$ possesses infinitely many pairs of roots, there are in fact infinitely many possibilities to solve (4.3.10), since for each pair one is free to choose which root will contribute to ζ_0 and which one to $\overline{\zeta_0}$.

Finally, in addition to (4.3.4) also the equations $\hat{M}_\infty = D\hat{M}_1D^{-1}$ and $\hat{M}_0 = D\hat{M}_\infty D^{-1}$ have to hold. An easy calculation shows, that this implies that only

$$p_0 = -\frac{\cos(\pi\mu)}{\sqrt{3}\sin(\pi\mu)} \tag{4.3.11}$$

can occur.

Altogether, by computing ω_0 from (3.4.9), we obtain T from (3.4.10) which will produce a solution $\tilde{H} = TH$ to $\mathrm{d}H = H\eta$, that generates a trinoid via the loop group method with unitary monodromy matrices $\hat{M}_j$ satisfying (4.2.9). The latter ones are explicitly given by

$$\hat{M}_0 = \cos(2\pi\mu)\cdot I + \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} -\cos(\pi\mu) & \overline{\zeta_0} \\ \zeta_0 & \cos(\pi\mu) \end{pmatrix}, \tag{4.3.12}$$

$$\hat{M}_1 = \cos(2\pi\mu)\cdot I + \frac{2i}{\sqrt{3}}\cos(\pi\mu)\begin{pmatrix} -\cos(\pi\mu) & e^{\frac{-2\pi i}{3}}\overline{\zeta_0} \\ e^{\frac{2\pi i}{3}}\zeta_0 & \cos(\pi\mu) \end{pmatrix}. \tag{4.3.13}$$

Remark 4.5. Note that $\hat{M}_j(\lambda = 1) = \pm I$ and $\partial_\lambda \hat{M}_j(\lambda)|_{\lambda=1} = 0$ hold for $j = 0, 1$. Thus, by theorem 2.5 together with remark 3.4, the new solution $\hat{H} = TH$ produces a CMC-immersion that descends to M.

Remark 4.6. Altogether, we have proved, that the solution $\tilde{H} = TH$ to $\mathrm{d}H = H\eta$ corresponding to a rotationally symmetric trinoid is obtained from H given in section 3.4 by dressing with a matrix T of the form above. Furthermore, we obtain monodromy matrices of the above form. As a consequence, we have classified all CMC-trinoids with monodromy matrices satisfying the "rotation relation" (4.2.9).

Computer experiments carried out by Kilian, Schmitt, Sterling [16] indicate that there exist trinoids produced by the given H and a T of the given form which are actually *not* rotationally symmetric. Such trinoids would only satisfy the "rotation relation" (4.2.9) for the monodromy matrices, but not the defining condition given in equation (4.1.1). It remains to study, in which cases the T given above will actually give rise to a solution $\tilde{H}$ generating a rotationally symmetric trinoid.

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