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Some Recent Developments in the study of minimal 2-spheres in spheres

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Abstract

We discuss recent progress in the study of the space of harmonic maps from the 2-sphere to the unit n-sphere in Euclidean $(n+1)$-space. We consider the structure of this space as an algebraic variety, the existence of non-manifold points in this space, and the relation between this question and the integrability of Jacobi fields along harmonic maps. One of the main tools used is that of the twistor lift of a harmonic map, which replaces a harmonic map by a holomorphic horizontal map into a Kähler manifold.

Key words: *Minimal surface; harmonic map; moduli space; infinitesimal deformation.*
Subject class: 53C42, 53C43.

1 Introduction

A smooth map $\phi : M \to W$ between Riemannian manifolds M and W is *harmonic* if it is an extremal of the *energy functional.* Here, the *energy* $\mathcal{E}(\phi)$ of a smooth map $\phi : M \to W$ between compact Riemannian manifolds is given by

$$\mathcal{E}(\phi) = \frac{1}{2}\int_M |d\phi|^2\omega\,, \tag{1}$$

where ω is the volume form on M and $|d\phi|$ is the Hilbert-Schmidt norm of $d\phi$ given at each point by

$$|d\phi_x|^2 = \sum_i \langle d\phi_x(e_i), d\phi_x(e_i)\rangle$$

for any orthonormal basis $\{e_i\}$ of the tangent space T_xM of M at x. Equivalently, the map ϕ is harmonic if it satisfies the Euler–Lagrange equations for the energy functional. These equations may be expressed as $\tau(\phi) = 0$, where $\tau(\phi)$ is a vector field along the map called the *tension field*, which is defined by $\tau(\phi) = \operatorname{trace}\nabla d\phi$. Here ∇ denotes the connection on the bundle $T^*M \otimes \phi^{-1}TW$ induced from the Levi-Civita connections on M and W.

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For more details and an extensive survey of harmonic maps, with many references, see the articles [16, 18].

¿From now on, we assume that M is 2-dimensional. In this case, $\mathcal{E}(\phi)$, and hence harmonicity of ϕ, depends only on the conformal structure of M, and, if ϕ is conformal, the energy is equal to the area of the image of ϕ. If the domain surface M is the unit sphere S^2 in $\mathbb{R}^3$, then an argument due to Hopf [32] involving holomorphic differentials shows that a non-constant harmonic map ϕ from S^2 is weakly conformal, and hence a map ϕ from S^2 is harmonic if and only if it is a minimal branched [30] immersion.

The case of harmonic maps from S^2 to the unit sphere S^m in $\mathbb{R}^{m+1}$ has a long history which contains many beautiful and interesting results (see, for example, [10, 13, 14, 2]). Although this is a special case of the more general case of harmonic maps of a Riemann surface into S^m, for reasons to do with the general theory of singularities of harmonic maps [42, 43], it is arguably the most important case. It also has a wealth of interesting features. For instance [13], the area of the image of a harmonic 2-sphere in S^m has area $4\pi d$ for some integer d. Further, if the map is *full*, that is to say its image is not contained in a proper vector subspace of $\mathbb{R}^{m+1}$, then $m = 2n$ for some integer n, and $d \geq n(n+1)/2$.

In 1975, Lawson [35] posed the problem of studying the structure of the space $\mathrm{Harm}_d(S^2, S^{2n})$ of harmonic maps of S^2 into S^{2n} with induced area $4\pi d$. In the present article, we shall give a brief survey of some recent results we have obtained in this area; it may be regarded as a sequel to [6], which appeared in the report of the first Mathematical Society of Japan International Research Institute held at Tohoku University in 1993.

It was conjectured in [6] that $\mathrm{Harm}_d(S^2, S^{2n})$ is a complex algebraic variety of dimension $2d + n^2$, and this was proved by Fernández in 2006. We give a brief account of the method of proof in Section 7.

At the 1993 MSJ conference, Leon Simon asked about the singular points of the algebraic variety $\mathrm{Harm}_d(S^2, S^{2n})$. It is not hard to show that a non-full harmonic 2-sphere which is the limit of a 1-parameter family of full ones is singular, but the question of whether any full harmonic maps are singular points remains. In [9], it is shown that the space $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$ of full harmonic 2-spheres of area $4\pi d$ in S^4 is a manifold for $d \leq 5$, while recent work of Bolton and Fernández, see Section 4, shows that $\mathrm{Harm}_6^{\mathrm{full}}(S^2, S^4)$ is also a manifold. As the case $d = 6$ is somewhat different from $d < 6$, see Section 8, this is perhaps rather a surprising result.

One way of understanding the space of harmonic maps is to look at their infinitesimal deformations, or *Jacobi fields*; in particular, if they are all integrable, the space of harmonic maps is a manifold with the Jacobi fields giving the tangent spaces. For $m = 4$, this has been recently addressed by Lemaire and Wood [38], and a brief account of this work is given in Section 8. The paper ends with applications of this to calculating the nullity of the energy, and a comparision with the nullity of the area functional.

Remark 1 Similar questions may be asked about the space of harmonic 2-spheres in complex space forms. This has been studied in [15, 36] for the case of harmonic 2-spheres in $\mathbb{C}P^2$. In this case, the components of this space consist of the holomorphic and antiholomorphic maps of degree $\pm d$ and energy $4\pi|d|$, together with harmonic maps of degree d and energy $4\pi E$, where $E = 3|d| + 4 + 2r$ for some non-negative integer r. It is shown in [36] that these components are smooth manifolds, of dimension $6|d|+4$ in the holomorphic and antiholomorphic cases and $2E + 8$ in the other cases; in [37] it is shown that the tangent bundle is given precisely by the Jacobi fields.

As in the talk on which it is based, the aim of this article is to give an overview and a flavour of the topic. The interested reader should refer to the papers cited in the text for further details.

2 Early results

It is clear from the characterization of harmonic 2-spheres in S^m as minimal branched immersions that all great 2-spheres are harmonic. Rather more interestingly, we recall that for each positive integer d, the space $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^{2n})$ of *full* harmonic 2-spheres in S^{2n} of area $4\pi d$ is non-empty for each $d \geq n(n+1)/2$.

In fact, some interesting special cases were studied in 1933 by Boruvka [10], who found full harmonic 2-spheres of constant curvature $K = \dfrac{2}{n(n+1)}$ in S^{2n}. The particular case of $n = 2$ gives the Veronese surface in S^4, given by

$$\phi(x,y,z) = \left(xy, xz, yz, \frac{1}{2}(x^2 - y^2), \frac{x^2 + y^2 - 2z^2}{2\sqrt{3}}\right), \quad x^2 + y^2 + z^2 = 3.$$

These Boruvka spheres all have the smallest possible area among full harmonic 2-spheres in S^{2n}, namely $4\pi n(n+1)/2$. However, in 1975 Barbosa [2] gave examples of full harmonic 2-spheres in S^{2n} of area $4\pi d$ for each $d \geq n(n+1)/2$. Barbosa also showed that if $d = n(n+1)/2$, then $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^{2n}) = \mathrm{O}(2n+1; \mathbb{C})$.

The space $\mathrm{Harm}_d(S^2, S^2)$ consists of those maps from S^2 to itself which are holomorphic ($d \geq 0$) or antiholomorphic ($d \leq 0$) of degree d, while there are no full harmonic 2-spheres in S^3. Thus the first case where there are full harmonic maps of interest is $\mathrm{Harm}_d(S^2, S^4)$, which may be studied using the the twistor fibration described in the next section.

3 The twistor fibration

We first recall the definition of the *twistor fibration* $\pi : \mathbb{C}P^3 \to S^4$. Regarding $\mathbb{H}^2$ as a left quaternionic vector space, this is obtained by composing the Hopf map $\rho : \mathbb{C}P^3 \to \mathbb{H}P^1$ given by

$$\rho([z_1, z_2, z_3, z_4]) = [z_1 + z_2 j, z_3 + z_4 j],$$

with the canonical identification of $\mathbb{H}P^1$ and $S^4 \subset \mathbb{H} \oplus \mathbb{R} = \mathbb{R}^5$ given by stereographic projection of S^4 from $(0,0,0,0,-1)$ onto the equatorial 4-plane $\mathbb{H}$ in $\mathbb{R}^5$ which is included in $\mathbb{H}P^1$ by $[q] \mapsto [q, 1]$. We recall [7, 11] that π is a Riemannian submersion when $\mathbb{C}P^3$ is given the Fubini-Study metric of constant holomorphic sectional curvature 1.

A map into $\mathbb{C}P^3$ is said to be *horizontal* if its image is everywhere orthogonal to the fibres of π, and *full* if its image is not contained in a totally geodesic $\mathbb{C}P^2$. It is easy to see that if $\psi : S^2 \to \mathbb{C}P^3$ is holomorphic and horizontal then $\pi \circ \psi$ is harmonic, but the crucial result, as formulated by Bryant [11], is that:

Theorem 1 *Every full harmonic map $\phi : S^2 \to S^4$ is given by*

$$\phi = \pm(\pi \circ \psi) \tag{2}$$

for some uniquely-determined full horizontal holomorphic map $\psi : S^2 \to \mathbb{C}P^3$. Every non-full (and hence totally geodesic) harmonic map $\phi : S^2 \to S^4$ is the projection of a unique horizontal totally geodesic $\mathbb{C}P^1$ in $\mathbb{C}P^3$.

We call the sign in (2) the *spin* of ϕ. In some sense, this result reduces the study of $\mathrm{Harm}(S^2, S^4)$ to that of the space $\mathrm{HHol}(S^2, \mathbb{C}P^3)$ of horizontal holomorphic maps $\psi : S^2 \to \mathbb{C}P^3$. As we shall see below, the latter space is much easier to work with, as it is contained in the projectivization of a finite-dimensional vector space.

With the above as motivation, we now give an elementary description of the elements of $\mathrm{HHol}(S^2, \mathbb{C}P^3)$. Regarding S^2 as $\mathbb{C} \cup \{\infty\}$ in the usual way, a map $\psi : S^2 \to \mathbb{C}P^3$ is holomorphic if and only if it may be written as

$$\psi(z) = [f_1(z), f_2(z), f_3(z), f_4(z)] \tag{3}$$

where $f_1(z), \ldots, f_4(z)$ are polynomials which we may assume have no common zeros. The *degree d* of ψ is then the maximum of the degrees of the polynomials $f_1(z), \ldots, f_4(z)$.

In this way, we identify the space $\mathrm{Hol}_d(S^2, \mathbb{C}P^3)$ of holomorphic 2-spheres of degree d in $\mathbb{C}P^3$ with the projectivization of a dense open subset V of the vector space $(\mathbb{C}[z]_d)^4$, where $\mathbb{C}[z]_d$ is the vector space of complex polynomials in z with degree less than or equal to d. It is easy to see [11] that a map of the form (3) is horizontal if and only if

$$f_1 f_2' - f_1' f_2 + f_3 f_4' - f_3' f_4 = 0, \tag{4}$$

in which case the corresponding harmonic map $\phi = \pi \circ \psi$ has area $4\pi d$.

4 The structure of $\mathrm{HHol}_d(S^2, \mathbb{C}P^3)$

It follows from the previous section that $\mathrm{HHol}_d(S^2, \mathbb{C}P^3)$, and hence $\mathrm{Harm}_d(S^2, S^4)$, may be given the structure of a complex algebraic variety in the projectivization of the vector space $(\mathbb{C}[z]_d)^4$. By counting the number of constraints imposed by the horizontality condition (4), one might expect that the dimension of this algebraic variety should be

$$4(d+1) - (2d-1) - 1 = 2d + 4.$$

This was confirmed independently by Verdier and Loo [39, 45, 46, 47], who both made a detailed study of this variety, and, in particular, proved the following.

Theorem 2 (Verdier 1985, Loo 1989) *For any positive integer d,* $\mathrm{Harm}_d(S^2, S^4)$ *is a connected algebraic variety of dimension $2d+4$. When $d = 1, 2$, it is irreducible; when $d \geq 3$, it has three irreducible components, namely the subset of non-full maps and the closures of the subsets of full maps of positive and negative spin.*

Of course, it is clear from the description above in terms of polynomials that $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$ is empty for $d = 1, 2$.

It is natural to ask if $\mathrm{Harm}_d(S^2, S^4)$ has any singular points. Non-full harmonic 2-spheres in S^4 which are the limits of a 1-parameter family of full ones are singular (see Section 8); on the other hand, it is shown in [9] that $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$ has no singular points for $d \leq 5$ and hence is a manifold. This uses the twistor correspondence described in Section 3 to identify $\mathrm{Harm}^{\mathrm{full}}(S^2, S^4)$ as a double cover of $\mathrm{HHol}^{\mathrm{full}}(S^2, \mathbb{C}P^3)$; in [5], it is shown that the compact-open topology on $\mathrm{Harm}^{\mathrm{full}}(S^2, S^4)$ coincides with that coming from the complex algebraic variety structure on $\mathrm{HHol}^{\mathrm{full}}(S^2, \mathbb{C}P^3)$. In fact, Lemaire and the third author [38, §2] have shown that the correspondence is real analytic.

We now outline a proof of the fact that $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$ has no singular points for $d \leq 5$, since the techniques will be useful later on. We let V_0 be the dense open subset of V consisting of quadruplets of linearly independent polynomials. The condition (4) for horizontality motivates our definition of

$$Q : V_0 \to \mathbb{C}[z]_{2d-2}$$

as

$$Q(f_1, \ldots, f_4) = f_1 f_2' - f_1' f_2 + f_3 f_4' - f_3' f_4 . \tag{5}$$

We hope to show that the zero polynomial in $\mathbb{C}[z]_{2d-2}$ is a *regular value* of Q, so that $Q^{-1}(0)$ is a manifold. Since $\mathrm{HHol}_d^{\mathrm{full}}(S^2, \mathbb{C}P^3)$ may be identified with the projectivization of $Q^{-1}(0)$, it then follows that $\mathrm{HHol}_d^{\mathrm{full}}(S^2, \mathbb{C}P^3)$, and hence its double cover $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$, is a manifold, in fact, by [38] a real-analytic submanifold of a suitable space of smooth mappings from S^2 to S^4.

However, the dimensions of the spaces involved are quite high! For instance, if $d = 5$ then the domain has dimension 24 and the codomain 9, so verifying that dQ has maximal rank at all points of $Q^{-1}(0)$ is quite daunting.

We now describe how we may simplify the problem by using two natural group actions on V_0. Firstly, the standard action of the *complexified symplectic group* $\mathrm{Sp}(2, \mathbb{C})$ on $\mathbb{C}^4$ induces a natural action on V_0 via $A\mathbf{f}(z) = A(\mathbf{f}(z))$, and Q is constant on the orbits of this action. Secondly, for each positive integer k, a Möbius transformation $\mu = (\alpha z + \beta)/(\gamma z + \delta)$ induces a diffeomorphism $\tilde{\mu} : \mathbb{C}[z]_k \to \mathbb{C}[z]_k$ given by

$$(\tilde{\mu} f)(z) = (\gamma z + \delta)^k (f(\mu(z))).$$

This, in turn, induces a diffeomorphism, also denoted $\tilde{\mu}$, from V_0 to V_0. It is easily checked that if $\mathbf{f} = (f_1, f_2, f_3, f_4) \in V_0$, then

$$Q(\tilde{\mu}\mathbf{f}) = (\alpha\delta - \beta\gamma)\tilde{\mu}(Q\mathbf{f}),$$

so that the rank of dQ at $\mathbf{f}$ is equal to the rank of dQ at $A\tilde{\mu}\mathbf{f}$.

This reduces the problem to showing that the rank of dQ is maximal at certain special elements of V_0. For instance, for $d = 4$ it is shown in [8] that if $\mathbf{f} \in V_0$ satisfies (4) then there exists a Möbius transformation μ and an element A of $\mathrm{Sp}(2, \mathbb{C})$ such that

$$A(\tilde{\mu}\mathbf{f})(z) = (1, 2z^4, -4z, z^3).$$

Hence it is enough to show that dQ has maximal rank at

$$(f_1(z), f_2(z), f_3(z), f_4(z)) = (1, 2z^4, -4z, z^3),$$

and this is easy to see.

For $d = 5$ it turns out to be sufficient to consider the case

$$(f_1(z), f_2(z), f_3(z), f_4(z)) = (a_0 + a_1 z, b_4 z^4 + b_5 z^5, c_1 z + c_2 z^2, d_3 z^3 + d_4 z^4),$$

where horizontality reduces to the system of equations:

$$2a_0 b_4 + c_1 d_3 = 0,$$

$$5a_0 b_5 + 3a_1 b_4 + 3c_1 d_4 + c_2 d_3 = 0,$$

$$2a_1 b_5 + c_2 d_4 = 0.$$

This was done by Bolton and Woodward [8], who thus showed that $\mathrm{Harm}_5^{\mathrm{full}}(S^2, S^4)$ is a manifold.

The third author of this article pointed out that the case $d = 6$ may be worth investigating because some harmonic 2-spheres of degree 6 in S^4 are the limits of sequences of harmonic 2-spheres which are full in S^6, and hence are not regular points of $\mathrm{Harm}_6(S^2, S^6)$. Taking up the challenge, and using similar methods (and, initially, Mathematica) the first two authors of this article have proved that $\mathrm{Harm}_6^{\mathrm{full}}(S^2, S^4)$ is a manifold. In line with the method used for $d = 4$ and $d = 5$, the crucial simplifying result is the following.

Proposition 1 *Let* $\mathbf{f} \in V_0$ *Then there exists a Möbius transformation* μ *and an element* A *of* $\mathrm{Sp}(2, \mathbb{C})$ *such that*

$$A(\tilde{\mu}\mathbf{f})(z) = (a_0 + a_1 z + a_2 z^2, b_4 z^4 + b_5 z^5 + b_6 z^6, c_1 z + c_2 z^2 + c_3 z^3, d_3 z^3 + d_4 z^4 + d_5 z^5),$$

or

$$A(\tilde{\mu}\mathbf{f})(z) = (a_0 + a_1 z + a_2 z^2, b_4 z^4 + b_5 z^5 + b_6 z^6, c_1 z + c_2 z^2 + c_4 z^4 + c_5 z^5, d_3 z^3),$$

with, in both cases, $a_0 b_6 \neq 0$*, and, in the second case,* $d_3 \neq 0$*, and where both right hand sides satisfy* (4).

5 Full harmonic maps from S^2 to even-dimensional spheres.

As mentioned earlier, if a harmonic maps from S^2 to a sphere is full, then the codomain sphere is even-dimensional [13]. The study of harmonic maps from S^2 to S^{2n} for general n has many common features with the case $n = 2$. The twistor fibration explained above is a particular case of the general construction that appeared in [13, 2]. Recall that the twistor space of the $2n$-sphere, denoted $\mathcal{Z}_n$, is defined as the subvariety of $\mathrm{Gr}(n, \mathbb{C}^{2n+1})$ (the Grassmanian of n-dimensional subspaces in $\mathbb{C}^{2n+1}$) consisting of *totally isotropic* subspaces with respect to the complex-bilinear extension of the usual dot product. In other words,

$$\mathcal{Z}_n = \{P \in \mathrm{Gr}(n, \mathbb{C}^{2n+1}) : (\mathbf{u}, \mathbf{v}) = 0 \ \ \forall\, \mathbf{u}, \mathbf{v} \in P\},$$

where $(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^{2n+1} u_i v_i$ for $\mathbf{u} = (u_1, \cdots, u_{2n+1})$ and $\mathbf{v} = (v_1, \cdots, v_{2n+1})$ in $\mathbb{C}^{2n+1}$.

There is a projection $\pi : \mathcal{Z}_n \to S^{2n}$ defined as follows: given $P \in \mathcal{Z}_n$, and $\{E_1, \ldots, E_n\}$ an orthonormal basis of P, $\pi(P)$ is defined as the (unique) real vector such that the basis of $\mathbb{C}^{2n+1}$ given by $\{\pi(P), E_1, \ldots, E_n, \overline{E}_1, \ldots, \overline{E}_n\}$ is orthonormal and positively oriented.

As in the $n = 2$ case, we have the following [2, 13, 27]:

- Given a harmonic and full map $\phi : S^2 \to S^{2n}$ there exists a unique holomorphic and horizontal map $\psi : S^2 \to \mathcal{Z}_n$ (the *twistor lift* of ϕ) such that $\pi \circ \psi$ is either ϕ or $-\phi$.
- Conversely, if $\psi : S^2 \to \mathcal{Z}_n$ is holomorphic, horizontal and full, then $\pi \circ \psi : S^2 \to S^{2n}$ is harmonic and full.
- The area of $\phi(S^2)$ is equal to $4\pi d$, where d is the algebraic degree of ψ (or equivalently, the image of $1 \in \mathbb{Z} \simeq H_2(S^2, \mathbb{Z})$ under the map $\psi_* : H_2(S^2, \mathbb{Z}) \to \mathbb{Z} \simeq H_2(\mathcal{Z}_n, \mathbb{Z})$).

An immediate consequence of this is that $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^{2n})$ (i.e. the set of harmonic, full maps from S^2 to S^{2n}) can be identified with two copies of $\mathrm{HHol}_d^{\mathrm{full}}(S^2, \mathcal{Z}_n)$, where $\mathrm{HHol}_d^{\mathrm{full}}(S^2, \mathcal{Z}_n)$ denotes the variety of holomorphic, horizontal, full maps of degree d from S^2 to $\mathcal{Z}_n$.

Therefore, from now on, we will concentrate in the study of $\mathrm{HHol}_d^{\mathrm{full}}(S^2, \mathcal{Z}_n)$. For the particular case of $n = 2$, recall that $\mathcal{Z}_2$ is just $\mathbb{C}P^3$, and that the horizontality condition, written in homogeneous coordinates in $\mathbb{C}P^3$, is given by equation (4).

For general n, it is certainly not the case that $\mathcal{Z}_n$ is isomorphic to a complex projective space. However, the variety $\mathcal{Z}_n$ is birationally equivalent to $\mathbb{C}P^{n(n+1)/2}$ (note that the dimension of $\mathcal{Z}_n$ is $n(n+1)/2$). The idea would then be: Fix a birational map b from $\mathbb{C}P^{n(n+1)/2}$ to $\mathcal{Z}_n$. Then, for each $\psi \in \mathrm{HHol}_d^{\mathrm{full}}(S^2, \mathcal{Z}_n)$, define the map $b^{-1} \circ \psi : S^2 \to \mathbb{C}P^{n(n+1)/2}$. This should give some variety of maps from S^2 into $\mathbb{C}P^{n(n+1)/2}$ satisfying some sort of 'horizontality' condition. Then, instead of studying $\psi \in \mathrm{HHol}_d^{\mathrm{full}}(S^2, \mathcal{Z}_n)$, study the set of such maps.

Of course this is all wishful thinking: the idea of the previous paragraph, although plausible, is full of obstacles. Several things can go wrong:

1. Since a birational map is only defined outside of a codimension 2 subvariety, the map $b^{-1}\circ\psi$ will not be defined at all if the image of ψ lies entirely in the subvariety where b^{-1} is not defined.

2. The horizontality condition in $\mathcal{Z}_n$ will translate into some condition for maps into $\mathbb{C}P^{n(n+1)/2}$. But this condition may be much harder to work with than the original.

3. Even if $b^{-1}\circ\psi$ is defined, we also have to take into account that we want the degree of maps to be preserved. In other words, if the degree of $b^{-1}\circ\psi$ is not the same as the degree of ψ we will not be able to study the variety $\mathrm{HHol}_d^{\mathrm{full}}(S^2,\mathcal{Z}_n)$.

Fortunately all the possible things that can go wrong either go right or not terribly wrong. But, before giving the answer to these questions, we need to give an explicit description of some birational maps between $\mathcal{Z}_n$ and $\mathbb{C}P^{n(n+1)/2}$.

Given an orthonormal basis (with respect to the canonical Hermitian product) $\beta = \{E_0, E_1, \ldots, E_n, \overline{E}_1, \ldots, \overline{E}_n\}$ of $\mathbb{C}^{2n+1}$, define a birational map $b_\beta : \mathbb{C}P^{n(n+1)/2} \longrightarrow \mathcal{Z}_n$ by

$$[s:\alpha_1:\cdots:\alpha_n:\tau_{12}:\cdots:\tau_{n-1,n}] \xrightarrow{b_\beta} \left\{ \begin{array}{c} n-\text{plane generated by the vectors} \\ \dfrac{\alpha_\ell}{s}E_0 + E_\ell + \displaystyle\sum_{k=1}^{n}\left(-\frac{\alpha_\ell\alpha_k}{2s^2}+\frac{\tau_{\ell k}}{2s}\right)\overline{E}_k,\ 1\le \ell\le n \end{array}\right\}.$$

Then, given $\psi \in \mathrm{HHol}_d^{\mathrm{full}}(S^2,\mathcal{Z}_n)$ the idea would be to define the map $\widetilde{\psi}_\beta = b_\beta^{-1}\circ\psi : S^2 \to \mathbb{C}P^{n(n+1)/2}$ and study its properties. The questions about what can go wrong are solved as follows:

1′. The image of ψ cannot lie in the subvariety of $\mathcal{Z}_n$ where b_β^{-1} is not defined. A complete proof of this appears in [24]. The key ingredient of the proof is that the map ψ is full.

2′. The fact that the map ψ is horizontal translates into the following relatively nice differential system:

Writing a map from S^2 to $\mathbb{C}P^{n(n+1)/2}$ as $[s:\alpha_1:\cdots:\alpha_n:\tau_{12}:\cdots:\tau_{n-1,n}]$ (in homogeneous coordinates), the fact that ψ is horizontal translates into the map $b_\beta^{-1}\circ\psi : S^2\to\mathbb{C}P^{n(n+1)/2}$ satisfying the differential system given by

$$\alpha_i'\alpha_j - \alpha_i\alpha_j' = s\tau_{ij}' - s'\tau_{ij}, \quad 1\le i,j\le n, \tag{6}$$

where, as usual, the dashes denote differentiation with respect to a conformal parameter on S^2. Note that this reduces to equation (4) when $n=2$.

This differential system was actually found by Bryant in [12], although in a different form. It also appears in [31] in the form presented here.

3′. There are examples for which the degree of $b_\beta^{-1}\circ\psi$ is not equal to the degree of ψ. Although for *most* maps the degree is the same, since we are trying to study the set of *all* holomorphic and horizontal maps into $\mathcal{Z}_n$, it seems that the original idea will not work. However, this problem can be overcome as follows.

Define the varieties

$$\mathrm{PD}_d^{\text{full}}(S^2, \mathbb{C}P^{n(n+1)/2}) = \Big\{\text{maps } [s:\alpha_1:\ldots\alpha_n:\tau_{12}:\cdots:\tau_{n-1,n}] : S^2 \to \mathbb{C}P^{n(n+1)/2}$$

$$\text{holomorphic of degree } d \text{ satisfying } \alpha_i'\alpha_j - \alpha_i\alpha_j' = s\tau_{ij}' - s'\tau_{ij}, \text{ and } \left(\frac{\alpha_i}{s}\right)' \text{independent}\Big\}.$$

Notice that, since these are maps from S^2 to $\mathbb{C}P^{n(n+1)/2}$ of degree d, each homogeneous component of one such map can be regarded as a polynomial of degree d in one complex variable z. We define the following open subset of $\mathrm{PD}_d^{\text{full}}(S^2, \mathbb{C}P^{n(n+1)/2})$:

$$\mathrm{PD}_{d,0}^{\text{full}}(S^2, \mathbb{C}P^{n(n+1)/2}) = \Big\{[s:\alpha_1:\ldots] \in \mathrm{PD}_d^{\text{full}}(S^2, \mathbb{C}P^{n(n+1)/2}) \text{ with}$$

$$s = \prod_{\ell=1}^{d}(z - s_\ell),\ s_\ell \text{ distinct, and } \alpha_1(s_\ell) \neq 0,\ \forall \ell\Big\}.$$

The following proposition gives the key for the subsequent study of $\mathrm{HHol}_d^{\text{full}}(S^2, \mathcal{Z}_n)$. The proof is long and technical; details can be found in [26].

Proposition 2 *For all $\psi_0 \in \mathrm{HHol}_d^{\text{full}}(S^2, \mathcal{Z}_n)$, there exists a birational map b_β and an open set $\mathcal{U}_\beta \subset \mathrm{HHol}_d^{\text{full}}(S^2, \mathcal{Z}_n)$ with $\psi_0 \in \mathcal{U}_\beta$ such that the map*

$$\psi \in \mathcal{U}_\beta \subset \mathrm{HHol}_d^{\text{full}}(S^2, \mathcal{Z}_n) \longrightarrow \widetilde{\psi}_\beta = b_\beta^{-1} \circ \psi \in \mathrm{PD}_{d,0}^{\text{full}}(S^2, \mathbb{C}P^{n(n+1)/2})$$

is an algebraic isomorphism.

In other words, although $\mathrm{PD}_{d,0}^{\text{full}}(S^2, \mathbb{C}P^{n(n+1)/2})$ is really not isomorphic to $\mathrm{HHol}_d^{\text{full}}(S^2, \mathcal{Z}_n)$, we can completely cover the latter variety with patches algebraically isomorphic to the former.

6 An algebraic construction of harmonic maps from S^2 to S^{2n}

In view of Proposition 2, in order to study local characteristics of $\mathrm{HHol}_d^{\text{full}}(S^2, \mathcal{Z}_n)$ we can study $\mathrm{PD}_{d,0}^{\text{full}}(S^2, \mathbb{C}P^{n(n+1)/2})$ instead. To this end, we have to analyse the system $\alpha_i'\alpha_j - \alpha_i\alpha_j' = s\tau_{ij}' - s'\tau_{ij}$ where s is a polynomial with d distinct roots s_m, $1 \leq m \leq d$, and α_i, τ_{jk} are polynomials of degree less than or equal to d, with $\alpha_1(s_m) \neq 0$ for all m.

The obvious idea would be to write the polynomials in the usual basis, substitute into system (6) and obtain algebraic equations on the coefficients. The equations obtained, however, turn out to be too entangled and they are too hard to analyse.

Instead, one can write the polynomials as follows: since s has distinct roots s_m, $1 \leq m \leq d$, the polynomials $\{s, \frac{s}{z-s_1}, \ldots, \frac{s}{z-s_m}\}$ form a basis for the space of polynomials of degree d. Thus we can write

$$s = \prod_{m=1}^{d} (z - s_m) , \qquad \alpha_i = a_{i0}s + \sum_{m=1}^{d} a_{im} \frac{s}{z - s_m} , \qquad \tau_{ij} = t_{ij0}s + \sum_{m=1}^{d} t_{ijm} \frac{s}{z - s_m} .$$

Using this representation, the system (6) turns into the following algebraic equations (see [26] for details):

$$a_{im} \sum_{k \neq m} \frac{a_{jk}}{(s_m - s_k)^2} - a_{jm} \sum_{k \neq m} \frac{a_{ik}}{(s_m - s_k)^2} = 0, \ 1 \leq m \leq d \tag{7}$$

and

$$\tau_{ij} = t_{ij0}s + s \int \frac{\alpha_i' \alpha_j - \alpha_i \alpha_j'}{s^2} \, \mathrm{d}z. \tag{8}$$

(Equation (7) guarantees that the integrand in equation (8) has no residues and τ_{ij} is a polynomial of degree at most d.)

It is useful to write (7) in the following matrix form:

$$\begin{pmatrix} \lambda_1 & \frac{1}{(s_1-s_2)^2} & \cdots & \frac{1}{(s_1-s_d)^2} \\ \frac{1}{(s_2-s_1)^2} & \lambda_2 & \cdots & \frac{1}{(s_2-s_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(s_d-s_1)^2} & \frac{1}{(s_d-s_2)^2} & \cdots & \lambda_d \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \cdots & a_{nd} \end{pmatrix} = 0, \tag{9}$$

where $\lambda_m = -\frac{1}{a_{1m}} \sum_{k \neq m} \frac{a_{1k}}{(s_m - s_k)^2}$.

This approach immediately gives an interesting result: it provides an algebraic 'recipe' to construct *any* linearly full harmonic map from S^2 to S^{2n} (and hence any harmonic map from S^2 to a sphere): First find s_m and a_{1m} so that the nullity of the matrix

$$\begin{pmatrix} \lambda_1 & \frac{1}{(s_1-s_2)^2} & \cdots & \frac{1}{(s_1-s_d)^2} \\ \frac{1}{(s_2-s_1)^2} & \lambda_2 & \cdots & \frac{1}{(s_2-s_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(s_d-s_1)^2} & \frac{1}{(s_d-s_2)^2} & \cdots & \lambda_d \end{pmatrix} \tag{10}$$

(where $\lambda_m = -\frac{1}{a_{1m}} \sum_{k \neq m} \frac{a_{1k}}{(s_m - s_k)^2}$) is at least n. Then find a_{im}, $2 \leq i \leq n$, $1 \leq m \leq d$, so that equation (9) is satisfied, and so that the second matrix in that equation has maximal rank (this will guarantee that the map is full). Then use equation (8) to find the τ_{ij} and follow the procedure above backwards to obtain a harmonic map. Of course, there will be some free choices on the way, such as the choice of a basis β as explained in Section 5.

7 $\mathbf{Harm}_d(S^2, S^{2n}(1))$ has dimension $2d + n^2$.

In this section we sketch the proof of the following:

Conjecture (Bolton-Woodward, 1993, First MSJ International Research Institute, Tohoku University [6]): *$Harm_d(S^2, S^{2n}(1))$ is an algebraic variety of dimension $2d + n^2$.*

The algebraic construction of the previous section allows for a very detailed analysis of the variety $\mathrm{PD}^{\mathrm{full}}_{d,0}(S^2, \mathbb{C}P^{n(n+1)/2})$. In particular, it is possible to give a constructive proof that there is a $2d+n^2$-dimensional variety inside $\mathrm{PD}^{\mathrm{full}}_{d,0}(S^2, \mathbb{C}P^{n(n+1)/2})$, which shows that the dimension of $\mathrm{PD}^{\mathrm{full}}_{d,0}(S^2, \mathbb{C}P^{n(n+1)/2})$, and hence of $\mathrm{Harm}^{\mathrm{full}}_d(S^2, S^{2n})$, is at least $2d+n^2$. The main steps are as follows:

1. Show that the variety of those $(s_1, \ldots, s_d, \lambda_1, \ldots, \lambda_d)$ such that the matrix (10) has nullity n has dimension at least $2d - n(n+1)/2$ [26].
2. Assuming that the nullity of the matrix (10) is n, it is not hard to see that the dimension of the set of solutions a_{im}, $1 \le i \le n$, $0 \le m \le d$, of equation (9), is $n^2 + n$.
3. Finally, the τ_{ij} are completely determined by (8), but each has one degree of freedom (the t_{ij0}), giving $n(n-1)/2$ dimensions more.
4. Add up: $2d - n(n+1)/2 + n^2 + n + n(n-1)/2 = 2d + n^2$, as desired. Hence $\dim(\mathrm{Harm}^{\mathrm{full}}_d(S^2, S^{2n})) \ge 2d + n^2$.

To finish the proof of the conjecture stated at the beginning of the section, it only remains to show that $\mathrm{Harm}^{\mathrm{full}}_d(S^2, S^{2n})$ has dimension at most $2d + n^2$. This is actually easier, and it was essentially known to Bolton and Woodward. It was also proved by Kotani (see the last corollary in [34]). Another proof of this fact, using different techniques, appears in [25] for the particular case $n = 3$.

The proof we sketch here is very similar to that in [34], but we use the algebraic construction explained above. The key point is to note that there are well-defined projections

$$p_n : \mathrm{PD}^{\mathrm{full}}_{d,0}(S^2, \mathbb{C}P^{n(n+1)/2}) \to \mathrm{PD}^{f}_{d,0}(S^2, \mathbb{C}P^{(n-1)n/2}),$$

defined by deleting the α_{n+1} and the $\tau_{i,n+1}$:

$$\begin{aligned} &p_n([s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-2,n-1} : \tau_{1,n} \cdots : \tau_{n-1,n}]) \\ &\quad = [s : \alpha_1 : \cdots : \alpha_{n-1} : \tau_{12} : \cdots : \tau_{n-2,n-1}]). \end{aligned}$$

This map has the following properties:

- Its image has codimension at least 1. This is expected but not quite trivial. See [26] for the detailed proof.

- The fibre over a generic point has dimension at most $2n$. This is not hard to see if we look at equation (9). The points in the fibre are essentially those α_n such that the vector $(a_{n1},\dots,a_{nd})$ is in the kernel of the matrix (9). Since this matrix has nullity n, we have n degrees of freedom. In addition, we have 1 degree of freedom from the choice of a_{n0} and $n-1$ degrees of freedom from the choice of t_{in0}, $1 \le i \le n-1$. Therefore the fibre has dimension $n+1+(n-1)=2n$.

Then proceed by induction on the dimension of $\mathrm{PD}^{\text{full}}_{d,0}(S^2, \mathbb{C}P^{n(n+1)/2})$. Note that $\mathrm{PD}^{f}_{d,0}(S^2, \mathbb{C}P^1)$, which corresponds to the case $n=1$, is the set of holomorphic maps from S^2 to $\mathbb{C}P^1$ of degree d; this has dimension $2d+1$, so of course the conjecture is true in this case.

If the conjecture is true at level $n-1$, then

$$\begin{aligned}
\dim(\mathrm{PD}^{\text{full}}_{d,0}(S^2, \mathbb{C}P^{n(n+1)/2})) &\le \dim(\text{Image of } p_n) + \dim(\text{Fibre of } p_n) \\
&\le (\dim(\mathrm{PD}^{f}_{d,0}(S^2, \mathbb{C}P^{(n-1)n/2})) - 1) + 2n \\
&\le 2d + (n-1)^2 - 1 + 2n \\
&= 2d + n^2,
\end{aligned}$$

as desired. Therefore we have proved the following.

Theorem 3 *The (pure) dimension of $Harm^{full}_d(S^2, S^{2n})$ is $2d+n^2$.*

Maybe the curious thing about these proofs is that numbers match very well, but it is not clear why things work (or at least the second author does not completely understand why they work). There are also many relationships between the last part of this section and integrability of Jacobi fields, as well as the *extra eigenfunctions* (see below) which correspond, in our setting, to maps for which the matrix (9) has nullity greater than n.

8 The role of Jacobi fields

A *Jacobi field* is an infinitesimal deformation of a harmonic map. We can make this more precise in two ways.

The first way is by means of the the *second variation* as follows. Let $\phi : M \to W$ be a harmonic map between compact Riemannian manifolds. Let v, w be *vector fields along ϕ*, i.e, $v, w \in \Gamma(\phi^{-1}TW)$. Choose a smooth two-parameter variation $\{\phi_{t,s}\}$ of ϕ with

$$\left.\frac{\partial \phi_{t,s}}{\partial t}\right|_{(t,s)=(0,0)} = v \quad \text{and} \quad \left.\frac{\partial \phi_{t,s}}{\partial s}\right|_{(t,s)=(0,0)} = w\,;$$

then the *second variation* or *Hessian of the energy at ϕ* is defined by

$$H_\phi(v,w) = \left.\frac{\partial^2}{\partial t\, \partial s} E(\phi_{t,s})\right|_{(t,s)=(0,0)}.$$

It is given by the *second variation formula* (see, for example, [17]):

$$H_\phi(v,w) = \int_M \langle J_\phi(v), w\rangle\,\omega_g$$

where $J_\phi : \Gamma(\phi^{-1}TW) \to \Gamma(\phi^{-1}TW)$ is the self-adjoint linear operator defined by

$$J_\phi(v) = \Delta^\phi v - \operatorname{Tr} R^W(d\phi, v)\, d\phi\,.$$

Here Δ^ϕ denotes the Laplacian on $\phi^{-1}TW$ and R^W the curvature operator of W (conventions as in [17]). The operator J_ϕ is called the *Jacobi operator*; a vector field $v \in \Gamma(\phi^{-1}TW)$ is called a Jacobi field if it satisfies the *Jacobi equation* $J_\phi(v) = 0$. By standard elliptic theory, the set $\ker J_\phi$ of Jacobi fields along a given harmonic map is a finite-dimensional vector subspace of $\Gamma(\phi^{-1}TW)$.

A second way to understand the Jacobi operator is as (minus) the *linearization* of the tension field as follows (see [37]).

Proposition 3 *Let $\phi : M \to W$ be harmonic and let $v \in \Gamma(\phi^{-1}TW)$. Let $\{\phi_t\}$ be a smooth (one-parameter) variation of ϕ which is* tangent to v, *i.e., with $\partial\phi_t/\partial t|_{t=0} = v$. Then*

$$J_\phi(v) = -\left.\frac{\partial}{\partial t}\tau(\phi_t)\right|_{t=0}, \tag{11}$$

i.e., the components of each side with respect to a local frame on $\phi^{-1}TW$ satisfy $J_\phi(v)^\alpha = -(\partial/\partial t)\tau(\phi_t)^\alpha|_{t=0}$ $(\alpha = 1,\ldots,m)$.

Thus v is a Jacobi field along ϕ if and only if

$$\tau(\phi) = 0 \quad \text{and} \quad \left.\frac{\partial}{\partial t}\tau(\phi_t)\right|_{t=0} = 0\,. \tag{12}$$

Note that equation (11) and condition (12) are independent of the local frame chosen. We shall call a smooth variation $\{\phi_t\}$ *harmonic to first order* if it satisfies condition (12). Thus *a smooth variation $\{\phi_t\}$ of a harmonic map ϕ is harmonic to first order if and only if it is tangent to a Jacobi field along ϕ.*

In particular, if $\{\phi_t\}$ is a smooth variation of ϕ with each ϕ_t harmonic, its variation vector field $v = \partial\phi_t/\partial t|_{t=0}$ is a Jacobi field. We now ask whether every Jacobi field arises this way; to discuss this, we make the following definition.

Definition 1 A Jacobi field v along a harmonic map $\phi : M \to W$ is called *integrable* if it is tangent to a smooth variation $\{\phi_t\}$ of ϕ through harmonic maps, i.e., there exists a one-parameter family $\{\phi_t\}$ of harmonic maps with $\phi_0 = \phi$ and $\partial\phi_t/\partial t|_{t=0} = v$.

Proposition 4 [1] *Let $\phi : M \to W$ be a harmonic map between compact real-analytic Riemannian manifolds. Then all Jacobi fields along ϕ are integrable if and only if the space of harmonic maps ($C^{2,\alpha}$-)close to ϕ is a smooth manifold whose tangent space at ϕ is exactly the space* $\ker J_\phi$ *of Jacobi fields along it.*

The converse is false: there are examples where the space of harmonic maps is a smooth manifold, but the space of Jacobi fields contains non-integrable ones which are not in a tangent space, see [37] and below.

Now, to analyse Jacobi fields along harmonic maps from S^2 to S^m, one idea is to use the twistor construction to replace them with infinitesimal deformations of the twistor lift. This works well in the case $m = 4$, as we now describe. Given a holomorphic map $\psi : S^2 \to \mathbb{C}P^3$, we call a vector field u along ψ an *infinitesimal horizontal holomorphic deformation* (*IHHD*) if it is *holomorphic*, i.e, tangent to a curve of holomorphic maps, and preserves horizontality 'to first order'. Representing ψ by a quadruplet of polynomials as in Section 3, $\mathbf{f} = (f_1, f_2, f_3, f_4) : \mathbb{C} \to \mathbb{C}^4 \setminus \{0\}$ so that u is represented by a holomorphic map $U : \mathbb{C} \to \mathbb{C}^4$, the latter condition reads

$$dQ_{\mathbf{f}}(U) = 0\,. \tag{13}$$

Given an infinitesimal horizontal holomorphic deformation u of ψ, it is easy to see from the composition law for harmonic maps [16, §4] that $v = d\pi \circ u$ is a Jacobi field along $\phi = \pi \circ \psi$. The inverse construction is harder because of the presence of branch points, however, we can show [38]:

Proposition 5 *Let $\phi : S^2 \to S^4$ be a full harmonic map with twistor lift $\psi : S^2 \to \mathbb{C}P^3$. Then setting $v = d\pi \circ u$ defines a one-to-one correspondence between IHHDs u of ψ and Jacobi fields v along ϕ.*

This reduces the problem of finding Jacobi fields along ϕ to solving equation (13). In particular, we see that, if Q has maximal rank at F, then, not only is the space of harmonic maps a smooth manifold at $\phi = \pi \circ \psi$, but also the Jacobi fields along ϕ are all integrable and form the tangent space to that manifold at ϕ. If Q does not have maximal rank, then there will be non-integrable Jacobi fields along ϕ.

As we saw earlier, *if $d \leq 6$ then Q is always submersive, so that all Jacobi fields are integrable and form the tangent space to the smooth manifold* $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$.

It is not known whether Proposition 5 generalizes to higher dimensions; the argument establishing extension over branch points is special to four dimensions. Note that for any $m \geq 4$, $d \geq 3$, the space of *all* (i.e. full and non-full) harmonic maps from S^2 to S^m is not a manifold — indeed, harmonic maps can collapse to a non-full harmonic map, see the work of N. Ejiri and M. Kotani [20, 21, 22, 34]. For $d \geq 3$, some non-full maps $S^2 \to S^4$ are the limits of a family of full harmonic maps into S^4, we shall call such maps *collapse points*; see [38] for an analysis of those, especially for $d < 6$. When $d \geq 6$, a non-full map might also occur as the limit of full maps into higher-dimensional spheres; see [34] for some results on collapsing in higher dimensions.

Let ϕ be a non-full (non-constant) harmonic map from S^2 to S^m with $m = 3$ or 4; we examine the Jacobi fields along ϕ. Note first that ϕ has image in a totally geodesic S^2. From this, it is easy to see that *the space of non-full maps is a smooth manifold.* Now any Jacobi field along such a map decomposes into components tangential and normal to the image S^2. The tangential component is a conformal vector field, so we concentrate on the normal component. This may be tangent to the space of non-full maps; if it is not, then it is called *extra*. Take a parallel basis for the normal bundle of the image S^2. Then the Jacobi equation assumes a simple form: a vector field along ϕ is Jacobi if and only if its $n-2$ components v_i satisfy the *generalized eigenvalue (Schrödinger) equation*:

$$\Delta v_i = |d\phi|^2 v_i\,.$$

The coordinate functions of ϕ considered as a map into $\mathbb{R}^3$ span a 3-dimensional space of trivial solutions to this equation; any other solution is called an *extra eigenfunction* (of ϕ). It is easy to see that a Jacobi field is extra if and only if at least one of its components is an extra eigenfunction.

Now, it can be shown that a non-full harmonic map from S^2 to S^4 has an extra Jacobi field v if and only if it is a collapse point. But then one of the components of v is an extra eigenfunction; this gives an extra Jacobi field of ϕ considered as a map into S^3 which cannot be integrable, since all harmonic maps into S^3 are non-full. So we see that *the space of harmonic maps from S^2 to S^3 is a smooth manifold, however those harmonic maps $S^2 \to S^3$ which are collapse points when considered as maps into S^4 have non-integrable Jacobi fields.*

Thus, *integrability of all Jacobi fields implies that the space of harmonic maps is a smooth manifold, but not conversely.*

9 Area and nullity

The *nullity* of (the energy) of a harmonic map is the *real* dimension of the space of Jacobi fields along it. Since the Jacobi fields are the solutions to equation (13), we obtain

Theorem 4 *Let $\phi : S^2 \to S^4$ be a harmonic map of twistor degree d. Then the nullity of ϕ is greater than or equal to $4d+8$ with equality if and only if ϕ is a regular point of Q.*

Recalling the results of Bolton–Woodward and Bolton–Fernández cited in Section 3, we deduce:

Corollary 1 *The nullity of a full harmonic map $\phi : S^2 \to S^4$ of degree ≤ 6 is exactly $4d+8$.*

We can consider instead the second variation of the *area*. This depends only on normal vector fields. Results of N. Ejiri and M. Micallef [23] imply that, *for any non-constant harmonic map from the 2-sphere, the map $v \mapsto$ the normal component of v is a surjective linear map from the space of Jacobi fields for the energy to the space of Jacobi fields for the area, with kernel the tangential conformal fields.*

S. Montiel and F. Urbano [40, Corollary 7] show that the nullity of the (second variation of the) area of a (full or non-full) minimal immersion of S^2 in S^4 of twistor degree d is exactly $4d+2$. Since the tangential conformal fields form a space of (real) dimension 6, the nullity of the energy is precisely $4d+8$; we deduce that *any immersive harmonic map is a regular, and so a smooth, point of* $\mathrm{Harm}_d(S^2, S^4)$.

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