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GLOBAL EXISTENCE OF STRONG SOLUTIONS FOR A SUPER-CRITICAL WAVE EQUATION

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This text represents the content of the second author's talk given at a RIMS conference on January 15, 2008. The exposé was aiming at describing the result of [3]. We thank the organizers of the conference on "Spectral and Scattering Theory and Related topics" for the kind invitation to present our work on random data nonlinear dispersive equations.

Denote by $\Theta = \{x \in \mathbb{R}^3 : |x| < 1\}$ the unit ball of the Euclidean space $\mathbb{R}^3$. Consider the semi-linear wave equation, posed on Θ,

$$(\partial_t^2 - \Delta)w + |w|^\alpha w = 0, \quad w : \mathbb{R} \times \Theta \to \mathbb{R}, \quad 2 < \alpha < 4, \tag{1}$$

with radial initial data

$$(w, \partial_t w)|_{t=0} = (f_1, f_2) \tag{2}$$

and subject to Dirichlet boundary condition

$$w|_{\mathbb{R}\times\partial\Theta} = 0. \tag{3}$$

The goal is to study the initial boundary value problem (1)-(2)-(3) for data

$$(f_1, f_2) \in H^s(\Theta) \times H^{s-1}(\Theta) \equiv \mathcal{H}^s(\Theta),$$

where $H^s(\Theta)$ denotes the Sobolev space defined as the domain of $(-\Delta_D)^{s/2}$, where Δ_D denotes the Dirichlet self-adjoint realisation of the Laplacian on Θ. In this text the index s will only take values between 0 and 1 and we have that $H^0 = L^2$ while H^1 is the space usually denoted by H_0^1 and defined as the completion of C_0^∞ with respect the H^1 norm (defined in the physical space). In the study of (1)-(2)-(3) a special role is played by the index s_{cr} defined by

$$s_{cr} \equiv \frac{3}{2} - \frac{2}{\alpha}$$

(observe that with our restrictions on α the number s_{cr} is between 1/2 and 1). The index s_{cr} is obtained by a very classical scaling analysis applied to (1). This scaling analysis is usually applied for (1) posed in the Euclidean space but it remains relevant in the case of bounded spatial domain since (one of) the main point in the scaling consideration is the existence of solutions concentrating at a point and this (local) feature remains valid in the case of (1) posed on a bounded domain.

We have the following two facts confirming the importance of s_{cr} in the analysis of (1)-(2)-(3) :

(1) If $1 \geq s \geq s_{cr}$ then the initial boundary value problem (1)-(2)-(3) is locally well-posed for radial initial data in $\mathcal{H}^s(\Theta)$ (the case $s = s_{cr}$ being more delicate).

(2) If $0 \leq s < s_{cr}$ then (1)-(2)-(3) is locally ill-posed for radial initial data in $\mathcal{H}^s(\Theta)$. More precisely the exists a sequence (f_n) with $f_n \in \mathcal{H}^s(\Theta) \cap C^\infty(\bar{\Theta})^2$, f_n radial, there exist $t_n > 0$ with $t_n \to 0$ such that the solutions u_n of (1)-(2)-(3) with data f_n exist on $[0, t_n]$ and

$$\lim_{n\to\infty} \|f_n\|_{\mathcal{H}^s(\Theta)} = 0$$

but

$$\lim_{n\to\infty} \|(u_n(t_n,\cdot), \partial_t u_n(t_n,\cdot)\|_{\mathcal{H}^s(\Theta)} = \infty.$$

The proof of the first fact is based on the Strichartz estimates for radial solutions of the linear wave equation posed on Θ. The proof of the second fact uses the scaling analysis discussed above.

In view of the above two facts we see that we reach the limit of the deterministic well-posedness theory for (1)-(2)-(3) with radial data with a critical threshold the Sobolev space $\mathcal{H}^{s_{cr}}(\Theta)$.

A very natural question is to ask what really happens for data in $\mathcal{H}^s(\Theta)$, $s < s_{cr}$. Our very basic idea is to suitably randomize the initial data, in order get a kind of well-posedness even for $s < s_{cr}$.

Let us denote by $(e_n(r))_{n=1}^\infty$ the orthonormal bases of the L^2 radial functions on Θ, formed by eigenfunctions of Δ satisfying $-\Delta e_n = (\pi n)^2 e_n$ and vanishing on the boundary of Θ. We even have an explicit representation of e_n (which does not play a role in our analysis).

Let us denote by $(h_n(\omega), l_n(\omega))_{n=1}^\infty$ a family of independent Gaussian variables in $\mathcal{N}(0,1)$. Then we set

$$f_1^\omega(r) = \sum_{n=1}^\infty \frac{h_n(\omega)}{\pi n} e_n(r), \quad f_2^\omega(r) = \sum_{n=1}^\infty l_n(\omega) e_n(r).$$

We have that for $s < 1/2$, $(f_1^\omega, f_2^\omega) \in \mathcal{H}^s(\Theta)$ almost surely but

$$p(\omega \, : \, (f_1^\omega, f_2^\omega) \in \mathcal{H}^{1/2}(\Theta)) = 0.$$

Therefore (f_1^ω, f_2^ω) is almost surely a super-critical initial data for (1), i.e. almost surely no existence result for (1) with data (f_1^ω, f_2^ω) is known. We however have the following global existence result for data (f_1^ω, f_2^ω).

Theorem 1. *Suppose that $\alpha < 3$. Let us fix a real number p such that $2\alpha < p < 6$. Then for every $s < 1/2$, almost surely in $\omega \in \Omega$, the problem (1)-(2)-(3) (with data (f_1^ω, f_2^ω)) has a unique global solution*

$$u^\omega \in C(\mathbb{R}, H^s(\Theta)) \cap L^p_{loc}(\mathbb{R}_t; L^p(\Theta)).$$

Moreover :

(1) *The solution is a perturbation of the* linear *solution*

$$u^\omega(t) = \cos(\sqrt{-\Delta}t) f_1^\omega + \frac{\sin\sqrt{-\Delta}}{\sqrt{-\Delta}} f_2^\omega + v^\omega(t),$$

where $v^\omega \in C(\mathbb{R}, H^\sigma(\Theta))$ for some $\sigma > 1/2$.

(2)

$$\|u^\omega(t)\|_{H^s(\Theta)} \leq C(\omega, s) \log(2 + |t|)^{\frac{1}{2}}.$$

Let us next give an important remark. In Theorem 1 one gets almost surely **global** well-posedness. If one is only interested on local well-posedness then a much more general situation may be considered (see [2]). More precisely, we can replace Θ by a general smooth domain of $\mathbb{R}^3$, the sequence $(1/\pi n, 1)_{n=1}^{\infty}$ by a general sequence and $(h_n, l_n)_{n=1}^{\infty}$ by a fairly general sequence of independent zero mean value random variables.

Let us now give the main ideas involved in the proof of Theorem 1. To get local solutions we combine random series estimates with the (deterministic) Strichartz estimates. The key point is that the random oscillations combined with estimates on the eigenfunctions e_n give better estimates than the usual Strichartz estimates (but the Strichartz estimates are still involved in the local theory). To get global solutions, we use invariant measure considerations similarly to the work by Bourgain (see [1]).

Let us next describe the Strichartz estimates, in the context of the linear wave equation, posed on Θ, we use. Denote by $S(t)$, the free wave evolution, namely

$$S(t)(f_1, f_2) \equiv \cos(\sqrt{-\Delta}t) f_1 + \frac{\sin\sqrt{-\Delta}}{\sqrt{-\Delta}} f_2 \, .$$

Then we have the following statement.

Proposition 1 (Strichartz estimate). *Let $4 < p < 6$ and $\sigma = \frac{3}{2} - \frac{4}{p}$. Then for every $T > 0$ there exists $C > 0$ such that for every radial $f \in \mathcal{H}^{\sigma}(\Theta)$,*

$$\|S(t)(f)\|_{L^p([0,T]\times\Theta)} \leq C\|f\|_{\mathcal{H}^{\sigma}(\Theta)} \, . \tag{4}$$

Let us compare the result of Proposition 1 with the Sobolev embedding. The Sobolev embedding yields

$$\|g\|_{L^p(\Theta)} \leq C\|g\|_{H^s(\Theta)}, \quad s = \frac{3}{2} - \frac{3}{p} > \frac{3}{2} - \frac{4}{p} = \sigma.$$

Therefore, the result of Proposition 1 may indeed be seen as an almost sure in time improvement of the Sobolev embedding theorem satisfied by $S(t)(f)$.

Let us next state the almost sure improvement of the Strichartz estimates involved in our argument.

Proposition 2 (An almost sure improvement of the Strichartz estimetes). *Let $4 < p < 6$. Then for every $T > 0$ there exists $C > 0$ such that for every $q \geq p$,*

$$\|S(t)(f_1^{\omega}, f_2^{\omega})\|_{L^q(\Omega; L^p([0,T]\times\Theta))} \leq c\sqrt{q} \, .$$

In particular $\|S(t)(f_1^{\omega}, f_2^{\omega})\|_{L^p([0,T]\times\Theta)} < \infty$ almost surely.

Let us explain why Proposition 2 is indeed an improvement of Proposition 1 almost surely. The regularity of $(f_1^{\omega}, f_2^{\omega})$ is typically $\mathcal{H}^s$, $s \sim 1/2$, i.e. we have that $S(t)(f_1^{\omega}, f_2^{\omega})$ is almost surely in $L^p([0,T] \times \Theta)$ for data of less regularity then the one involved in Proposition 1 (observe that in Proposition 1 σ is always greater than 1/2).

Using Propositions 1, 2, we get the following local existence result.

Proposition 3 (Almost sure local well-posedness). *There exists $\sigma > 1/2$ such that almost surely in ω there exists $T_\omega > 0$ and a unique solution of (1) with data (f_1^ω, f_2^ω) in*

$$S(t)(f_1^\omega, f_2^\omega) + C([0, T_\omega]; \mathcal{H}^\sigma(\Theta)).$$

More precisely there exists $C > 0$ and $\alpha > 0$ such that for every $\delta > 0$ there exist $\Omega_\delta \subset \Omega$, $p(\Omega_\delta) \geq 1 - \delta$ such that $\forall \omega \in \Omega_\delta$,

$$T_\omega = \frac{C}{(\log(1/\delta))^\alpha}.$$

Let us finally roughly explain the idea of the globalisation argument involved in the proof of Theorem 1. For the rigorous analysis we refer to [3]. The map $\omega \longmapsto (f_1^\omega, f_2^\omega)$ defines a measure on $\mathcal{H}^s$, $s < 1/2$. We denote this measure by μ. We control the nonlinear evolution with respect to μ thanks to Gibbs measure consideration, the Gibbs measure being absolutely continuous with respect to μ. Let us fix a time $T > 0$, $\delta > 0$ and a set Ω_δ provided by Proposition 3. To arrive at time T, we need to iterate $\sim (\log(1/\delta))^\alpha T$ times the local existence result of Proposition 3. The **key step** is that thanks to measure invariance arguments at each step we eliminate a set of Ω of probability only about $\sim \delta$. Therefore to arrive at time T the eliminated set is of probability $\sim T\delta(\log(1/\delta))^\alpha$ which tends to zero as $\delta \to 0$. Let us denote this set by $\tilde{\Omega}_{\delta,T}$. We then define Σ_T as the union for $\delta = 2^{-j}$, $j = 1, 2, 3, \cdots$ of $\tilde{\Omega}_{\delta,T}$. Then $p(\Sigma_T) = 1$ and for $\omega \in \Sigma_T$ the solutions exist up to time T. Finally we define Σ as the intersection of Σ_T for $T = 2^j$, $j = 1, 2, 3, \cdots$. Then $p(\Sigma) = 1$ and for $\omega \in \Sigma$, we have global existence.

References

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