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The Dicke-Type Transition for Non-Commutative Harmonic Oscillators in the Light of Cavity QED

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1 Introduction

In this paper we explore the possibility whether we can find superradiance [2, 28] in the spectrum of the differential operators describing a non-commutative harmonic oscillator [19, 20, 25, 26, 27], where the superradiance is a physical phenomenon. Before illustrating the spectral property arising in the superradiance, let us explain the differential operator that we consider in this paper. The differential operator as a free Hamiltonian is given in the following: Let a non-commutative quadratic form be defined by

$$Q(p,q)=A_{11}p^2+A_{12}pq+A_{21}qp+A_{22}q^2 \tag{1.1}$$

for variables p, q into which some operators are inserted, where $A_{k\ell}\in M_2(\mathbb{C})$, the set of all 2×2 matrices with complex coefficients $(k,\ell=1,2)$. By the canonical quantization we mean replacement of p and q by the differential operator $-id/dx$ and the multiplication operator $x\times$ respectively in Eq.(1.1). Thus, a canonical quantization $Q(-id/dx, x\times)$ is a kind of matrix-valued Schrödinger operator [1, 7, 10] as $A_{11}\neq 0$. The differential operator $Q(-id/dx, x\times)$ is the *free part* of the differential operator describing non-commutative harmonic oscillator [19, 20, 25, 26, 27] setting $A_{k\ell}$ $(k,\ell=1,2)$ as:

$$A_{11}=A_{22}=\frac{\alpha+\beta}{4}\sigma_0+\frac{\alpha-\beta}{4}\sigma_3 \tag{1.2}$$

and $A_{12}=A_{21}=0$. We denote the Pauli matrices by

$$\sigma_0:=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma_1:=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2:=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

throughout this paper, where $i := \sqrt{-1}$.

As one of spectral properties caused by superradiance, we are interested in the following phenomena: Let us consider a differential operator H_0 which has only discrete spectrum (i.e., eigenvalues). We denote them by $E_0(0) < E_1(0) < \cdots < E_n(0) < \cdots$, $n \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Introduce by $\lambda \in \mathbb{R}$ the strength of the coupling operator (or an interaction operator). Then, λ is called the *coupling constant*. When $|\lambda|$ is so small that the regular perturbation theory is available for an interaction operator λV, each eigenvalue $E_n(\lambda)$ of $H(\lambda) := H_0 + \lambda V$ sits near its original position $E_n(0)$ [35, Theorem XII.13], so that $E_0(\lambda) < E_1(\lambda) < \cdots < E_n(\lambda) < \cdots$. On the other hand, as in [23] some crossings among eigenvalues take place for a model of the non-commutative harmonic oscillator. Moreover, the phase transition of the Dicke superradiance tells us about a possibility that $E_1(\lambda)$ is less than $E_0(\lambda)$. Then, it may have a chance to become a new ground state energy for some strong $|\lambda|$. As $|\lambda|$ becomes much stronger, even $E_n(\lambda)$ may be less than $E_0(\lambda)$ because of superradiant energy loss [2]. The *Dicke-type crossing* is how we describe this phenomenon among eigenvalues in this paper. Moreover, $E_n(\lambda)$ may usurp the position of the ground state energy. We call such a new ground state energy the *superradiant ground state energy*. We will give differential operators $H(\lambda)$ of a non-commutative harmonic oscillator with the interaction in a class, which have such the Dicke-type crossing. Also we will investigate dynamics of the eigenvalues of $H(\lambda(t))$ by controlling the strength of $|\lambda(t)|$ with a parameter $t \geq 0$. We will explain the physical background of our mathematical set-ups in §3.

Here we introduce some notations that we will use in this paper. The sets $\{0\} \cup \mathbb{N}$ and $[0\,,\,\infty)$ are respectively denoted by $\mathbb{Z}_+$ and $\mathbb{R}_+$. For a Banach space $\mathcal{H}$, we denote its norm by $\|\ \|_{\mathcal{H}}$. In the case where $\mathcal{H}$ is a Hilbert space, we denote its inner product by $(\ ,\ )_{\mathcal{H}}$, and the norm induced by the inner product is also expressed by $\|\ \|_{\mathcal{H}}$. We use the symbol Id for the identity operator on $\mathcal{H}$. The symbol $D(A)$ stands for the domain of an operator A. For a closable operator A we denote by $\overline{A}$ its closure. For a closed operator, $\sigma(A)$ (or $\rho(A)$) is the spectrum (or resolvent set). For a self-adjoint operator A, in particular, we mean the essential spectrum of A by $\sigma_{\mathsf{ess}}(A)$. So, the discrete spectrum $\sigma_{\mathsf{dis}}(A)$ is given by $\sigma_{\mathsf{dis}}(A) := \sigma(A) \setminus \sigma_{\mathsf{ess}}(A)$. In addition, we call $\inf \sigma(A)$ the *ground state energy* of A (i.e., the infimum of $\sigma(A)$) if $\inf \sigma(A) > -\infty$. By an *excited state energy* of A we mean an eigenvalue of A which is more than the ground state energy.

2 Mathematical Results

In this section some results are stated without their proofs. All the proofs are in Ref.[16].

Let $\omega > 0$ be an arbitrary constant, which is the angular frequency. Then, the Hamiltonian $h_{\rm os}^{\omega}$ of quantum harmonic oscillator with ω is defined by the following 1-dimensional Schrödinger operator:

$$h_{\rm os}^{\omega} := -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega^2}{2}x^2.$$

We define a differential operator H_0 acting in $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ by

$$H_0 := \overline{\left(\frac{\alpha+\beta}{2}\sigma_0 + \frac{\alpha-\beta}{2}\sigma_3\right) \otimes h_{\rm os}^{\omega}} \tag{2.1}$$

for $\alpha, \beta > 0$. Then, H_0 is a self-adjoint operator acting in $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ since $\{2^{-1}\alpha(\sigma_0+\sigma_3) + 2^{-1}\beta(\sigma_0-\sigma_3)\} \otimes h_{\rm os}^{\omega}$ is essentially self-adjoint on the domain $\mathbb{C}^2 \otimes \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$. This H_0 is the differential operator used by Ichinose, Parmeggiani, and Wakayama [19, 20, 25, 26, 27] picking ω so that $\omega = 1$. The Hamiltonian H_0 is the free part of the differential operator describing the non-commutative harmonic oscillator.

We assume that

$$\beta \geq \alpha \quad \text{or} \quad \alpha \geq 3\beta \tag{2.2}$$

throughout this paper.

The spectrum $\sigma(H_0)$ of H_0 consists of only isolated eigenvalues of H_0 with each multiplicity less than 2:

$$\sigma(H_0) = \left\{ \left(n+\frac{1}{2}\right)\alpha\omega,\ \left(n+\frac{1}{2}\right)\beta\omega \,\middle|\, n \in \mathbb{Z}_+ \right\}.$$

We call n a *quantum number.*

We consider the following type of interaction: Let $V_\infty(t)$ and $W(t)$ be differential operators acting in $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ for every $t \in \mathbb{R}_+$, and let $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a continuous function with $\lim_{t\to\infty}\gamma(t) = 0$. Then, our interaction operator $V(t)$ is given by

$$V(t) := V_\infty(t) + \gamma(t)W(t)$$

for every $t \in \mathbb{R}_+$. We call $V_\infty(t)$ and $\gamma(t)W(t)$ the *source potential* and *error potential* for the Dicke-type crossing, respectively.

Let $\lambda : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a continuous function satisfying that $\lambda(0) = 0$ and that $|\lambda(t)|$ is strictly increasing in t as $t \to \infty$, i.e., $|\lambda(s)| < |\lambda(t)|$ if $s < t$. Then, the differential operator $H_{\lambda,\gamma}(t)$ that we consider in this paper is given by

$$H_{\lambda,\gamma}(t) := H_0 + \lambda(t)V(t)$$

for every $t \in \mathbb{R}_+$.

Definition 2.1 We say an eigenvalue $E_{\mathrm{des}}(t)$ of $H_{\lambda,\gamma}(t)$ is a *descendant of the ground state energy of* H_0 if $E_{\mathrm{des}}(t)$ is a continuous function of $t \in \mathbb{R}_+$ so that $E_{\mathrm{des}}(0) = \inf \sigma(H_0)$.

Definition 2.2 Let us assume all eigenvalues of $H_{\lambda,\gamma}(t)$ are continuous functions of $t \in \mathbb{R}_+$. Then, we say the family $\{H_{\lambda,\gamma}(t)\}_{t\in\mathbb{R}_+}$ of the differential operators has *the Dicke-type crossing* if there exists an eigenvalue $E_{\lambda,\gamma}(t)$ of $H_{\lambda,\gamma}(t)$ for every $t \in \mathbb{R}_+$ such that the following (DC1) and (DC2) hold:

(DC1) $E_{\lambda,\gamma}(0)$ is an excited state energy of H_0;

(DC2) there exists a certain $t_* > 0$ such that $E_{\lambda,\gamma}(t_*) < E_{\mathrm{des}}(t_*)$ for all descendants $E_{\mathrm{des}}(t)$ of the ground state energy of H_0.

In addition, we call such an $E_{\lambda,\gamma}(t_*)$ a *superradiant ground state energy* if $E_{\lambda,\gamma}(t_*) = \inf \sigma(H_{\lambda,\gamma}(t_*))$.

Definition 2.3 Let $E_n(t)$ be an eigenvalue of $H_{\lambda,\gamma}(t)$ for each $n \in \mathbb{Z}_+$ and every $t \in \mathbb{R}_+$ so that $E_n(t)$ is a continuous function of $t \in \mathbb{R}_+$ and

$$E_n(0) = \left(n + \frac{1}{2}\right)\alpha\omega \text{ or } \left(n + \frac{1}{2}\right)\beta\omega.$$

We say the family $\{E_n(t)\}_{n\in\mathbb{Z}_+,t\in\mathbb{R}_+}$ of eigenvalues is *asymptotically harmonic with* $\omega_\infty > 0$ *for sufficiently large quantum numbers* if there is a sequence $t_n > 0$ for each $n \in \mathbb{Z}_+$ so that $t_n \longrightarrow \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{E_n(t)}{n} = \omega_\infty \quad \text{for} \quad t \in [t_n\,,\,\infty)\,.$$

In particular, we just say the family $\{E_n(t)\}_{n\in\mathbb{Z}_+,t\in\mathbb{R}_+}$ of eigenvalues is *harmonic with* $\omega_\infty > 0$ *for sufficiently large quantum numbers* if there is a subset $\mathcal{T} \subset \mathbb{R}_+$ so that $\mathcal{T}$ is independent of any $n \in \mathbb{Z}_+$, and

$$\lim_{n\to\infty} \frac{E_n(t)}{n} = \omega_\infty \quad \text{for} \quad t \in \mathcal{T}.$$

Then, we call ω_∞ the *angular frequency of* $\{E_n(t)\}_{n\in\mathbb{Z}_+, t\in\mathbb{R}_+}$ if

$$\lim_{n\to\infty}\Big\{E_{n+1}(t) - E_n(t)\Big\} = \omega_\infty \quad \text{for} \quad t\in\mathcal{T}.$$

On the other hand, we say the family $\{E_n(t)\}_{n\in\mathbb{Z}_+, t\in\mathbb{R}_+}$ of eigenvalues is *anharmonic at* $t = t_0$ if eigenvalues $E_n(t_0)$ get dense or sparse in the following sense, respectively:

$$\lim_{n\to\infty}\{E_{n+1}(t_0) - E_n(t_0)\} = 0 \quad \text{or} \quad \pm\infty.$$

Although Ichinose, Parmeggiani, and Wakayama have studied the interaction operator

$$V_{\mathrm{IPW}} = -i\sigma_2 \otimes \left(x\frac{d}{dx} + \frac{1}{2}\right), \tag{2.3}$$

we give a differential operator $V_\infty(t)$ as a source potential in the following. For given functions $c_\ell^k : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $k, \ell = 0, 1$, we define $V_\infty(t)$ by

$$V_\infty(t) := \sum_{k,\ell=0,1} i^k c_\ell^k(t)\sigma_{\tau(\ell+1)} \otimes x^{1-k}\left(\frac{d}{dx}\right)^k \tag{2.4}$$

for every $t \in \mathbb{R}_+$, where τ is the permutation of $\{1, 2\}$ defined as $\tau(1) = 2$ and $\tau(2) = 1$.

For simplicity we denote $H_{\lambda,0}(t)$ (i.e., $H_{\lambda,\gamma}(t)$ with $\gamma(t) \equiv 0$) by $H_\lambda(t)$:

$$H_\lambda(t) := H_{\lambda,0}(t) = H_0 + \lambda(t)V_\infty(t)$$

for every $t \in \mathbb{R}_+$. We employ the following $c_\ell^k(t)$, $k, \ell = 0, 1$ given by

$$\begin{pmatrix} c_0^0(t) + ic_0^1(t) \\ \\ c_1^0(t) + ic_1^1(t) \end{pmatrix} := \begin{pmatrix} \cos t & -\sin t \\ \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} i\sqrt{1/2\omega} \\ \\ \sqrt{\omega/2} \end{pmatrix} \tag{2.5}$$

for every $t \in \mathbb{R}_+$. Namely,

$$c_0^0(t) = -\sqrt{\frac{\omega}{2}}\sin t, \qquad c_0^1(t) = \sqrt{\frac{1}{2\omega}}\cos t,$$
$$c_1^0(t) = \sqrt{\frac{\omega}{2}}\cos t, \qquad c_1^1(t) = \sqrt{\frac{1}{2\omega}}\sin t.$$

Thus, $H_\lambda(t)$ is the canonical quantization of $Q(p,q)$ with A_{11} as in Eq.(1.2), $A_{22} = \omega^2 A_{11}$, $A_{12} = A_{21} = A_{33} = 0$,

$$A_{13} + A_{31} = -\sqrt{\frac{1}{2\omega}}\Big\{\sin t\,\sigma_1 + \cos t\,\sigma_2\Big\}\lambda(t),$$

and

$$A_{23} + A_{32} = \sqrt{\frac{\omega}{2}}\Big\{\cos t\,\sigma_1 - \sin t\,\sigma_2\Big\}\lambda(t).$$

Under Eqs.(2.4) and (2.5), we have the following lemma for $W(t)$:

Lemma 2.4 *If there is a constant $b_1 > 0$ independent of $t \in \mathbb{R}_+$ such that*

$$\|W(t)\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} \le b_1\|H_0\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} + b_2(t)\|\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})}$$

for all $\Psi \in D(H_0)$, all $t \in \mathbb{R}_+$, and some $b_2(t) > 0$, then for arbitrary $\varepsilon > 0$ and $t \in \mathbb{R}_+$ there exists a certain $C_t(\varepsilon) > 0$ such that

$$\|W(t)\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} \le b_1(1+\varepsilon)\|H_\lambda(t)\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} + (b_1 C_t(\varepsilon) + b_2(t))\|\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})}$$

for all $\Psi \in D(H_\lambda(t))$.

In the above lemma, it is important that we can take $b_1(1+\varepsilon)$ independently of t to obtain Theorem 2.5 below.

Example 2.1 *As an example of $W(t)$, let us define $W_\pm(t)$ by*

$$W_\pm(t) := \sum_{k,\ell=0,1} (\pm 1)^{\ell+1}(\mp i)^k c_\ell^k(t)\sigma_{\tau(\ell+1)} \otimes x^{1-k}\left(\frac{d}{dk}\right)^k,$$

where $c_\ell^k(t)$, $k,\ell = 0,1$, as in Eq.(2.5), and

$$(\pm 1)^{\ell+1}(\mp i)^k := \begin{cases}(-1)^{\ell+1}(+i)^k,\\ (+1)^{\ell+1}(-i)^k.\end{cases}$$

Then, for arbitrary δ with $0 < \delta < 1$ and every $\Psi \in D(H_0)$

$$\|W_\pm(t)\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} \le \delta\|H_0\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} + \frac{1}{2}\left(\frac{1}{2\min\{\alpha,\beta\}\omega\delta} + 1\right)\|\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})}.$$

Our statement on the asymptotic harmonicity and the Dicke-type crossing is the following:

Theorem 2.5 *Assume the condition (2.2). Let* $V_\infty(t)$ *be as in Eq.(2.4). Set* $c_\ell^k(t)$, $k, \ell = 0, 1$, *as in Eq.(2.5). Assume that the error potential* $\gamma(t)W(t)$ *satisfies (A1)–(A3):*

(A1) $W(t)$ *is a symmetric operator acting in* $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ *with* $D(W(t)) \supset D(H_0)$ *for every* $t \in \mathbb{R}_+$*;*

(A2) there is a constant $b_1 > 0$ *such that*

$$\|W(t)\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} \leq b_1\|H_0\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})} + b_2(t)\|\Psi\|_{\mathbb{C}^2\otimes L^2(\mathbb{R})}$$

for all $\Psi \in D(H_0)$, *all* $t \in \mathbb{R}_+$, *and some* $b_2(t) > 0$*;*

(A3) $\sup_{t\in\mathbb{R}_+} |\lambda(t)\gamma(t)| < b_1^{-1}$.

Then, the following (i) – (v) hold:

(i) *The differential operator* $H_{\lambda,\gamma}(t)$ *is self-adjoint on* $D(H_0)$ *and bonded from below for every* $t \in \mathbb{R}_+$.

(ii) $\sigma(H_{\lambda,\gamma}(t)) = \sigma_{\text{dis}}(H_{\lambda,\gamma}(t))$. *Namely, the spectrum* $\sigma(H_{\lambda,\gamma}(t))$ *of* $H_{\lambda,\gamma}(t)$ *consists entirely of isolated eigenvalues* $E_n(t)$, $n \in \mathbb{Z}_+$, *with finite multiplicities and their corresponding eigenfunctions,* $\Phi_n(t)$, $n \in \mathbb{Z}_+$, *make a complete orthonormal system of* $\mathbb{C}^2 \otimes L^2(\mathbb{R})$. *Each* $E_n(t)$ *sits near an eigenvalues of* $H_\lambda(t)$.

(iii) $\lim_{n\to\infty} E_n(t) = \infty$ *for every* $t \in \mathbb{R}_+$.

In addition, assume there is a constant $T > 0$ *so that* $|\lambda(t)\gamma(t)|$ *is decreasing in* $t > T$, *and* $\lim_{t\to\infty} |\lambda(t)\gamma(t)| = 0$. *Then, the following (iv) holds:*

(iv) *Some of* $E_n(t)$*s,* $n \in \mathbb{Z}_+$, *make a family of eigenvalues being asymptotically harmonic with* $\alpha\omega$ *or* $\beta\omega$ *for sufficiently large quantum numbers.*

Moreover, assume $\lim_{t\to\infty} |\lambda(t)|^3|\gamma(t)| = 0$ *and* $\sup_{t\in\mathbb{R}_+} |b_2(t)| < \infty$. *Then, the following (v) holds:*

(v) *The family* $\{H_{\lambda,\gamma}(t)\}_{t\in\mathbb{R}_+}$ *of the differential operators has the Dicke-type crossing. Moreover, there is a superradiant ground state energy sitting near that of* $H_\lambda(t)$.

Remark 2.1 As the source potential $V_\infty(t)$ is $V_{\rm IPW}$ given by Eq.(2.3), crossings among eigenvalues take place for $H_\lambda(t) = H_0 + \lambda(t)V_{\rm IPW}$ [23]. As for this $H_\lambda(t)$, (A3) is satisfied since $\gamma(t) \equiv 0$. In Theorem 2.5 we also assumed (A3) to show the Dicke-type crossing for the source potential $V_\infty(t)$ given by Eq.(2.4). However, we can give an example of crossings among eigenvalues of $H_{\lambda,\gamma}(t)$ even if (A3) is not satisfied: We now consider the canonical quantization of $Q(p,q)$ with

$$A_{11} = \frac{\alpha+\beta}{4}\sigma_0 - \frac{\lambda(t)}{2\omega}\sigma_1 + \frac{\alpha-\beta}{4}\sigma_3,$$
$$A_{22} = \frac{\alpha+\beta}{4}\omega^2\sigma_0 + \frac{\omega\lambda(t)}{2}\sigma_1 + \frac{\alpha-\beta}{4}\omega^2\sigma_3,$$

and $A_{12} = A_{21} = \frac{1}{2}\sigma_2$. *Assume $\alpha \geq \beta$ and $\gamma(t) \equiv 1$. Let $V_\infty(t)$ and $W(t)$ be respectively given by*

$$V_\infty(t) := V_{\rm IPW} = -i\sigma_2 \otimes \left(x\frac{d}{dx} + \frac{1}{2}\right),$$
$$W(t) := W \equiv \frac{1}{\omega}\sigma_1 \otimes \left(\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega^2}{2}x^2\right).$$

Then, $H_{\lambda,1}(t) = H_0 + \lambda(t)\left(V_{\rm IPW} + W\right)$ has eigenvalues:

$$\widetilde{E}_n^\pm(t) := \left(n-\frac{3}{2}\right)\beta\omega + \frac{1}{2}\left\{(\alpha-\beta)n + \frac{\alpha+3\beta}{2}\right\}\omega$$
$$\pm\frac{1}{2}\sqrt{\left\{(\alpha-\beta)n + \frac{\alpha+3\beta}{2}\right\}^2\omega^2 + 4\frac{n!}{(n-2)!}\lambda(t)^2}$$

for each $n \in \mathbb{Z}_+$ with $n \geq 2$. Therefore, in the same way we will show crossings in Proposition 2.7, we can show that *$H_{\lambda,1}(t)$ with the source potential $V_{\rm IPW}$ has crossings among $\widetilde{E}_n^\pm(t)$ as $t \to \infty$.* It is easy to check the following:

(i) *For every $t \in \mathbb{R}_+$ satisfying $|\lambda(t)| \neq \sqrt{\alpha\beta}\,\omega$, the family $\{\widetilde{E}_n^\pm(t)\}_{n\in\mathbb{Z}_+, t\in\mathbb{R}_+}$ is harmonic for sufficiently large quantum numbers. The angular frequency is*

$$\lim_{n\to\infty}\frac{\widetilde{E}_n^\pm(t)}{n} = \lim_{n\to\infty}\left\{\widetilde{E}_{n+1}^\pm(t) - \widetilde{E}_n^\pm(t)\right\} = \frac{\alpha+\beta}{2}\omega \pm \frac{1}{2}\sqrt{(\alpha-\beta)^2\omega^2 + 4\lambda(t)^2}.$$

(ii) *If there is a number $t_{\alpha,\beta} \in \mathbb{R}_+$ so that $|\lambda(t_{\alpha,\beta})| = \sqrt{\alpha\beta}\,\omega$, then family $\{\widetilde{E}_n^-(t)\}_{n\in\mathbb{Z}_+, t\in\mathbb{R}_+}$ is anharmonic at $t = t_{\alpha,\beta}$.*

Because our main theorem says that each eigenvalue of $H_{\lambda,\gamma}(t)$ sits near an eigenvalue of $H_\lambda(t)$ under our assumptions, the Dicke-type crossing for $H_{\lambda,\gamma}(t)$ is determined by that of $H_\lambda(t)$. Therefore, it is important to understand the behavior of eigenvalues of $H_\lambda(t)$.

Proposition 2.6 *Let $V_\infty(t)$ be as in Eq.(2.4). Set $c_\ell^k(t)$, $k,\ell = 0,1$, as in Eq.(2.5). Then, $H_\lambda(t)$ is self-adjoint on $D(H_0)$ and $\sigma_{\mathrm{ess}}(H_\lambda(t)) = \sigma_{\mathrm{ess}}(H_0)$ for every $t \in \mathbb{R}_+$.*

Let $E_n^-(\lambda;\alpha,\beta)$ be a real number for every $\lambda \in \mathbb{R}$ and each $n \in \mathbb{Z}_+$, defined by

$$E_n^-(\lambda;\alpha,\beta) := \begin{cases} \dfrac{(\alpha+\beta)}{2}\left(n+\dfrac{1}{2}\right)\omega + \dfrac{\beta\omega}{2} - \dfrac{1}{2}\Omega_n^-(\lambda;\alpha,\beta) & \text{if } \beta \geq \alpha, \\[2ex] \dfrac{(\alpha+\beta)}{2}\left(n+\dfrac{1}{2}\right)\omega - \dfrac{\alpha\omega}{2} - \dfrac{1}{2}\Omega_n^-(\lambda;\alpha,\beta) & \text{if } 3\beta \leq \alpha, \end{cases} \tag{2.6}$$

where

$$\Omega_n^-(\lambda;\alpha,\beta) = \sqrt{\left\{(\beta-\alpha)\left(n+\frac{1}{2}\right)+\beta\right\}^2\omega^2 + 4(n+1)\lambda^2}$$

if $\beta \geq \alpha$, and

$$\Omega_n^-(\lambda;\alpha,\beta) = \begin{cases} -\dfrac{\alpha+\beta}{2}\omega & \text{for } n = 0, \\[2ex] \sqrt{\left\{(\alpha-\beta)(n-1)+\dfrac{\alpha-3\beta}{2}\right\}^2\omega^2 + 4n\lambda^2} & \text{for } n \in \mathbb{N} \end{cases}$$

if $3\beta \leq \alpha$. Similarly $E_n^+(\lambda;\alpha,\beta)$ be a real number for every $\lambda \in \mathbb{R}$ and each $n \in \mathbb{Z}_+$, defined by

$$E_n^+(\lambda;\alpha,\beta) := \begin{cases} \dfrac{(\alpha+\beta)}{2}\left(n+\dfrac{1}{2}\right)\omega - \dfrac{\alpha\omega}{2} + \dfrac{1}{2}\Omega_n^+(\lambda;\alpha,\beta) & \text{if } \beta \geq \alpha, \\[2ex] \dfrac{(\alpha+\beta)}{2}\left(n+\dfrac{1}{2}\right)\omega + \dfrac{\beta\omega}{2} + \dfrac{1}{2}\Omega_n^+(\lambda;\alpha,\beta) & \text{if } 3\beta \leq \alpha, \end{cases} \tag{2.7}$$

where

$$\Omega_n^{+}(\lambda;\alpha,\beta) = \sqrt{\left\{(\beta-\alpha)\left(n+\frac{1}{2}\right)+\alpha\right\}^2 \omega^2 + 4n\lambda^2}$$

if $\beta \geq \alpha$, and

$$\Omega_n^{+}(\lambda;\alpha,\beta) = \sqrt{\left\{(\alpha-\beta)n+\frac{\alpha-3\beta}{2}\right\}^2 \omega^2 + 4(n+1)\lambda^2}$$

if $3\beta \leq \alpha$.

We note that

$$E_n^{-}(0;\alpha,\beta) = \begin{cases} \left(n+\dfrac{1}{2}\right)\alpha\omega & \text{if } \beta \geq \alpha, \\ \\ \left(n+\dfrac{1}{2}\right)\beta\omega & \text{if } 3\beta \leq \alpha, \end{cases} \qquad \text{for each } n \in \mathbb{Z}_+, \tag{2.8}$$

and

$$E_n^{+}(0;\alpha,\beta) = \begin{cases} \left(n+\dfrac{1}{2}\right)\beta\omega & \text{if } \beta \geq \alpha, \\ \\ \left(n+\dfrac{1}{2}\right)\alpha\omega & \text{if } 3\beta \leq \alpha, \end{cases} \qquad \text{for each } n \in \mathbb{Z}_+, \tag{2.9}$$

in the case where $\lambda = 0$.

Set $D_m(\alpha,\beta;\theta)$ for each $m \in \mathbb{Z}_+$ and every $\theta > 0$ as

$$D_m(\alpha,\beta;\theta) := \frac{(\beta-\alpha)(m+1/2)+\beta}{\alpha+\beta+\theta/\omega}. \tag{2.10}$$

For $H_\lambda(t)$ we can exactly understand the behavior of the Dicke-type crossing as in the following proposition:

Proposition 2.7 *Assume the condition (2.2). Let $V_\infty(t)$ be as in Eq.(2.4). Set $c_\ell^k(t)$, $k, \ell = 0, 1$, as in Eq.(2.5). Then, the following (i)–(iv) hold:*

(i) *For every $t \in \mathbb{R}_+$ the spectrum $\sigma(H_\lambda(t))$ of $H_\lambda(t)$ consists of only eigenvalues of $H_\lambda(t)$:*

$$\sigma(H_\lambda(t)) = \left\{E_n^{-}(\lambda(t);\alpha,\beta),\ E_n^{+}(\lambda(t);\alpha,\beta)\,\middle|\, n \in \mathbb{Z}_+\right\}.$$

(ii) *A non-trivial crossing take place in the spectrum $\sigma(H_\lambda(t))$: Pick $\theta > 0$ arbitrarily. If $m, n \in \mathbb{Z}_+$ and t, α, β satisfy $m < n$, $\alpha \leq \beta$, and*

$$\left(\frac{\alpha+\beta}{2}\omega + \theta\right)\sqrt{m+n+2+2\sqrt{mn+m+n+1+D_m(\alpha,\beta;\theta)^2}} < |\lambda(t)|,$$

then

$$E_m^-(\lambda(t);\alpha,\beta) > E_n^-(\lambda(t);\alpha,\beta) + (n-m)\theta.$$

(iii) *The family $\{H_\lambda(t)\}_{t\in\mathbb{R}_+}$ of the differential operators has the Dicke-type crossing, and there is a superradiant ground state energy.*

(iv) *Two families $\{E_n^-(\lambda(t);\alpha,\beta)\}_{n\in\mathbb{Z}_+,t\in\mathbb{R}_+}$ and $\{E_n^+(\lambda(t);\alpha,\beta)\}_{n\in\mathbb{Z}_+,t\in\mathbb{R}_+}$ are harmonic for sufficiently large quantum numbers. The angular frequency of both families is $\alpha\omega$ or $\beta\omega$:*

$$\lim_{n\to\infty}\frac{E_n^-(\lambda;\alpha,\beta)}{n} = \lim_{n\to\infty}\left\{E_{n+1}^-(\lambda;\alpha,\beta) - E_n^-(\lambda;\alpha,\beta)\right\} = \begin{cases}\alpha\omega & \text{if } \beta\geq\alpha,\\ \beta\omega & \text{if } 3\beta\leq\alpha,\end{cases}$$

$$\lim_{n\to\infty}\frac{E_n^+(\lambda;\alpha,\beta)}{n} = \lim_{n\to\infty}\left\{E_{n+1}^+(\lambda;\alpha,\beta) - E_n^+(\lambda;\alpha,\beta)\right\} = \begin{cases}\beta\omega & \text{if } \beta\geq\alpha,\\ \alpha\omega & \text{if } 3\beta\leq\alpha.\end{cases}$$

Remark 2.2 Eq.(2.8) says that

$$E_m^-(\lambda(0);\alpha,\beta) \equiv E_m^-(0;\alpha,\beta) < E_n^-(0;\alpha,\beta) \equiv E_n^-(\lambda(0);\alpha,\beta)$$

if $m < n$. Thus, Proposition 2.7(ii) means a crossing between $E_m^-(\lambda(t);\alpha,\beta)$ and $E_n^-(\lambda(t);\alpha,\beta)$.

We will see the behavior of the crossings in detail in the case where $\alpha \leq \beta$. In addition, we will find which $E_n^-(\lambda;\alpha,\beta)$ becomes a superradiant ground state energy in this case.

For $m, n \in \mathbb{Z}_+$ with $m < n$, set λ_1, λ_2 as

$$\lambda_1 \equiv \lambda_1(m,n) := \left\{\alpha\left(n+\frac{1}{2}\right) - \beta\left(m+\frac{1}{2}\right)\right\}\frac{\omega}{\sqrt{m}+\sqrt{n+1}} \tag{2.11}$$

and

$$\lambda_2 \equiv \lambda_2(m,n) := \sqrt{\frac{(\alpha+\beta)^2}{4}(n+1-m)^2 - \alpha^2}\;\frac{\omega}{2\sqrt{m}}. \tag{2.12}$$

Assume $0 < \alpha < \beta$. Then, we can define $x_{\alpha,\beta}(\lambda) \in \mathbb{R}$ for every $\lambda \in \mathbb{R}$ by

$$x_{\alpha,\beta}(\lambda) := \frac{\lambda(\alpha+\beta)\sqrt{4\lambda^2+2(\beta^2-\alpha^2)\omega^2} - \sqrt{\alpha\beta}\left\{4\lambda^2+(\beta^2-\alpha^2)\omega^2\right\}}{2\sqrt{\alpha\beta}(\beta-\alpha)^2\omega^2}.$$

Lemma 2.8 *If $\alpha, \beta \in \mathbb{R}$ and $m, n \in \mathbb{Z}_+$ satisfy $0 < \alpha \leq \beta$, $m < n$, and*

$$\frac{1}{2}(\beta - \alpha) < \alpha n - \beta m, \tag{2.13}$$

then $0 < \lambda_1 < \lambda_2$.

Lemma 2.9 *If α, β and ℓ satisfy $0 < \alpha < \beta$ and*

$$\sqrt{\sqrt{\alpha\beta}(\beta - \alpha)\left\{(\alpha + \beta) + 2\ell(\beta - \alpha)\right\}}\,\frac{\omega}{\sqrt{2}(\sqrt{\beta} - \sqrt{\alpha})} < |\lambda|,$$

then $\ell < x_{\alpha,\beta}(\lambda)$.

By Lemma 2.9 we can respectively define numbers $n_1(\alpha, \beta), n_2(\alpha, \beta) \in \mathbb{Z}_+$ by

$$n_1(\alpha, \beta) := \max\left\{n \in \mathbb{Z}_+ \,|\, n + 1 < x_{\alpha,\beta}(\lambda)\right\},$$
$$n_2(\alpha, \beta) := \min\left\{n \in \mathbb{Z}_+ \,|\, x_{\alpha,\beta}(\lambda) \leq n + 1\right\},$$

and then, $0 \leq n_1(\alpha, \beta) < n_2(\alpha, \beta)$ if

$$\sqrt{\sqrt{\alpha\beta}(\beta - \alpha)\left\{3\beta - \alpha\right\}}\,\frac{\omega}{\sqrt{2}(\sqrt{\beta} - \sqrt{\alpha})} < \lambda.$$

When $\alpha \leq \beta$, trivial crossings take place and a superradiant ground state energy appears in the following:

Proposition 2.10 *Let $V_\infty(t)$ be as in Eq.(2.4). Set $c_\ell^k(t)$, $k, \ell = 0, 1$, as in Eq.(2.5). Assume $\alpha \leq \beta$. Then, the following (i) and (ii) hold:*

(i) *A trivial crossing between $E_m^+(\lambda(t); \alpha, \beta)$ and $E_n^-(\lambda(t); \alpha, \beta)$ occurs in the following. Let m and n be in $\mathbb{Z}_+$ with $m < n$. Then, there exists a number $\lambda_0 \equiv \lambda_0(m, n)$ satisfying $0 < \lambda_1 < \lambda_0 < \lambda_2$ such that*

$$\begin{aligned} &E_m^+(\lambda(t); \alpha, \beta) < E_n^-(\lambda(t); \alpha, \beta) \quad \text{if } |\lambda(t)| < \lambda_0, \\ &E_m^+(\lambda_0; \alpha, \beta) = E_n^-(\lambda_0; \alpha, \beta), \\ &E_m^+(\lambda(t); \alpha, \beta) > E_n^-(\lambda(t); \alpha, \beta) \quad \text{if } |\lambda(t)| > \lambda_0. \end{aligned}$$

(ii) *A superradiant ground state energy appears in the following. Let $\ell \in \mathbb{N}$. If α, β, t, and ℓ satisfy $0 < \alpha < \beta$ and*

$$\sqrt{\sqrt{\alpha\beta}(\beta - \alpha)\left\{(\alpha + \beta) + 2\ell(\beta - \alpha)\right\}}\,\frac{\omega}{\sqrt{2}(\sqrt{\beta} - \sqrt{\alpha})} < |\lambda(t)|,$$

then

$$\begin{aligned} \inf \sigma(H_\lambda(t)) &= \min\{E_{n_1(\alpha,\beta)}^-(\lambda(t); \alpha, \beta)\,,\, E_{n_2(\alpha,\beta)}^-(\lambda(t); \alpha, \beta)\} \\ &< E_\nu^-(\lambda(t); \alpha, \beta) < 0, \quad \nu = 0, \cdots, \ell - 1. \end{aligned}$$

Since Proposition 2.10(ii) does not work in the case where $\alpha = \beta$. So, we investigate the superradiant ground state energy in this case from now on.

We define real numbers $E_n^-(\lambda)$ and $E_n^+(\lambda)$ for every $\lambda \in \mathbb{R}$ and each $n \in \mathbb{Z}_+$ by

$$E_n^-(\lambda) := E_n^-(\lambda;\alpha,\alpha) = (n+1)\alpha\omega - \frac{\sqrt{\alpha^2\omega^2 + 4(n+1)\lambda^2}}{2},$$

and

$$E_n^+(\lambda) := E_n^+(\lambda;\alpha,\alpha) = n\alpha\omega + \frac{\sqrt{\alpha^2\omega^2 + 4n\lambda^2}}{2},$$

respectively. We note that all eigenvalues of H_0 are degenerate with each multiplicity equal to 2:

$$E_n^-(0) = \left(n + \frac{1}{2}\right)\alpha\omega = E_n^+(0).$$

Thus, $E_0^-(0) = E_0^+(0) < E_1^-(0) = E_1^+(0) < \cdots < E_n^-(0) = E_n^+(0) < \cdots$.

We define $x_{\alpha,\alpha}(\lambda) \in \mathbb{R}$ for every $\lambda \in \mathbb{R}$ by

$$x_{\alpha,\alpha}(\lambda) := \frac{\lambda^4 - \alpha^4\omega^4}{4\alpha^2\omega^2\lambda^2}.$$

Then, the following lemma holds immediately:

Lemma 2.11 *If* $|\lambda| > \sqrt{2+\sqrt{5}}\,\alpha\omega$, *then* $1 < x_{\alpha,\alpha}(\lambda)$.

This lemma secures the existence of the following non-negative integers n_1 and n_2 defined by

$$n_1 := \max\left\{n \in \mathbb{Z}_+ \,|\, n+1 < x_{\alpha,\alpha}(\lambda)\right\},$$

$$n_2 := \min\left\{n \in \mathbb{Z}_+ \,|\, x_{\alpha,\alpha}(\lambda) \le n+1\right\}.$$

Then, it follows from Lemma 2.11 that $0 \le n_1 < n_2$ if $|\lambda| > \sqrt{2+\sqrt{5}}\,\alpha\omega$.

Let $m, n \in \mathbb{Z}_+$ satisfy $m < n$ now. We define λ_1, λ_2 by taking $\alpha = \beta$ in Eqs.(2.11) and (2.12):

$$\lambda_1 \equiv \lambda_1(m,n) := \frac{(n-m)\alpha\omega}{\sqrt{m} + \sqrt{n+1}},$$

$$\lambda_2 \equiv \lambda_2(m,n) := \sqrt{\frac{(n-m)(n+2-m)}{m}}\,\frac{\alpha\omega}{2}.$$

We note that $0 < \lambda_1 < \lambda_2$ by Lemma 2.8.

We can also exactly understand the behavior of the Dicke-type crossing in the case where $\alpha = \beta$:

Proposition 2.12 *Let $V_\infty(t)$ be as in Eq.(2.4). Set $c_\ell^k(t)$, $k, \ell = 0, 1$, as in Eq.(2.5). Assume $\alpha = \beta$. Then, the family of the differential operator $\{H_\lambda(t)\}_{t\in\mathbb{R}_+}$ has the Dicke-type crossing in the following:*

(i) *(degenerate eigenvalues) If $t \in \mathbb{R}_+$ satisfies*

$$\lambda(t) = \sqrt{m+n+2+\sqrt{4mn+4m+4n+5}}\,\alpha\omega$$

for $m, n \in \mathbb{Z}_+$ with $m \neq n$, then

$$E_m^-(\lambda(t)) = E_n^-(\lambda(t)).$$

(ii) *(superradiant ground state energy I) Assume that $t \in \mathbb{R}_+$ satisfies $|\lambda(t)| > \sqrt{2+\sqrt{5}}\,\alpha\omega$. Then,*

$$\inf \sigma(H_\lambda(t)) = \min\left\{E_{n_1}^-(\lambda(t))\,,\, E_{n_2}^-(\lambda(t))\right\} \leq E_\nu^-(\lambda(t)) < 0,$$

for $\nu = 0, 1, 2, 3$.

(iii) *(superradiant ground state energy II) Let n be in $\mathbb{N}$. If $t \in \mathbb{R}_+$ satisfies $\sqrt{2n+\sqrt{4n^2+1}}\,\alpha\omega < |\lambda(t)| \leq \sqrt{2(n+1)+\sqrt{4(n+1)^2+1}}\,\alpha\omega$, then*

$$\inf \sigma(H_\lambda(t)) = \min\{E_{n-1}^-(\lambda(t))\,,\, E_n^-(\lambda(t))\} < 0.$$

Remark 2.3 We note the following points: Let $\alpha = \beta$ now.

(1) For so small $|\lambda(t)|$ that [35, Theorem XII.13] works, $E_n^-(\lambda(t))$ and $E_n^+(\lambda(t))$ are the only eigenvalues near $E_n^-(0) = E_n^+(0)$ and thus $E_0^-(\lambda(t)) < E_0^+(\lambda(t)) < E_1^-(\lambda(t)) < E_1^+(\lambda(t)) < \cdots < E_n^-(\lambda(t)) < E_n^+(\lambda(t)) < \cdots$ by Proposition 2.7(i). On the other hand, Proposition 2.10(i) and Proposition 2.12 say that sufficiently large $|\lambda(t)|$ breaks this order among eigenvalues determined by regular perturbation theory. That is when non-perturbative phenomenon appears in the spectrum of $H_\lambda(t)$ because of the Dicke-type crossings.

(2) Proposition 2.12 says that the ground state energy gets degenerate again for a certain $\lambda(t)$. See Example 2.2 below.

Example 2.2 We give concrete examples for Proposition 2.12(iii) in the following:

(1) Let $\lambda(t) = \sqrt{6}\,\alpha\omega$. Then, $\sqrt{2+\sqrt{15}}\,\alpha\omega < \lambda(t) < \sqrt{4+\sqrt{65}}\,\alpha\omega$ and

$$\begin{aligned}\inf\sigma(H_\lambda(t)) &= E_0^-(\lambda(t)) = E_1^-(\lambda(t)) = -3\alpha\omega/2\\ &< E_0^+(\lambda(t)) = \alpha\omega/2.\end{aligned}$$

Thus, in this case a superradiant ground state energy has not appeared yet.

(2) Let $\lambda(t) = \sqrt{10}\,\alpha\omega$. Then, $\sqrt{4+\sqrt{25}}\,\alpha\omega < \lambda(t) < \sqrt{6+\sqrt{145}}\,\alpha\omega$ and

$$\begin{aligned}\inf\sigma(H_\lambda(t)) &= E_1^-(\lambda(t)) = E_2^-(\lambda(t)) = -5\alpha\omega/2\\ &< E_0^-(\lambda(t)) = -(2^{-1}\sqrt{41}-1)\alpha\omega\\ &< E_0^+(\lambda(t)) = \alpha\omega/2.\end{aligned}$$

Thus, in this case superradiant ground state energy, $E_1^-(\lambda(t)) = E_2^-(\lambda(t)) = 5\alpha\omega/2$, has appeared.

3 Physical Backgrounds

Several problems around *energy level crossing* attract many mathematicians (see [4, 8, 12, 23] and the literatures in their references). Though the Dicke-type crossing is a kind of energy level crossing, our point of view is different from that in Ref.[4, 8, 12].

Our mathematical set-up in this paper comes from the following physical background. Superradiance appears in a resonant atom-*cavity* system [24], in quantum optics [2, 28], in the Bose-Einstein condensation [29, 30, 31], and in quantum field theory [14, 15], The standard atom-light coupling in nature is not so strong that we expect in this paper. But, in experimental physics of cavity quantum electrodynamics (QED) [9, §10], the present technology can make the coupling strength strong (see [6, 21, 32, 37] and the literatures in their references). Namely, the so-called atom-cavity interaction is able to become strong for an atom in a certain cavity when we irradiate an 1-mode laser to the atom with enough *interaction time* [5, 11, 17, 18]. The model which this paper deals with describes a two-level atom coupled with the 1-mode laser. The two-level atom is in one of two states $\binom{1}{0}$ and $\binom{0}{1}$ if there is no interaction. The energy of the 1-mode laser is expressed by the 1-dimensional Schrödinger operator with harmonic oscillator because an 1-mode laser is a monochromatic light. We are interested in the observation of the spectrum of the two-level atom in the cavity, so we employ the condition (2.5), the so-called the *rotating wave approximation* (RWA), for $\sqrt{2\omega}\,\sigma_1 \otimes x$, because we observe the physical performance in an experimental equipment. Namely, because of the detection limit of the equipment, we cannot observe some parts of interactions. Thus, we can omit the parts from the full interaction (see [13, Remark on p.346-p.347]). In this physical situation physicists are theoretically and experimentally interested in dynamics of $E_n(\lambda(t))$ for each $n \in \mathbb{Z}_+$ (see [24] and the literatures in their references). Our interest in this paper consists in the stability of occurrence of the Dicke-type transition for $E_n(\lambda(t))$ under a class of error potentials in an experiment. Because each error potential plays a role to break the effect coming from RWA.

Here we briefly give a physical meaning to H_0 in Eq.(2.1). We denote by $H_{\mathrm{las}}(\alpha \geq \beta)$ (or $H_{\mathrm{las}}(\alpha \leq \beta)$) the Hamiltonian of a 1-mode laser with the frequency $\beta\omega$ (or $\alpha\omega$), and by H_{ato} the Hamiltonian of a two-level atom with the energy gap $\alpha\omega/2$. Following [28, (4.62)], the Hamiltonian of a 1-mode laser with the frequency $\beta\omega$ (or $\alpha\omega$) is defined by $H_{\mathrm{las}}(\alpha \geq \beta) := \sigma_0 \otimes \beta\omega N$ (or $H_{\mathrm{las}}(\alpha \leq \beta) := \sigma_0 \otimes \alpha\omega N$). The Hamiltonian of a two-level atom with the energy gap $\alpha\omega/2$ is $H_{\mathrm{ato}} := (\alpha\omega/2)\sigma_3 \otimes \mathsf{Id}$. Then, H_0 can be expressed as:

$$H_0 = \begin{cases} H_{\mathrm{las}}(\alpha \geq \beta) + H_{\mathrm{ato}} + H_{\mathrm{int}}(\alpha \geq \beta) & \text{if} \quad \alpha \geq \beta, \\ \\ H_{\mathrm{las}}(\alpha \leq \beta) + H_{\mathrm{ato}} + H_{\mathrm{int}}(\alpha \leq \beta) & \text{if} \quad \alpha \leq \beta \end{cases} \tag{3.1}$$

with interaction Hamiltonians $H_{\mathrm{int}}(\alpha \geq \beta)$ and $H_{\mathrm{int}}(\alpha \leq \beta)$ between the two-level atom and the 1-mode laser:

$$\begin{cases} H_{\mathrm{int}}(\alpha \geq \beta) := \dfrac{\sigma_0 + \sigma_3}{2} \otimes (\alpha - \beta)\omega N + \dfrac{\sigma_0 - \sigma_3}{2} \otimes \dfrac{\alpha + \beta}{2}\,\omega\,\mathsf{Id}, \\ \\ H_{\mathrm{int}}(\alpha \leq \beta) := \dfrac{\sigma_0 - \sigma_3}{2} \otimes \left\{ (\beta - \alpha)\omega N + \dfrac{\alpha + \beta}{2}\,\omega \right\}. \end{cases}$$

In Proposition 2.7 we showed the transition among the eigenvalues of $H_\lambda(t)$ in detail, though the possibility of such a transition was briefly stated in [28, p.91], [24], and [15, §3][1].

[1]Here we note that the minus sign in front of σ_3 in $H(k)$ of [15, §3] should be corrected to the plus sign.

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