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# Knots and minimal surfaces

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We show that every smooth orientable surface in 3-space with boundary is isotopic to a strictly stable minimal surface. We also show that every ribbon link in the 3-sphere bounds strictly stable minimal disks in the 4-ball.

## 1. Introduction

A *link* is smoothly embedded circles in the 3-space $\mathbb{R}^3$ (or the 3-sphere $\mathbb{S}^3$). A *knot* is a link with one connected component. Let $\Sigma$ be a smooth orientable surface with a boundary. The surface $\Sigma$ is said to be a *minimal surface* if the mean curvature is identically zero. Let $M$ be a closed, orientable, irreducible 3-manifold. W. Meeks III, L. Simon, and S. T. Yau showed that every incompressible surface in $M$ is isotopic to a globally least area minimal surface by geometric measure theory [3]. We consider the following problem.

**Problem.**
(1) What kind of an embedded minimal surface does a link bound in 3-space?
(2) Which links bound embedded minimal disks in the 4-ball?

A *Seifert surface* for a link in $\mathbb{R}^3$ is a compact oriented 2-manifold $S$ embedded in $\mathbb{R}^3$ such that the boundary of $S$ is $L$ as an oriented link and $S$ does not have any closed components. It is well-known that there exists a Seifert surface for any oriented link in $\mathbb{R}^3$. An invariant of a link, the *genus* of a link $L$, can be defined by the minimal genus among all Seifert surfaces of $L$. We show the following.

**Theorem 1.1.** *Every Seifert surface in* $\mathbb{R}^3$ *for a link is isotopic to a strictly stable minimal surface.*

**Corollary 1.2.** *Every link bounds a strictly stable minimal surface such that it realizes the genus of the link.*

**Remark 1.3.** It is well-known that every minimal genus Seifert surface is incompressble.

Let $L$ be a link in $S^3$ and an arc $b$ connecting two different components of $L$, i.e. $b$ is smoothly embedded in $S^3$ and intersects $L$ only at its end points (orthogonally), choose a normal vector field $\mu$ along $b$ which is normal to $L$ at both endpoints of $b$. With the proper orientation of $b$, one can perform the connected sum of the two components of $L$ along $b$ (just use the orthogonal complement of $\mu$ in a tubular neighborhood of $b$ as the connecting tube). The resulting link $F(L)$ is a link with one less component than $L$ and is called the *fusion* of $L$ along the band $B = \{\mu \cup b\}$. One can perform more than one fusion to a link along a collection of bands $\{B_i\}$, thus obtaining a sequence of fusions $F_1(L), \ldots, F_k(L)$. Then $F(L) = F_k(L)$ is called the fusion of a link along the bands $\{B_i\}$. A link which is obtained from a trivial link by a sequence of fusions is called a *ribbon link*. J. Hass showed that a knot is a ribbon knot if and only if the knot bounds an embedded minimal surface [1]. We show the following result.

**Theorem 1.4.** *Every ribbon link in the 3-sphere bounds strictly stable minimal disks in the 4-ball.*

This paper is organized as follows. In Section 2, we shall introduce the bridge principle for minimal surfaces and recall a result of B. White about the principle. In Section 3, we shall prove Theorem 1.1 and Theorem 1.4.

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## 2. The bridge principle

W. Meeks III and S. T. Yau explained about the bridge principle for minimal surfaces as follows [2]: the bridge principle is related to a physical property of soap films. This principle can be illustrated by the following experiment. Suppose two soap film surfaces are bounded by two bent steel wires. We can change this wire configuration by joining these wires by parallel wire segments which are close to each other. The experiment shows that usually one can form a soap film surface bounded by this new configuration and the new surface is close to the old surfaces joined together with a soap film bridge joining the old surfaces. Since soap films correspond to strictly stable minimal surfaces, the bridge principle can be reformulated using the concept of stable minimal surfaces.

Here, we quote a result of B. White about the bridge principle. Let $S$ be a two dimensional minimal surface in $R^N$, and let $P \subset R^N$ be a thin curved rectangle whose two short sides lie along the boundary of $S$ and that is otherwise disjoint from $S$. Typically $S$ will have two connected components, and $P$ will join one to the other. The bridge principle for minimal surfaces is the principle that it should usually be possible to deform $S \cup P$ slightly to make a minimal surface with boundary $\partial(S \cup P)$. In this paper, we will show that it is possible provided, roughly speaking, that $S$ is smooth and strictly stable, that P is sufficienty thin, and that, at each end of $P$, the angle between P and S is strictly between 0 and $2\pi$. ("Strictly stable" means "stable and having no nonzero Jacobi fields that vanish on the boundary" or, equivantly, "having index 0 and nullity 0 as a critical point for the area functional".)

B. White showed the following theorem [4].

**Theorem 2.1.** *Let $C$ be a compact smooth embedded $(m-1)$ manifold in $R^N$, and let $\mathcal{S}$ be a finite set of smooth, embedded, strictly stable minimal surfaces, each of which has boundary $C$. Let $\Gamma$ be a smooth curve joining two points of $C$ in such a way that for every $S \in \mathcal{S}$,*
(1) $\Gamma \cap S = \partial\Gamma$, *and*
(2) *at each of its two endpoints, $\Gamma$ makes a nonzero angle with the tangent half-plane to $S$ at that endpoint.*
*Then there exists a sequence $P_n$ of bridges on $C$ that shrink nicely to $\Gamma$. Given such a sequence, for all sufficiently large $n$ and for all $S \in \mathcal{S}$, there exists a strictly stable minimal surface $S_n$ and a diffeomorphism $f_n : S \cup P_n \longrightarrow S_n$ such that:*
(1) $f_n(x) = x$ *for $x \in \partial S_n$ (so that, in particular, $S_n$ and $S \cup P_n$ have the same boundary),*
(2) $\sup\{|x - f_n(x)| : x \in S_n\} = O(w_n)$ *where $w_n$ is the width of the bridge $P_n$,*
(3) *$f_n$ converges smoothly to the identity map $S \to S$ on compact subsets of $S \setminus \Gamma$,*
*and*
(4) $\mathrm{area}(S_n) \to \mathrm{area}(S)$ *as $n \to \infty$.*
*Furthermore, if $M$ is a smooth manifold that contains $C \cup \Gamma$, then we can choose the $P_n$ to lie in $M$.*

**Remark 2.2.** For the definition of "shrink nicely" in the statement of Theorem 2.1, see [4].

## 3. Proofs

First, we recall a construction of a Seifert surface for a link. Let $L$ be a link in $\mathbb{R}^3$ and assume that $L$ is oriented. We take a regular projection of $L$. Near each crossing, delete the over- and under-crossings, and replace them by arcs as in Figure 1.

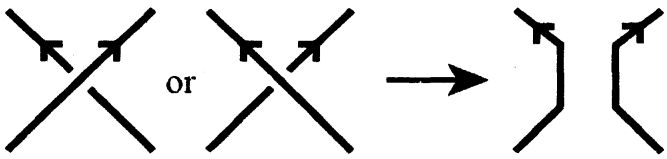


FIGURE 1

Then we have a disjoint collection of simple closed curves bounding disks, possibly nested. These disks can be made disjoint by pushing their interiors slightly off the plane. Now, let us connect them together at the old crossing with half-twisted strips to have a new surface as in Figure 2.

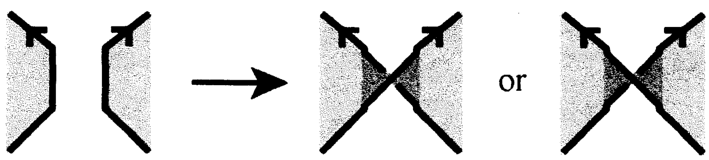


FIGURE 2

In this way, we have at least one Seifert surface for a link $L$ which can be constructed from disks by attaching some strips (1-handles) to the disks. In general, every Seifert surface is ambient isotopic to a surface which is obtained from disks by attaching 1-handles to the disks. In fact, we can start the construction with a single disk as follows. Let $S$ be a Seifert surface for a link $L$. Then, shortening a strip, and bringing any two connected disks together we join them reducing their number by one. Let us repeat this procedure until we end up with a single disk for each component of $S$. In the case where we have more than two components, we join the components by tubes. Reducing the size of the first disk and shortening the first tube, and next bringing the two first disks together we join them reducing their number by one. Each such operation creates a hole with strips inside the second disk. Pushing the hole out of the interior of the disk, we obtain a "standard" disk with a large number of strips. Let us keep on repeating the procedure until all the tubes disappear. Then we denote the resulting surface by $S_C$.

*Proof of Theorem 1.1.* We consider a (flat) unit disk $D_0$ on the $(x, y)$-plane. In particular, it is embedded (strictly stable) minimal disk in $\mathbb{R}^3$. By Theorem 2.1, we can construct an embedded strictly stable minimal surface which is isotopic to $S_C$ since $S_C$ can be obtained from $D_0$ by attaching some 1-handles. (We can easily construct a sequence of bridges that shrink nicely to the core of a 1-handle.)

*Proof of Theorem 1.4.* Let $n$ be a sufficiently large integer and $\epsilon_i = \frac{i}{n}$ $(i = 1, \ldots, m)$. Let
$\mathbb{B}^4 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 \leq 1\}$,
$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 = 1\}$,
$R_{\epsilon_i} = \{(x, y, z, w) \in \mathbb{R}^4 | w = \epsilon_i\}$.
Note that $R_{\epsilon_i} \cap \mathbb{B}^4$ is a 3-ball, denoted by $B^3_{\epsilon_i}$. Let $D_{\epsilon_i} = \{(x, y, z, w) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 \leq 1, z = 0, w = \epsilon\} \subset B^3_{\epsilon_i}$. We denote the boundary of the (embedded strictly stable) minimal disk by $S_{\epsilon_i}$. Let $L$ be a ribbon link in $\mathbb{S}^3$. Now, we can construct embedded disks with boundary $L$ by attaching some strips to $S_{\epsilon_i}$'s in $\mathbb{S}^3$. Then, by Theorem 2.1, we obtain the result as in the proof of Theorem 1.1.

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