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Real forms of complex surfaces of constant mean curvature

Shimpei Kobayashi
School of Information Environment
Tokyo Denki University Japan
shimpei@sie.dendai.ac.jp

1 Introduction

This is a summary of the paper [10]. The goal is to give a unified theory for *integrable surfaces* using real forms of the *complex extended framings* of complex CMC-immersions and the generalized Weierstraß type representation for complex CMC-immersions.

It is well known that a surface in $\mathbb{R}^3$ has nonzero constant mean curvature (CMC for short) if and only if there exists a moving frame with spectral parameter, an element in $SU(2)$ loop group, which satisfies the certain condition (see [5]). Such moving frame is called the *extended framing* of a CMC-immersion.

The extended framing of a CMC-immersion in $\mathbb{R}^3$ has a natural complexification, which is called the *complex extended framing* ([3]). Moreover in [4], we considered a holomorphic immersion in $\mathbb{C}^3$ associated with the complex extended framing. It turned out that the holomorphic immersion had nonzero complex constant mean curvature, which was called a *complex* CMC*-immersion*. Then a CMC-immersion in $\mathbb{R}^3$ can be obtained from a real form of the complex extended framing of a complex CMC-immersion.

It is known that a CMC-immersion in $\mathbb{R}^3$ has the parallel immersion with constant Gauß curvature (CGC for short) $K > 0$ in $\mathbb{R}^3$. Similar to the real case, a holomorphic immersion with complex constant Gauß curvature $K \in \mathbb{C}^*$ (CGC for short) will be obtained as the parallel immersion of a complex CMC-immersion. Thus a CGC-immersion with $K > 0$ in $\mathbb{R}^3$ also can be obtained from a real form of the complex extended framing. Then it is natural to ask whether other classes of *real surfaces* exist from real forms of the complex extended framing of a complex CMC-immersion or a complex CGC-immersion.

In this summary, we show that there are seven classes of surfaces as real forms of the complex extended framing, which are called *integrable surfaces*. These are CGC-immersions with $K > 0$ (or $K < 0$) in $\mathbb{R}^3$ and their parallel CMC-immersions, spacelike (or timelike) CGC-immersions with $K > 0$ (or $K < 0$) in $\mathbb{R}^{2,1}$ and their parallel CMC-immersions, and CMC-immersions with mean curvature $H < 1$ in H^3 (see Theorem 3.1 and Corollary 3.2). Some of these classes of surfaces were considered from harmonic maps and integrable systems points of views (see [9], [6], [12], [8] and [1]).

The generalized Weierstraß type representation for complex CMC-immersions is a procedure to construct complex CMC-immersions in $\mathbb{C}^3$ (see Section 4.1 for more details): **1.** Define pairs of holomorphic potentials, which are pairs of holomorphic 1-forms $\check{\eta} = (\eta, \tau)$ with $\eta = \sum_{j\geq -1}^{\infty} \eta_j \lambda^j$ and $\tau = \sum_{-\infty}^{j\leq 1} \tau_j \lambda^j$. Here λ is the complex parameter, the so-called "spectral parameter", η_j and τ_j are diagonal (resp. off-diagonal) holomorphic 1-forms depending only on one complex variable if j is even (resp. j is odd). **2.** Solve the pair of ODE's $d(C, L) = (C, L)\check{\eta}$ with some initial condition $(C(z_*), L(w_*))$, and perform the generalized Iwasawa decomposition (Theorem A.1) for (C, L), giving $(C, L) = (F, F)(V_+, V_-)$. It is known that $F \cdot l$ is the complex extended framing of some complex CMC-immersion (Theorem 4.1), where l is some λ-independent diagonal matrix. **3.** Form a complex CMC-immersion by the Sym formula Ψ via the complex extended framing $F \cdot l$ (Theorem 2.4).

Since each class of integrable surfaces is defined by the real form of a complex extended framing, there exists a unique semi-linear involution ρ corresponding to each class of integrable surfaces. Then these semi-linear involutions naturally define the pairs of semi-linear involutions on pairs of holomorphic potentials $\check{\eta} = (\eta, \tau)$. It follows that the generalized Weierstraß type representation for each class of integrable surfaces can be formulated by the above construction via a pair of holomorphic potentials which is invariant under a pair of semi-linear involutions (Theorem 4.2). In this way we will give a unified theory for all integrable surfaces.

2 Preliminaries

In this preliminary section, we give a brief review of the basic results for holomorphic null immersions, complex CMC-immersions and complex CGC-immersions.

Throughout this paper, $\mathbb{C}^3$ is identified with $\mathfrak{sl}(2, \mathbb{C})$ as follows:

$$(a, b, c)^t \in \mathbb{C}^3 \leftrightarrow -\frac{ia}{2}\sigma_1 - \frac{ib}{2}\sigma_2 - \frac{ic}{2}\sigma_3 \in \mathfrak{sl}(2, \mathbb{C}) \ , \qquad (2.0.1)$$

where σ_j $(j = 1, 2, 3)$ are Pauli matrices as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad (2.0.2)$$

2.1 Holomorphic null immersions in $\mathbb{C}^3$

In this subsection, we show the basic results for holomorphic immersions in $\mathbb{C}^3$. We give natural definitions of complex mean curvature (Definition 1) and complex Gauß curvature (Definition 2) for a holomorphic immersion analogous to the mean curvature and the Gauß curvature of a surface in $\mathbb{R}^3$. We refer to [4] for more details.

Let $\mathcal{M}$ be a simply connected 2-dimensional Stein manifold, and let $\Psi : \mathcal{M} \to \mathfrak{sl}(2, \mathbb{C})$ be a holomorphic immersion, i.e. the complex rank of $d\Psi$ is two. We consider the following bilinear form on $\mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}^3$:

$$\langle a, b\rangle = -2\mathrm{Tr}\, ab\ , \tag{2.1.1}$$

where $a, b \in \mathfrak{sl}(2, \mathbb{C})$. We note that the bilinear form (2.1.1) is a $\mathbb{C}$-bilinear form on $\mathbb{C}^3$ by the identification (2.0.1). Then it is known that, for a neighborhood $\widetilde{\mathcal{M}_p} \subset \mathcal{M}$ around each point $p \in \mathcal{M}$, the bilinear form (2.1.1) induces a holomorphic Riemannian metric on $\widetilde{\mathcal{M}_p}$, i.e. a holomorphic covariant symmetric 2-tensor g (see [11] and [4]). From [4], it is also known that there exist special coordinates $(z, w) \in \mathfrak{D}^2 \subset \mathbb{C}^2$ such that a holomorphic Riemannian metric g can be written as follows:

$$g = e^{u(z,w)} dz dw\ , \tag{2.1.2}$$

where $u(z, w) : \mathfrak{D}^2 \to \mathbb{C}$ is some holomorphic function. The special coordinates defined above are called *null coordinates*. From now on, we always assume a holomorphic immersion $\Psi : \mathcal{M} \to \mathfrak{sl}(2, \mathbb{C})$ has null coordinates. A holomorphic immersion with null coordinates is also called the *holomorphic null immersion*.

From [4], we quote the following theorem:

Theorem 2.1 ([4]). *Let $\Psi : \mathcal{M} \to \mathbb{C}^3 (\cong \mathfrak{sl}(2, \mathbb{C}))$ be a holomorphic null immersion. Then there exists a $SL(2, \mathbb{C})$ matrix F such that the following equations hold:*

$$\begin{aligned} F_z &= FU, \\ F_w &= FV, \end{aligned} \tag{2.1.3}$$

where

$$\begin{cases} U = \begin{pmatrix} \frac{1}{4}u_z & -\frac{1}{2}He^{u/2} \\ Qe^{-u/2} & -\frac{1}{4}u_z \end{pmatrix}, \\ \\ V = \begin{pmatrix} -\frac{1}{4}u_w & -Re^{-u/2} \\ \frac{1}{2}He^{u/2} & \frac{1}{4}u_w \end{pmatrix}, \end{cases} \tag{2.1.4}$$

with $Q := \langle \Psi_{zz}, N\rangle$, $R := \langle \Psi_{ww}, N\rangle$ and $H := 2e^{-u}\langle \Psi_{zw}, N\rangle$.

We call $F : \mathcal{M} \to SL(2, \mathbb{C})$ *the moving frame* of Ψ. Then the compatibility condition for the equations in (2.1.3) is

$$U_w - V_z + [V, U] = 0. \tag{2.1.5}$$

A direct computation shows that the equation (2.1.5) can be rephrased as follows:

$$\begin{cases} u_{zw} - 2RQe^{-u} + \frac{1}{2}H^2e^u = 0, \\ Q_w - \frac{1}{2}H_z e^u = 0, \\ R_z - \frac{1}{2}H_w e^u = 0. \end{cases} \tag{2.1.6}$$

The first equation in (2.1.6) will be called the *complex Gauß equation*, and the second and third equations in (2.1.6) will be called the *complex Codazzi equations.*

We now define a vector $N \in \mathfrak{sl}(2, \mathbb{C})$ as follows:

$$N := 2ie^{-u}[\Psi_w,\ \Psi_z]\ . \tag{2.1.7}$$

It is easy to verify that $\langle \Psi_z, N\rangle = \langle \Psi_w, N\rangle = 0$ and the $\langle N, N\rangle = 1$. Thus N is a transversal vector to $d\Psi$. Therefore it is natural to call N the *complex Gauß map* of Ψ.

Using the functions u, Q, R and H defined in (2.1.2) and (2.1.4) respectively, the symmetric quadratic form $II := -\langle d\Psi, dN\rangle$ can be represented as follows:

$$II := -\langle d\Psi, dN\rangle = Qdz^2 + e^u H dz dw + Rdw^2\ . \tag{2.1.8}$$

The symmetric quadratic form II is called the *second fundamental form* for a holomorphic null immersion Ψ. Then the complex mean curvature and the complex Gauß curvature for a holomorphic null immersion Ψ are defined as follows.

Definition 1. *Let $\Psi : \mathcal{M} \to \mathbb{C}^3$ be a holomorphic null immersion. Then the function $H = 2e^{-u}\langle\Psi_{zw}, N\rangle$ will be called the* complex mean curvature *of Ψ.*

Definition 2. *Let $\tilde{I}$ (resp. $\widetilde{II}$) be the coefficient matrix of the holomorphic metric g (resp. the second fundamental form II). Then the function $K = \det(\tilde{I}^{-1} \cdot \widetilde{II})$ will be called the* complex Gauß curvature *of Ψ.*

2.2 Complex CMC and CGC immersions in $\mathbb{C}^3$

In this subsection, we give characterizations of complex constant mean curvature immersions via loop groups (see Appendix A for the definitions of loop groups). There is a useful formula representing complex CMC-immersions, which is a generalization of the Sym formula for CMC-immersions in $\mathbb{R}^3$ (see also [3]). There is also a formula for complex CGC-immersions given by the parallel holomorphic immersions of complex CMC-immersions with $H \in \mathbb{C}^*$.

The notions of a complex CMC-immersion and a CGC-immersion are defined analogous to the notions of a CMC-immersion and a CGC-immersion in $\mathbb{R}^3$ (see also [4]).

Definition 3. *Let $\Psi : \mathcal{M} \to \mathbb{C}^3$ be a holomorphic null immersion, and let H (resp. K) be its complex mean curvature (resp. Gauß curvature). Then Ψ is called a complex constant mean curvature (*CMC *for short) immersion (resp. a complex constant Gauß curvature (*CGC *for short) immersion) if H (resp. K) is a complex constant.*

Remark 2.2. *Since we are interested in complexifications of* CMC *(resp.* CGC*) surfaces with nonzero mean curvature $H \in \mathbb{R}^*$ (resp. Gauß curvature $K \in \mathbb{R}^*$), from now on, we always assume that the complex mean curvature H (resp. the complex Gauß curvature K) is a nonzero constant.*

From [4], we quote the following characterizations of a complex CMC-immersion:

Lemma 2.3. *Let $\mathcal{M}$ be a connected 2-dimensional Stein manifold, and let $\Psi : \mathcal{M} \to \mathbb{C}^3(\cong \mathfrak{sl}(2,\mathbb{C}))$ be a holomorphic null immersion. Further, let Q, R, H and N be the complex functions defined in* (2.1.4) *and the Gauß map defined in* (2.1.7)*, respectively. Then the following statements are equivalent:*

1. *H is a nonzero constant;*
2. *Q depends only on z and R depends only on w;*
3. *$N_{zw} = \rho N$, for some holomorphic function $\rho : \mathcal{M} \to \mathbb{C}$.*
4. *There exists $\tilde{F}(z,w,\lambda) \in \Lambda SL(2,\mathbb{C})_\sigma$ such that*

$$\tilde{F}(z,w,\lambda)^{-1} d\tilde{F}(z,w,\lambda) = \tilde{U}dz + \tilde{V}dw,$$

where

$$\begin{cases} \tilde{U} = \begin{pmatrix} \frac{1}{4}u_z & -\frac{1}{2}\lambda^{-1}He^{u/2} \\ \lambda^{-1}Qe^{-u/2} & -\frac{1}{4}u_z \end{pmatrix}, \\ \tilde{V} = \begin{pmatrix} -\frac{1}{4}u_w & -\lambda Re^{-u/2} \\ \frac{1}{2}\lambda He^{u/2} & \frac{1}{4}u_w \end{pmatrix}, \end{cases}$$

and $\tilde{F}(z,w,\lambda = 1) = F(z,w)$ is the moving frame of Ψ in (2.1.3).

The $\tilde{F}(z,w,\lambda)$ defined in (4) of Lemma 2.3 is called the *complex extended framing* of a complex CMC-immersion Ψ. From now on, for simplicity, the symbol $F(z,w,\lambda)$ (resp. $U(z,w,\lambda)$ or $V(z,w,\lambda)$) is used instead of $\tilde{F}(z,w,\lambda)$ (resp. $\tilde{U}(z,w,\lambda)$ or $\tilde{V}(z,w,\lambda)$).

There is an immersion formula for a complex CMC-immersion using the complex extended framing $F(z,w,\lambda)$ for a complex CMC-immersion Ψ, the so-called "Sym formula" (see [4]). We show a similar immersion formula for a complex CGC-immersion using the same complex extended framing $F(z,w,\lambda)$ of a complex CMC-immersion Ψ.

Theorem 2.4. *Let $F(z,w,\lambda)$ be the complex extended framing of some complex CMC-immersion defined as in Lemma 2.3, and let H be its nonzero complex constant mean curvature. We set*

$$\begin{cases} \Psi &= -\frac{1}{2H}\left(i\lambda\partial_\lambda F(z,w,\lambda)\cdot F(z,w,\lambda)^{-1} + \frac{i}{2}F(z,w,\lambda)\sigma_3 F(z,w,\lambda)^{-1}\right), \\ \Phi &= -\frac{1}{2H}\left(i\lambda\partial_\lambda F(z,w,\lambda)\cdot F(z,w,\lambda)^{-1}\right), \end{cases} \tag{2.2.1}$$

where σ_3 has been defined in (2.0.2). Then Ψ (resp. Φ) is, for every $\lambda \in \mathbb{C}^*$, a complex constant mean curvature immersion (resp. complex constant Gaußian curvature immersion, possibly degenerate) in $\mathbb{C}^3$ with complex mean curvature $H \in \mathbb{C}^*$ (resp. complex Gauß curvature $K = 4H^2 \in \mathbb{C}^*$), and the Gauß map of Ψ (resp. Φ) can be described by $\frac{i}{2}F(z,w,\lambda)\sigma_3 F(z,w,\lambda)^{-1}$.

3 Real forms of complex CGC-immersions

In this section, we show that "integrable surfaces" obtained from the real forms of the twisted $\mathfrak{sl}(2,\mathbb{C})$ loop algebra $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$.

3.1 Integrable surfaces as real forms of complex CGC-immersions

Let $F(z,w,\lambda) \in \Lambda SL(2,\mathbb{C})_\sigma$ be the complex extended framing of some complex CGC-immersion Φ. And let $\alpha(z,w,\lambda) = F(z,w,\lambda)^{-1}dF(z,w,\lambda)$ be the Maurer-Cartan form of $F(z,w,\lambda)$. From the forms of U and V defined as in Lemma 2.3, we set α_i ($i \in \{-1,0,1\}$) as follows:

$$\alpha(z,w,\lambda) = F^{-1}dF = Udz + Vdw = \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1\ , \tag{3.1.1}$$

where

$$\begin{cases} \alpha_{-1} = \begin{pmatrix} 0 & -\frac{1}{2}He^{u/2}dz, \\ Qe^{-u/2}dz & 0 \end{pmatrix}, \\ \alpha_0 = \begin{pmatrix} \frac{1}{4}u_z dz - \frac{1}{4}u_w dw & 0 \\ 0 & -\frac{1}{4}u_z dz + \frac{1}{4}u_w dw \end{pmatrix}, \\ \alpha_1 = \begin{pmatrix} 0 & -Re^{-u/2}dw \\ \frac{1}{2}He^{u/2}dw & 0 \end{pmatrix}. \end{cases} \tag{3.1.2}$$

We denote the space of $\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma$ valued 1-forms by $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma)$. It is clear that $\alpha(z,w,\lambda)$ defined in (3.1.1) is an element in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma)$. Then it is also clear that the following automorphisms define involutions on $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma)$:

$$\begin{cases} \tilde{\mathfrak{c}}_1 : g(\lambda) \mapsto -\overline{g(-1/\bar{\lambda})}^t, \\ \tilde{\mathfrak{c}}_2 : g(\lambda) \mapsto \overline{g(-1/\bar{\lambda})}, \\ \tilde{\mathfrak{c}}_3 : g(\lambda) \mapsto -\overline{g(1/\bar{\lambda})}^t, \\ \tilde{\mathfrak{c}}_4 : g(\lambda) \mapsto -\mathrm{Ad}\begin{pmatrix} 1/\sqrt{i} & 0 \\ 0 & \sqrt{i} \end{pmatrix} \overline{g(i/\bar{\lambda})}^t, \end{cases} \qquad \begin{cases} \tilde{\mathfrak{s}}_1 : g(\lambda) \mapsto -\overline{g(-\bar{\lambda})}^t, \\ \tilde{\mathfrak{s}}_2 : g(\lambda) \mapsto \overline{g(-\bar{\lambda})}, \\ \tilde{\mathfrak{s}}_3 : g(\lambda) \mapsto -\overline{g(\bar{\lambda})}^t. \end{cases} \tag{3.1.3}$$

Then the real forms of $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},j)})$ are defined as follows:

$$\begin{aligned} \Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},j)}) &= \{g(\lambda) \in \Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma) \mid \tilde{\mathfrak{c}}_j \circ g(\lambda) = g(\lambda)\}, \\ \Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{s},j)}) &= \{g(\lambda) \in \Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma) \mid \tilde{\mathfrak{s}}_j \circ g(\lambda) = g(\lambda)\}. \end{aligned} \tag{3.1.4}$$

From now on, for simplicity, we use the symbols $\mathfrak{c}_j$ and $\mathfrak{s}_j$ instead of $\tilde{\mathfrak{c}}_j$ and $\tilde{\mathfrak{s}}_j$ respectively. We now consider the following conditions on $\alpha(z,w,\lambda)$:

- **Almost Compact cases** (C,j): $\alpha(z,w,\lambda)$ is an element in one of the real forms $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},j)})$ for $j \in \{1,2,3,4\}$.

- **Almost Split cases** (S,j): $\alpha(z,w,\lambda)$ is an element in one of the real forms $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{s},j)})$ for $j \in \{1,2,3\}$.

We now set the following formulas $\Phi^{(\mathfrak{c},j)}$ for $j \in \{1,2,3,4\}$ (resp. $\Phi^{(\mathfrak{s},j)}$ for $j \in \{1,2,3\}$) analogous to the second formula in (2.2.1):

$$\Phi^{(\mathfrak{c},j)} = -\frac{1}{2|H|}\left(i\lambda\partial_\lambda F^{(\mathfrak{c},j)}(z,\bar{z},\lambda)\cdot F^{(\mathfrak{c},j)}(z,\bar{z},\lambda)^{-1}\right)\Big|_{\lambda\in S^1} \quad \text{for } j \in \{1,2,3\}, \tag{3.1.5}$$

$$\Phi^{(\mathfrak{c},4)} = \frac{1}{2}\left(F^{(\mathfrak{c},4)}(z,\bar{z},\lambda)\begin{pmatrix} e^{q/2} & 0 \\ 0 & e^{-q/2} \end{pmatrix}(F^{(\mathfrak{c},4)}(z,\bar{z},\lambda))^*\right)\Big|_{\lambda\in S^r}, \tag{3.1.6}$$

$$\Phi^{(\mathfrak{s},j)} = -\frac{1}{2|H|}\left(\lambda\partial_\lambda F^{(\mathfrak{s},j)}(x,y,\lambda)\cdot F^{(\mathfrak{s},j)}(x,y,\lambda)^{-1}\right)\Big|_{\lambda\in\mathbb{R}^*} \quad \text{for } j \in \{1,2,3\}, \tag{3.1.7}$$

where $\lambda = \exp(it) \in S^1$ or $\lambda = \exp(q/2+it) \in S^r$ for (3.1.5) or (3.1.6) (resp. $\lambda = \pm\exp(t) \in \mathbb{R}^*$ for (3.1.7)) with $t, q \in \mathbb{R}$, and where $*$ denotes $X^* = \bar{X}^t$ for $X \in M_{2\times 2}(\mathbb{C})$. Then, for each $\lambda \in S^1$ or $\lambda \in S^r$ (resp. $\lambda \in \mathbb{R}^*$), the formula $\Phi^{(\mathfrak{c},j)}$ (resp. $\Phi^{(\mathfrak{s},j)}$) defines a map into one of the following spaces:

$$\begin{cases} \mathfrak{su}(1,1) \cong \mathbb{R}^{1,2} & \text{for the } (C,1) \text{ and } (S,1) \text{ cases,} \\ \mathfrak{sl}_*(2,\mathbb{R}) \cong \mathbb{R}^{1,2} & \text{for the } (C,2) \text{ and } (S,2) \text{ cases,} \\ \mathfrak{su}(2) \cong \mathbb{R}^3 & \text{for the } (C,3) \text{ and } (S,3) \text{ cases,} \\ SL(2,\mathbb{C})/SU(2) \cong H^3 & \text{for the } (C,4) \text{ case,} \end{cases}$$

where $\mathfrak{sl}_*(2,\mathbb{R}) = \{g \in \mathfrak{sl}(2,\mathbb{C}) \mid g = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, a \in \mathbb{R},\ b, c \in i\mathbb{R}\}$, which is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Here $\mathbb{R}^{1,2}$ and $\mathbb{R}^3$ can be identified with $\mathfrak{su}(1,1)$, $\mathfrak{sl}_*(2,\mathbb{R})$ and $\mathfrak{su}(2)$

analogous to the identification (2.0.1). Minkowski space $\mathbb{R}^{3,1}$ can be identified with $\mathrm{Herm}(2) := \{X \in M_{2\times 2}(\mathbb{C}) \mid \bar{X}^t = X\}$ via the map

$$(x_1, x_2, x_3, x_0) \mapsto \frac{1}{2}\begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix},$$

then $H^3 \subset \mathbb{R}^{3,1}$ can be identified with Herm(2) with the determinant 1/4. Then the inner product for $\mathfrak{su}(1,1) \cong \mathbb{R}^{1,2}$ (resp. $\mathfrak{sl}_*(2,\mathbb{R}) \cong \mathbb{R}^{1,2}$ or $\mathfrak{su}(2) \cong \mathbb{R}^3$) can be defined by $\langle a, b\rangle = -2\mathrm{Tr}\,(ab)$ for $a, b \in \mathfrak{su}(1,1)$ (resp. $a, b \in \mathfrak{sl}_*(2,\mathbb{R})$ or $a, b \in \mathfrak{su}(2)$). The inner product for $\mathrm{Herm}(2) \cong \mathbb{R}^{3,1}$ can be defined by $\langle a, b\rangle = -2\mathrm{Tr}\,(a\sigma_2 b^t \sigma_2)$ for $a, b \in \mathrm{Herm}(2)$, where σ_2 is defined in (2.0.2). From now on, we always assume that the spectral parameter λ is in S^1 or S^r for the almost compact cases and λ is in $\mathbb{R}^*$ for the almost split cases, respectively. Then we have the following theorem:

Theorem 3.1. *Let $F(z, w, \lambda)$ be the complex extended framing of some complex* CGC*-immersion Φ. Then the following statements hold:*

$(C,1)$ *If $F^{-1}dF$ is in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},1)})$, then for each $\lambda \in S^1$ the Sym formula in* (3.1.5) *defines a* spacelike constant negative Gaußian curvature surface *in* $\mathbb{R}^{2,1}$.

$(C,2)$ *If $F^{-1}dF$ is in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},2)})$, then for each $\lambda \in S^1$ the Sym formula in* (3.1.5) *defines a* timelike constant negative Gaußian curvature surface *in* $\mathbb{R}^{2,1}$.

$(C,3)$ *If $F^{-1}dF$ is in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},3)})$, then for each $\lambda \in S^1$ the Sym formula in* (3.1.5) *defines a* constant positive Gaußian curvature surface *in* $\mathbb{R}^3$.

$(C,4)$ *If $F^{-1}dF$ is in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{c},4)})$, then for each $\lambda \in S^r$ the Sym formula in* (3.1.6) *defines a* constant mean curvature surface *with mean curvature $|H^{(\mathfrak{c},4)}| < 1$ in H^3.*

$(S,1)$ *If $F^{-1}dF$ is in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{s},1)})$, then for each $\lambda \in \mathbb{R}^*$ the Sym formula in* (3.1.7) *defines a* spacelike constant positive Gaußian curvature surface *in* $\mathbb{R}^{2,1}$.

$(S,2)$ *If $F^{-1}dF$ is in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{s},2)})$, then for each $\lambda \in \mathbb{R}^*$ the Sym formula in* (3.1.7) *defines a* timelike constant positive Gaußian curvature surface *in* $\mathbb{R}^{2,1}$.

$(S,3)$ *If $F^{-1}dF$ is in $\Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma^{(\mathfrak{s},3)})$, then for each $\lambda \in \mathbb{R}^*$ the Sym formula in* (3.1.7) *defines a* constant negative Gaußian curvature surface *in* $\mathbb{R}^3$.

Definition 4. *Let $F^{(\mathfrak{c},j)}(z, \bar{z}, \lambda)$ for $j \in \{1,2,3,4\}$ (resp. $F^{(\mathfrak{s},j)}(x, y, \lambda)$ for $j \in \{1,2,3\}$) be the complex extended framings, which are elements in $\Lambda SL(2,\mathbb{C})_\sigma{}^{(\mathfrak{c},j)}$ (resp. $\Lambda SL(2,\mathbb{C})_\sigma{}^{(\mathfrak{s},j)}$). Then $F^{(\mathfrak{c},j)}(z, w, \lambda)$ (resp. $F^{(\mathfrak{s},j)}(x, y, \lambda)$) is called the* extended framing for the immersion *$\Phi^{(\mathfrak{c},j)}$ (resp. $\Phi^{(\mathfrak{s},j)}$).*

It is known that for three classes of surfaces in the above seven classes, there exist parallel constant mean curvature surfaces in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$ (see also [8] and [9]).

Surfaces class	Gauß curvature	Gauß curvature	Parallel CMC
Surfaces in $\mathbb{R}^3$	$K^{(s,3)} = -4\|H\|^2$	$K^{(c,3)} = 4\|H\|^2$	$H^{(c,3)} = \|H\|$
Spacelike surfaces in $\mathbb{R}^{2,1}$	$K^{(s,1)} = 4\|H\|^2$	$K^{(c,1)} = -4\|H\|^2$	$H^{(c,1)} = \|H\|$
Timelike surfaces in $\mathbb{R}^{2,1}$	$K^{(c,2)} = -4\|H\|^2$	$K^{(s,2)} = 4\|H\|^2$	$H^{(s,2)} = \|H\|$
Surfaces in H^3			$H^{(c,4)} = \tanh(q)$

Table 1: Integrable surfaces

Corollary 3.2. *We retain the assumptions in Theorem 3.1. Then we have the following:*

$(C, 1M)$ *For the $(C, 1)$ case in Theorem 3.1, there exists a parallel spacelike constant mean curvature surface with mean curvature $H^{(c,1)} = |H|$ in $\mathbb{R}^{2,1}$.*

$(C, 3M)$ *For the $(C, 3)$ case in Theorem 3.1, there exists a parallel constant mean curvature surface with mean curvature $H^{(c,3)} = |H|$ in $\mathbb{R}^3$.*

$(S, 2M)$ *For the $(S, 2)$ case in Theorem 3.1, there exists a parallel timelike constant mean curvature surface with mean curvature $H^{(s,2)} = |H|$ in $\mathbb{R}^{2,1}$.*

Definition 5. *The surfaces defined in Theorem 3.1 and Corollary 3.2 are called the* integrable surfaces.

Remark 3.3. *For the three classes of surfaces in Theorem 3.1, which are spacelike constant positive Gaußian curvature surfaces in $\mathbb{R}^{2,1}$, constant negative Gaußian curvature surfaces in $\mathbb{R}^3$ and timelike constant negative Gaußian curvature surfaces in $\mathbb{R}^{2,1}$, there never exist parallel constant mean curvature surfaces.*

4 The generalized Weierstraß type representation for integrable surfaces

The generalized Weierstraß type representation for complex CMC-immersions (or equivalently CGC-immersions as the parallel immersions) is the procedure of a construction of complex CMC-immersions from a pair of holomorphic potentials (see [4]). In the previous section, we obtained integrable surfaces according to the real forms of $\Lambda\mathfrak{sl}(2, \mathbb{C})_\sigma$. In this section, we show how all integrable surfaces are obtained from the pairs of holomorphic potentials in the generalized Weierstraß type representation.

4.1 Integrable surfaces via the generalized Weierstraß type representation

The generalized Weierstraß type representation for complex CMC-immersions (or equivalently CGC-immersions as the parallel immersions) is divided into the following 4 steps (see also [4] for more details):

Step1 Let $\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda))$ be a pair of holomorphic potentials of the following forms:

$$\check{\eta} = (\eta(z,\lambda),\ \ \tau(w,\lambda)) = \left(\sum_{k=-1}^{\infty} \eta_k(z)\lambda^k,\ \ \sum_{m=-\infty}^{1} \tau_m(w)\lambda^m\right), \tag{4.1.1}$$

where $(z,w) \in \mathfrak{D}^2$ and where $\mathfrak{D}^2$ is some holomorphically convex domain in $\mathbb{C}^2$, $\lambda \in \mathbb{C}^*$, $|\lambda| = r$ $(0 < r < 1)$, and η_k and τ_m are $\mathfrak{sl}(2,\mathbb{C})$-valued holomorphic differential 1-forms. Moreover $\eta_k(z)$ and $\tau_k(w)$ are diagonal (resp. off-diagonal) matrices if k is even (resp. odd). We also assume that the upper right entry of $\eta_{-1}(z)$ and the lower left entry $\tau_1(w)$ do not vanish for all $(z,w) \in \mathfrak{D}^2$.

Step2 Let C and L denote the solutions to the following linear ordinary differential equations

$$dC = C\eta \ \text{ and } \ dL = L\tau \ \text{ with } \ C(z_*,\lambda) = L(w_*,\lambda) = \mathrm{id}, \tag{4.1.2}$$

where $(z_*, w_*) \in \mathfrak{D}^2$ is a fixed base point.

Step3 We factorize the pair of matrices (C, L) via the generalized Iwasawa decomposition of Theorem A.1 as follows:

$$(C,\ L) = (F,\ F)(\mathrm{id},\ W)(V_+,\ V_-), \tag{4.1.3}$$

where $V_\pm \in \Lambda^\pm SL(2,\mathbb{C})_\sigma$.

Theorem 4.1 ([4]). *Let F be a $\Lambda SL(2,\mathbb{C})_\sigma$-loop defined by the generalized Iwasawa decomposition in (4.1.3). Then there exists a λ-independent diagonal matrix $l(z,w) \in SL(2,\mathbb{C})$ such that $F \cdot l$ is a complex extended framing of some complex CMC-immersion (or equivalently the complex CGC-immersion as the parallel immersion).*

Step4 The Sym formula defined in (2.2.1) via $F(z,w,\lambda)l(z,w)$ represents a complex CMC-immersion and a CGC-immersion in $\mathfrak{sl}(2,\mathbb{C}) \cong \mathbb{C}^3$.

Let $\mathfrak{c}_j$ for $j \in \{1,2,3,4\}$ and $\mathfrak{s}_j$ for $j \in \{1,2,3\}$ be the involutions defined in (3.1.3), respectively. Then we define the following pairs of involutions on $\check{\eta} = (\eta,\tau) \in \Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma) \times \Omega(\Lambda\mathfrak{sl}(2,\mathbb{C})_\sigma)$:

$$\mathfrak{r}_j : (\eta,\tau) \longmapsto (\mathfrak{c}_j\tau,\ \mathfrak{c}_j\eta) \ \text{ and } \ \mathfrak{d}_j : (\eta,\tau) \longmapsto (\mathfrak{s}_j\eta,\ \mathfrak{s}_j\tau). \tag{4.1.4}$$

We now prove the following theorem.

Theorem 4.2. *Let $\check{\eta} = (\eta(z,\lambda), \tau(w,\lambda))$ be a pair of holomorphic potentials defined as in* (4.1.1), *and let $\mathfrak{r}_j$ for $j \in \{1,2,3,4\}$ and $\mathfrak{d}_j$ for $j \in \{1,2,3\}$ be the pairs of involutions defined in* (4.1.4). *Then the following statements hold:*

$(C,1)$ *If $\mathfrak{r}_1(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are spacelike constant negative Gaußian curvature surfaces in $\mathbb{R}^{2,1}$.*

$(C,2)$ *If $\mathfrak{r}_2(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are timelike constant negative Gaußian curvature surfaces in $\mathbb{R}^{2,1}$.*

$(C,3)$ *If $\mathfrak{r}_3(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are constant positive Gaußian curvature surfaces in $\mathbb{R}^3$.*

$(C,4)$ *If $\mathfrak{r}_4(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are constant mean curvature surfaces with mean curvature $|H^{(c,4)}| < 1$ in H^3.*

$(S,1)$ *If $\mathfrak{d}_1(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are spacelike constant positive Gaußian curvature surfaces in $\mathbb{R}^{2,1}$.*

$(S,2)$ *If $\mathfrak{d}_2(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are timelike constant positive Gaußian curvature surfaces in $\mathbb{R}^{2,1}$.*

$(S,3)$ *If $\mathfrak{d}_3(\check{\eta}) = \check{\eta}$, then the resulting immersions given by the generalized Weierstraß type representation are constant negative Gaußian curvature surfaces in $\mathbb{R}^3$.*

Remark 4.3. *From the forms of pairs of involutions $\mathfrak{r}_j$ for $j \in \{1,2,3,4\}$ defined in* (4.1.4), *the pairs of holomorphic potentials $\check{\eta}$ for (C,j) cases in Theorem 4.2 are generated by a single potential, i.e. $\check{\eta} = (\eta,\tau) = (\eta, \mathfrak{c}_j(\eta))$, where $\mathfrak{c}_j$ for $j \in \{1,2,3,4\}$ are involutions defined in* (3.1.4).

A Double loop groups and the generalized Iwasawa decompositions

In this subsection, we give the basic notations and results for double loop groups (see [7] for more details). Let $D_r := \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$ be an open disk and denote the closure of D_r by $\overline{D_r} := \{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}$. Also, let $A_r = \{\lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}$ be an open annulus containing S^1, and denote the closure of A_r by $\overline{A_r}$. Furthermore, let $E_r = \{\lambda \in \mathbb{C} \mid r < |\lambda|\} \cup \{\infty\}$ be an exterior of the circle C_r.

We recall the definitions of the *twisted plus r-loop group* and the *minus r-loop group* of $\Lambda SL(2,\mathbb{C})_\sigma$ as follows:

$$\Lambda^+_{r,B}SL(2,\mathbb{C})_\sigma := \left\{ W_+ \in \Lambda_r SL(2,\mathbb{C})_\sigma \;\middle|\; \begin{array}{r} W_+(\lambda) \text{ extends holomorphically} \\ \text{to } D_r \text{ and } W_+(0) \in \boldsymbol{B}. \end{array} \right\},$$

$$\Lambda^-_{r,B}SL(2,\mathbb{C})_\sigma := \left\{ W_- \in \Lambda_r SL(2,\mathbb{C})_\sigma \;\middle|\; \begin{array}{r} W_-(\lambda) \text{ extends holomorphically} \\ \text{to } E_r \text{ and } W_-(\infty) \in \boldsymbol{B}. \end{array} \right\},$$

where $\boldsymbol{B}$ is a subgroup of $SL(2,\mathbb{C})$. If $\boldsymbol{B} = \{\mathrm{id}\}$ we write the subscript $*$ instead of $\boldsymbol{B}$, if $\boldsymbol{B} = SL(2,\mathbb{C})$ we abbreviate $\Lambda^+_{r,B}SL(2,\mathbb{C})_\sigma$ and $\Lambda^-_{r,B}SL(2,\mathbb{C})_\sigma$ by $\Lambda^+_r SL(2,\mathbb{C})_\sigma$ and $\Lambda^-_r SL(2,\mathbb{C})_\sigma$, respectively. From now on we will use the subscript $\boldsymbol{B}$ as above only if $\boldsymbol{B} \cap SU(2) = \{\mathrm{id}\}$ holds. When $r = 1$, we always omit the 1.

We set the product of two loop groups:

$$\mathcal{H} = \Lambda_r SL(2,\mathbb{C})_\sigma \times \Lambda_R SL(2,\mathbb{C})_\sigma \ ,$$

where $0 < r < R$. Moreover we set the subgroups of $\mathcal{H}$ as follows:

$$\mathcal{H}_+ = \Lambda^+_r SL(2,\mathbb{C})_\sigma \times \Lambda^-_R SL(2,\mathbb{C})_\sigma,$$

$$\mathcal{H}_- = \left\{ (g_1, \ g_2) \in \mathcal{H} \;\middle|\; \begin{array}{r} g_1 \text{ and } g_2 \text{ extend holomorphically} \\ \text{to } A_r \text{ and } g_1|_{A_r} = g_2|_{A_r} \end{array} \right\},$$

We then quote Theorem 2.6 in [7].

Theorem A.1. *$\mathcal{H}_- \times \mathcal{H}_+ \to \mathcal{H}_-\mathcal{H}_+$ is an analytic diffeomorphism. The image is open and dense in $\mathcal{H}$. More precisely*

$$\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_- w_n \mathcal{H}_+ \ ,$$

where $w_n = \left(\mathrm{id}, \ \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}\right)$ *if* $n = 2k$ *and* $\left(\mathrm{id}, \ \begin{pmatrix} 0 & \lambda^n \\ -\lambda^{-n} & 0 \end{pmatrix}\right)$ *if* $n = 2k+1$.

The proof of the theorem above is almost verbatim the proof given in the basic decomposition paper [2] (see also [3]).

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