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INVERSE SPECTRAL PROBLEMS WITH DATA ON A HYPERSURFACE

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1. Introduction and main results

In this paper we consider some inverse spectral problems on a compact connected Riemannian manifold (M, g). The first motivation to consider inverse problems on Riemannian manifolds comes from spectral geometry. The famous problem here, posed by Bochner and formulated by Kac in the paper "Can one hear the shape of a drum?", [7], is the problem of identifiability of the shape of a 2−dimensional domain from the eigenvalues of its Dirichlet Laplacian. More generally, the question is to find the relations between the spectrum of a Riemannian manifold (M, g), $\dim(M) = n \geq 2$, i.e. the spectrum of the Laplace-Beltrami operator $-\Delta_g$ on it, and geometry of this manifold. In particular, one can ask, following Bochner-Kac, if the spectrum of $-\Delta_g$ determines the geometry. However, already in 1964, it was shown by Milnor [10] that in higher dimensions, the answer to this question is negative. As for the original Bochner-Kac problem in dimension 2, the answer was found only in early 90th. Namely, in 1985 Sunada [11] introduced a method of producing examples of non-isometric isospectral compact Riemannian manifolds. Although in this paper Sunada did not give the answer to the Bochner-Kac problem, in 1992 Gordon, Webb and Wolpert [4] extended Sunada's method and settled in the negative this famous problem by constructing two non-isometric but isospectral plane domains. Further results in this direction can be found in [5] and [13].

It is clear from the above that, to determine geometry of a Riemannian manifold, further spectral information is needed. To understand the nature of this information, let us look at inverse boundary problems, e.g. for the Laplacian with Neumann boundary condition. In this case, the inverse data is the trace on ∂M of its resolvent. Depending on whether this resolvent is given for one or many values of the spectral parameter λ, these inverse boundary problems were originally posed by Calderon [2] and Gel'fand [3]. There are currently two rather general approaches to these problems and their generalizations, see pioneering works [12] and [1], correspondingly, with detailed expositions of substantial further developments in [6] and [8].

In this paper we consider two inverse problems with data on a hypersurface $\Sigma \subset M$ which are intrinsically related to the Gel'fand inverse problem. To describe them, we first reformulate the Gel'fand problem in an equivalent form which, however, has a more "spectral" flavor. Namely, let μ_j and ψ_j be the eigenvalues and normalized eigenfunctions of the Laplace operator with Neumann boundary condition,

$$(-\Delta_g - \mu_j)\psi_j = 0 \quad \text{in} \quad M, \quad \partial_\nu \psi_j|_{\partial M} = 0; \quad (\psi_j, \psi_k)_{L^2(M)} = \delta_{jk}, \tag{1.1}$$

where ∂_ν is the normal derivative to ∂M. Then the Gel'fand problem [3] is equivalent to the determination of (M, g) from the *boundary spectral data*, i.e. $\{\partial M, \ (\mu_j, \psi_j|_{\partial M})_{j=1}^\infty\}$.

Therefore, it seems that a natural extension of this problem to manifolds with a pointed closed hypersurface $\Sigma \subset M$, $\dim(\Sigma) = n - 1$, would be a problem of identifying (M, g) having in our disposal the *Dirichlet spectral data*

$$\{\Sigma, (\lambda_j, \phi_j|_\Sigma)_{j=1}^\infty\}, \tag{1.2}$$

where (λ_j, ϕ_j) are the eigenpairs of the Laplace operator $-\Delta_g$ on M with some, say Dirichlet or Neumann, boundary condition on ∂M, if $\partial M \neq \emptyset$.

However, a closer look at the nature of the Gel'fand boundary spectral data for a manifold with boundary shows that, due to the Neumann boundary condition on ∂M for ψ_j, we do actually know the whole Cauchy data on ∂M of the eigenfunctions of the Neumann Laplacian, i.e. $\psi_j|_{\partial M}$, $\partial_\nu \psi_j|_{\partial M}$. Clearly, the same is true for the Dirichlet Laplacian on M. Thus, a more straightforward generalization of the Gel'fand inverse problems to manifolds with a pointed hypersurface would be the inverse problem of identifying (M, g) having in our disposal the *Cauchy spectral data*

$$\{\Sigma, (\lambda_j, \phi_j|_\Sigma, \partial_\nu \phi_j|_\Sigma)_{j=1}^\infty\}. \tag{1.3}$$

Actually, in this paper we consider the inverse spectral problems, under some different conditions on Σ, for the both sets of data, i.e. the Cauchy spectral data and the Dirichlet spectral data. In this connection, further exposition is structured into two section, the first devoted to the inverse problem with the Cauchy data, and second devoted to the inverse problem with the Dirichlet data. In this paper we provide only the ideas of the proofs. For the complete exposition, please consult with [9].

To complete this introduction, we note that prescribing data over a hypersurface is natural for various physical applications when sources and receivers are located over some surface in space rather then are scattered over an $n-$dimensional region or put on, probably remote, boundary of M. Such localization is used e.g. in radars, sonars, and in medical ultrasound imaging when a single antenna array is used to produce the wave and to measure the scattered wave. It is typical also in geosciences/seismology where sources and receivers are often located over the surface of the Earth or an ocean.

2. Inverse problem with Cauchy spectral data

Assume that it is known *a priori* that Σ divides M into two relatively open subsets, M_+, M_- such that

$$\overline{M}_+ \cap \overline{M}_- = \Sigma, \quad \overline{M}_+ \cup \overline{M}_- = M,$$

where each of $M_\pm$ may consist of several components.

Then, the following result is valid

Theorem 2.1. *The Cauchy spectral data (1.3) determine the manifold (M, g) up to an isometry.*

Let us sketch the idea of the proof of Theorem 2.1. Denote by $\mu_k^\pm$, $\psi_k^\pm$ the eigenpairs for the Laplace operators $-\Delta_g$ in $M_\pm$ with the Neumann conditions on Σ and the boundary conditions on the remaining part of $\partial M_\pm$ inherited from the original Laplacian. The principal idea of the proof is to show that the Cauchy spectral data (1.3) determine the Gel'fand boundary spectral data $\{\Sigma, (\mu_k^\pm, \psi_k^\pm|_\Sigma)_{k=1}^\infty\}$ on a part Σ of the boundary $\partial M_\pm$. It is then standard for the boundary control method, see [8], that $\{\Sigma, (\mu_k^\pm, \psi_k^\pm|_\Sigma)_{k=1}^\infty\}$ uniquely determine $(M_\pm, g_\pm)$ which, by gluing along Σ, describe M.

To determine $\{\Sigma, (\mu_k^\pm, \psi_k^\pm|_\Sigma)_{k=1}^\infty\}$ consider the transmission problem

$$\begin{aligned} &(-\Delta_g - \lambda)u := -g^{-1/2}\partial_i(g^{1/2}g^{ij}\partial_j u) - \lambda u = 0 \quad \text{in} \quad M \setminus \Sigma, \\ &[u] = f \quad \text{on} \quad \Sigma, \quad [\partial_\nu u] = h \quad \text{on} \quad \Sigma, \quad f, h \in C^\infty(\Sigma,) \end{aligned} \tag{2.1}$$

where $[u]$ and $[\partial_\nu u]$ are the jumps of u and its normal derivative across Σ, $g = \det(g_{ij})$ and the tensor (g^{ij}) is the inverse to the metric tensor (g_{ij}), $i, j = 1, \dots, n$.

When $\lambda \neq \lambda_j$, (2.1) has a unique solution, $u = u_\lambda^{f,h}$ This defines the operator

$$R_\lambda(f, h) = u_\lambda^{f,h}|_{\Sigma+},$$

where the rhs stand for the value of $u_\lambda^{f,h}$ on Σ when approaching from M_+.

Using spectral arguments, we first observe that

$$R_\lambda(f, h) = \sum_{j=1}^\infty a_j^\lambda(h)\phi_j|_\Sigma - \sum_{j=1}^\infty b_j^\lambda(f)\phi_j|_\Sigma - \frac{1}{2}f, \tag{2.2}$$

where

$$a_j^\lambda(h) = \frac{1}{\lambda - \lambda_j}\int_\Sigma \phi_j(y)h(y)dS_g(y), \quad b_j^\lambda(f) = \frac{1}{\lambda - \lambda_j}\int_\Sigma \partial_\nu\phi_j(y)f(y)dS_g(y).$$

Note that the first sum in rhs of (2.2) converges in $H^{1/2}(\Sigma)$, while the second one diverges and should be properly regularized.

Now let $\sigma(-\Delta_+^D)$ be the spectrum of the Laplace operators $-\Delta_g$ in M_+ with the Dirichlet conditions on Σ and the boundary conditions on the remaining part of ∂M_+ inherited from the original Laplacian.

Let now $\lambda \neq \mu_k^-$, where μ_k^- are defined as in (1.1) with, however, M_- instead od M, and $\lambda \notin \sigma(-\Delta_+^D)$. Our second observation is that, for such λ and any $h \in C^\infty(\Sigma)$ there is a unique solution $f = f_\lambda(h)$ to the equation

$$R_\lambda(f, h) = 0.$$

Moreover, the corresponding $u_\lambda^{f,h}$ satisfies $u_\lambda^{f,h}(x) = 0$ in M_+.

These two observations show that the Cauchy spectral data (1.3) determine, for $\lambda \neq \mu_k^-$, $\lambda \neq \lambda_j$ and $\lambda \notin \sigma(-\Delta_+^D)$, the Neumann-to-Dirichlet map, namely,

$$\Lambda_\lambda^-(h) = -f.$$

Here $\Lambda_\lambda^-(h) = u_\lambda^-(h)|_\Sigma$, $u_\lambda^-(h)$ being the solution to

$$(-\Delta - \lambda)u = 0 \text{ in } M_-, \quad \partial_\nu u|_\Sigma = -h.$$

Similar, we can obtain, from the Cauchy spectral data (1.3), the Neumann-to-Dirichlet map Λ_λ^+.

It then follows from [8] that $\Lambda_\lambda^\pm$, $\lambda \neq \mu_k^\pm$, $\lambda \neq \lambda_j$ and $\lambda \notin \sigma(-\Delta_\mp^D)$, determine $\{\mu_k^\pm, \psi_k^\pm|_\Sigma\}$.

3. Inverse problem with Dirichlet spectral data

The inverse problem with only Dirichlet spectral data (1.2) contains much less information and, to solve it, we impose further restrictions onto domains $M_\pm$. We will assume that M_- consists of two relatively open subsets M_-^1, M_-^2, $\overline{M_-^1} \cap \overline{M_-^2} = \emptyset$, $M_- = M_-^1 \cup M_-^2$. Therefore, $\Sigma = \Sigma^1 \cup \Sigma^2$, with $\Sigma^i = \partial M_-^i$. In the future, it is convenient for us to introduce five subsets, N_i, $i = 1, \dots, 5$, in M. They are $M_-^{1,2}, M_+$ and $M \setminus \overline{M_-^{1,2}}$. Denote by Δ_i the Laplacian in N_i with the Dirichlet condition on $\widetilde{\Sigma}_i = \Sigma \cap \overline{N_i}$ and, if $\partial N_i \cap \partial M \neq \emptyset$, with additional boundary condition on $\partial N_i \cap \partial M$ inherited from Δ_g. Observe that, for any i, $M \setminus \overline{N_i}$, is among $N_j, j \neq i$ and we denote it by N_i^c.

Condition 3.1. For any $i \neq j$, $i, j = 1, \dots, 5$,

$$\sigma(-\Delta_i) \cap \sigma(-\Delta_j) = \emptyset,$$

where $\sigma(-\Delta_i)$ is the spectrum of $-\Delta_i$.

Theorem 3.1. *Assume that the manifold (M, g) and $\Sigma = \Sigma^1 \cup \Sigma^2$ satisfy condition 3.1. Then the Dirichlet spectral data (1.2) determine the manifold up to an isometry.*

The crucial ingredient of the proof is an approximate controllability result which is of its own interest. To formulate it, consider the following transmission problem, cf. (2.1)

$$\begin{aligned} &(\partial_t^2 - \Delta_g)u^h = 0 \quad \text{in } (M \setminus \widetilde{\Sigma}_i) \times \mathbb{R}, \\ &[u^h] = 0 \quad \text{on } \widetilde{\Sigma}_i \times \mathbb{R}, \quad [\partial_\nu u^h] = h \quad \text{on } \widetilde{\Sigma}_i \times \mathbb{R}, \quad u^h|_{t<t_h} = 0, \end{aligned} \tag{3.1}$$

where $h \in C_+^\infty(\widetilde{\Sigma}_i \times \mathbb{R})$. This space consists of $C^\infty-$ smooth functions equal to 0 for sufficiently large negative t, i.e.

$$h = 0 \quad \text{for } t < t_h.$$

Theorem 3.2. *Let $\sigma(-\Delta_i) \cap \sigma(-\Delta_i^c) = \emptyset$. Then the set*

$$Y_i = \{Wh := u^h(0);\ h \in C_+^\infty(\widetilde{\Sigma}_i \times \mathbb{R})\} \tag{3.2}$$

is dense in $H^1(M)$.

Remark 3.2. We note that Theorem 3.2 is not valid for arbitrary $\widetilde{\Sigma}_i$. Indeed, if M is just the Hopf double, with its metric, of N_i then all the solutions to (3.1) would be symmetric with respect to $\widetilde{\Sigma}_i$.

Together with the Blagovestchenskii identity, which makes it possible to evaluate, using the Dirichlet spectral data, the Fourier coefficients

$$u^h(x,t) = \sum_j u_j^h(t)\phi_j(x), \tag{3.3}$$

this theorem provides a possibility to constructively define the Hilbert spaces *of generalized sources*, D_i with the norm

$$|h|_i^2 := \|Wh\|_{H^1(M)}^2 = \sum_{j=1}^\infty (\lambda_j + 1)|u_j^{0,h}(0)|^2, \quad \text{supp}(h) \subset \widetilde{\Sigma}_i \times \mathbb{R}_-. \tag{3.4}$$

Moreover, for any $h \in D_i$, we can find the restriction $u^h(0)|_\Sigma$ and, in particular, to construct a subspace D_i^0 which consists of $h \in D_i$ with $Wh|_{\widetilde{\Sigma}_i} = 0$.

Observe that

$$\left(\nabla_g(Wf), \nabla_g(Wh)\right)_{L^2(M)} = \sum_{j=1}^\infty \lambda_j u_j^f(0)\overline{u_j^h(0)}. \tag{3.5}$$

Therefore, solving the min-max problem for (3.5) with $f, h \in D_i^0$, it is possible to find the eigenvalues and the Fourier coeffients (3.3) of the eigenfunctions of the operator $-\Delta_i \oplus (-\Delta_i^c)$. Utilizing condition 3.1, we can find the eigenvalues $\mu_{i,k}$, $k = 1, \dots$ of the subdomains N_i, $i = 1, \dots, 5$, and the Fourier coeffients $a_{i,k,j}$ of the corresponding eigenfunctions, $\psi_{i,k}$

$$\psi_{i,k} = \sum_j a_{i,k,j}\phi_j \tag{3.6}$$

of all operators $-\Delta_i$.

To proceed further, consider a pair of initial boundary value problems in N_i and N_i^c,

$$\begin{aligned} &(\partial_t^2 - \Delta_g) w_F = 0, \quad \text{in } N \times \mathbb{R}_+, \\ &w_F|_{\widetilde{\Sigma}_i \times \mathbb{R}_+} = F, \quad w_F|_{t=0} = 0,\, \partial_t w_F|_{t=0} = 0, \end{aligned} \tag{3.7}$$

with $F \in C_0^\infty(\widetilde{\Sigma}_i \times \mathbb{R}_+)$ and N being N_i or N_i^c. The pair (w_F, w_F^c) which provides the solution to (3.7) in N_i, N_i^c, correspondingly, is a solution to the transmission problem (3.1). Moreover, the corresponding h can be found uniquely and constructively from the equation

$$u^h|_{\widetilde{\Sigma}_i \times \mathbb{R}} = F.$$

Recall that $\{\psi_{i,k}\}_{k=1}^\infty$, $\{\psi_{i,l}^c\}_{l=1}^\infty$ together form an orthonormal basis in $L^2(M)$. Combining this with (3.3), (3.6), we obtain the representation

$$u^h(t) = \sum_{k=1}^\infty w_{F,k}(t)\psi_{i,k} + \sum_{l=1}^\infty w_{F,l}^c(t)\psi_{i,l}^c = w_F(t) + w_F^c(t).$$

In particular, this implies, for any $F \in C_0^\infty(\widetilde{\Sigma}_i \times \mathbb{R}_+)$ and $t \geq 0$, the Dirichlet spectral data (1.2) determine the L^2- norm $|w_F(t)|^2$. A slight modifications of the arguments in [8, sec. 4.2] makes it possible to reconstruct $(N_i, g|_{N_i})$ and, therefore, (M, g).

This completes the reconstruction of (M, g) from the Dirichlet spectral data.

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