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Lagrangian for Collective Matter-Field Couplings

The University of Tokyo, Graduate School of Frontier Sciences
Zensho Yoshida

1 Introduction

It is well known that an ideal fluid obeys a "non-canonical" Hamiltonian system of PDEs –the generator of the infinite-dimensional dynamics (Lie group) constitutes a degenerate Poisson bracket, and its "defect" represents a topological constraint (Casimir invariant) on the "helicity" of the fluid vorticity [1, 2, 3]. In the theory of linear dynamics, where the generator is represented by a linear operator, the scope expands from the category of self-adjoint (Hermitian) operators, through that of normal operators, and to the most general class of non-normal operators that may have kernels. In an infinite-dimension space, the theory is already stuck at general non-self-adjoint operators. A non-canonical Hamiltonian system is a nonlinear version of a non-normal generator, which, however, may assume a Lie-algebraic structure. The aim of this paper is to show how the kernel (Casimir) may be created (to allow a finite helicity) and separated to "normalize" the Hamiltonian structure.

The use of the Clebsch parameterization of the flow field (with a finite vorticity) is known to formulate a set of canonical equations that seems to be equivalent to the fluid equations [4]. A Lagrangian that is capable to introduce the Clebsch parameters and a finite vorticity was suggested by Lin [6], while the physical interpretation of the "parasite variable" (Lagrange multipliers and constraints to create vorticity) has not been established. On the other hand, a Lagrangian representation of motion, unlike the Eulerian view, is freed from the vorticity (helicity) problem, and the Lagrangian may be naturally constructed (reviewed in Sec. 2). In this paper, we will show the "equivalence" of the Lagrangians both in the Eulerian and Lagrangian views, and reveal that the "constraints" needed in the Eulerian formulation are nothing but the Lagrangian labels (initial positions) of fluid elements. Since the Lagrangian view attributes the origin of vorticity to the "initial condition", its "diachronic" analysis of vortex dynamics is unaware of the "topological defects" associated with the vorticity. The Eulerian view finds singularities in formulating Poisson brackets with "synchronic" variables.

2 Fluid Mechanics in Lagrangian View

The Lagrangian for the collective matter-field couplings may be naturally derived by generalizing the Lagrangian of a particle (mass m and charge e) in the presence of an EM field (we consider the non-relativistic limit for simplicity):

$$L = L_P + L_{EM}, \tag{1}$$

$$L_P = \boldsymbol{P} \cdot \boldsymbol{v} - H, \tag{2}$$

$$L_{EM} = \int \mathcal{L}_{EM} dx = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} dx, \tag{3}$$

where $H = p^2/(2m) + e\phi$ is the Hamiltonian, $\boldsymbol{P} = \boldsymbol{p} + (e/c)\boldsymbol{A}$ is the canonical momentum, and $\boldsymbol{p}$ is the mechanical momentum. The vector and the scalar potentials define the four potential $A^\mu = (\phi, \boldsymbol{A})$ whose curl is the Faraday (field strength) tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The velocity $\boldsymbol{v}$ is related to the particle orbit $\boldsymbol{q}(t)$ (along which L_P is to be evaluated) by

$$\boldsymbol{v} = \dot{\boldsymbol{q}}. \tag{4}$$

The relation (4) is the essential input that "causes" the motion of the particle. Indeed, if we calculate the variation of $\int L\ dt$ for general $\delta \boldsymbol{v}$, we find $\boldsymbol{P} = 0$, but, for $\delta \dot{\boldsymbol{q}}$, with fixed end points $\boldsymbol{q}(t_0)$ and $\boldsymbol{q}(t_1)$, the well-known Lagrange's equation of motion follows.

Generalizing the single particle orbit $\boldsymbol{q}(t)$ to a diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$ in $\mathbf{R}^3$ ($\boldsymbol{x}_0$ being the initial position of each streamline), we construct the fluid Lagrangian

$$L_F = \int \mathcal{L}_F dx = \int (\boldsymbol{P} \cdot \boldsymbol{V} - H_F)\, \rho dx \tag{5}$$

by replacing the single-particle velocity $\dot{\boldsymbol{q}}$ in (4) by the flow velocity

$$\boldsymbol{V}(\boldsymbol{x}, t) = \dot{\boldsymbol{Q}}|_{\boldsymbol{x},t} = \frac{d}{dt} \boldsymbol{Q}(\boldsymbol{x}_0(\boldsymbol{x}, t), t). \tag{6}$$

Here, $\boldsymbol{x}_0(\boldsymbol{x}, t)$ is the initial position of the streamline being connected to the time-space position $(\boldsymbol{x}, t)$. To relate $(\boldsymbol{x}, t)$ to $(\boldsymbol{x}_0, 0)$, one needs the inverse map $\boldsymbol{x}_0(\boldsymbol{x}, t) \equiv \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$ of the diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$, which traces-back the streamlines (the relations among $\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{x}_0, t)$, $\boldsymbol{x}_0 = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$, and $\boldsymbol{V}$ will be discussed more explicitly in sec. 5).

The fluid Hamiltonian H_F consists of the kinetic and potential energies plus an "internal (thermal) energy" $\varepsilon(\rho)$:

$$H_F = H + \varepsilon(\rho). \tag{7}$$

The fluid density ρ is defined by

$$\rho(\boldsymbol{x}, t) = \rho_0(\boldsymbol{x}_0(\boldsymbol{x}, t)) \cdot \frac{D(\boldsymbol{x}_0)}{D(\boldsymbol{x})}. \tag{8}$$

When we formally take $\rho_0(\boldsymbol{x}_0) = \delta(\boldsymbol{q}_0 - \boldsymbol{x}_0)$, we recover (1) with $\rho dx dt$ giving the integral along the orbit $\boldsymbol{q}(t)$.

In what follows we denote

$$D_t f = \partial_t f + \boldsymbol{V} \cdot \nabla f, \tag{9}$$

$$D_t^* f = \partial_t f + \nabla \cdot (\boldsymbol{V} f). \tag{10}$$

The mass conservation law

$$D_t^* \rho = 0 \tag{11}$$

is a direct consequence of (8). By the criticality of the action $\int (L_F + L_{EM}) dt$, fixing the space-time boundaries, we obtain, from the variation $\delta \boldsymbol{p}$, $\boldsymbol{p} = m\boldsymbol{V}$ $(= m\dot{\boldsymbol{Q}})$, and from $\delta \boldsymbol{Q}$, the equation of motion

$$mD_t\boldsymbol{V} = -\nabla h + e\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{V} \times \boldsymbol{B}\right), \tag{12}$$

where $\boldsymbol{E} = -\nabla\phi - \partial_t \boldsymbol{A}/c$, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and h is the enthalpy density defined by $h = \partial(\rho\varepsilon)/\partial\rho = \varepsilon + \mathcal{P}/\rho$ ($\mathcal{P}$ is the pressure; $\rho\nabla h = \nabla\mathcal{P}$). Maxwell's equations with the currents $(e\rho, e\rho\boldsymbol{V})$ follow from δA_μ.

3 Fluid Mechanics in Eulerian View

In this section, we formulate an Eulerian representation of the fluid Lagrangian. In the Eulerian case where no *a priori* relation between the fluid velocity $\boldsymbol{V}$ and the streamlines is assumed, and the unrestricted variation $\delta\boldsymbol{V}$ yields $\boldsymbol{P} = 0$. To reproduce properly the evolution equations, we must appropriately "constrain" $\boldsymbol{V}$.

The measure $\rho dx dt$, defined by (8), is the generalization of the integral along a single orbit to collective orbits. It was argued, then, that imposing a physically motivated "restriction" on ρ that leads to the conservation law (11), must be a step in the right direction. Serrin, in a pioneering paper,[5] proposed the Lagrangian density

$$\mathcal{L}_F = \rho\,(\boldsymbol{P} \cdot \boldsymbol{V} - H_F) + S D_t^* \rho, \tag{13}$$

in which the variation of the Lagrange multiplier S does exactly that.

The Serrin Lagrangian, however, is inadequate because the momentum $\boldsymbol{P} = \nabla S$ (obtained by the variation $\delta\boldsymbol{V}$) describes only an "irrotational" flow representing a relatively small area of general fluid mechanics. We will see later that this restriction is due to a "canonical structure" premised by (13).

To derive flows with vorticity, Lin[6] imposed another constraint by including a term $\rho\alpha(\partial_t\beta + \boldsymbol{V} \cdot \nabla\beta)$ to the integrand of (13). The resulting flow $\boldsymbol{P} = \nabla S +$

$\alpha\nabla\beta$, corresponding to the so-called Clebsch parameterization of a vector field, does acquire a finite "vorticity" ($\nabla\times\boldsymbol{P} = \nabla\alpha\times\nabla\beta$). However, the Clebsch representation (with a single pair of α and β) is not a global representation for arbitrary flows with non-zero helicity.

We propose a generalized and rearranged Serrin-Lin-type Lagrangian density

$$\mathcal{L}_F = \left[\boldsymbol{P}\cdot\boldsymbol{V} - H_F - (D_t S + \lambda^j D_t \sigma_j)\right]\rho. \tag{14}$$

The variational principle $\delta\int(\mathcal{L}_F + \mathcal{L}_{EM})\, dxdt = 0$ yields a complete set of determining equations:

$$\delta\boldsymbol{V} \quad \Rightarrow \quad \boldsymbol{P} = \nabla S + \lambda^j \nabla\sigma_j, \tag{15}$$

$$\delta\boldsymbol{p} \quad \Rightarrow \quad \boldsymbol{p} = m\boldsymbol{V}, \tag{16}$$

$$\delta S \quad \Rightarrow \quad D_t^*\rho = 0, \tag{17}$$

$$\delta\sigma_j \quad \Rightarrow \quad D_t^*(\rho\lambda^j) = 0 \quad \Rightarrow \quad D_t\lambda^j = 0, \tag{18}$$

$$\delta\lambda^j \quad \Rightarrow \quad D_t\sigma_j = 0, \tag{19}$$

$$\delta\rho \quad \Rightarrow \quad D_t S = \boldsymbol{P}\cdot\boldsymbol{V} - (H + h), \tag{20}$$

and, by δA_μ, Maxwell's equations with the currents $(e\rho, e\rho\boldsymbol{V})$. The vector field $\boldsymbol{P} = \nabla S + \lambda^j\nabla\sigma_j$ is now "complete" to represent any three-dimensional vectors if we take (at least) three independent scalars σ_j. The fluid (plasma) equation follows from

$$D_t\boldsymbol{P} = D_t(\nabla S + \lambda^j\nabla\sigma_j) = -\nabla(e\phi + h) + \frac{e}{c}\left[\boldsymbol{V}\times\boldsymbol{B} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{A}\right],$$

which is equivalent to (12).

The role of S is best understood by referring to the original Serrin form ($\lambda^j = \sigma_j = 0$). Though Serrin's S is a Lagrange multiplier that imposes mass conservation (11), we proffer a different interpretation. By moving (by integrating by parts) D_t^* from ρ to S, one may think of ρ as a Lagrange multiplier demanding that $\mathcal{L}_F$ must be a complete derivative (evaluated through each streamline of $\boldsymbol{V}$) of some scalar field S —this is nothing but Hamilton's principle demanding the criticality of the action integral with S as the "action". Indeed, if the thermal energy ε is neglected and $\lambda^j = \sigma_j = 0$ in (15) and (20), we obtain the well-known Hamilton-Jacobi equations $\partial_t S = -H(\boldsymbol{x}, \boldsymbol{P}, t)$ and $\nabla S = \boldsymbol{P}$.

The role and meaning of the additional fields λ^j and σ_j will be discussed in the following sections.

4 Non-canonical Poisson Bracket and Clebsch Parameterization

We have to specify what we have called "non-canonical". A canonical Hamiltonian mechanics is endowed with a regular symplectic 2-form $\mathcal{A}_{ij}du^i \wedge du^j$. Denoting $\mathcal{A}_{ij}^{-1} = \mathcal{A}^{ij}$ and $\{F,G\} = -(\partial_{u^i}F)\mathcal{A}^{ij}(\partial_{u^j}G)$ (Poisson bracket), the corresponding equation of motion is $\dot{u}^i = \{H, u^i\}$. If there exists C such that $\{F, C\} \equiv 0$ for all F, f^{ij} has a "kernel", and hence, $\mathcal{A}_{ij}$ is singular. Then, we say that the system is non-canonical. Such a C is called a "Casimir invariant" (because $\{H, C\} = 0$, C is a constant of motion).

The fluid-mechanics equations (11)-(12) do have Casimir invariants [1, 2, 7]. Choosing $\boldsymbol{u} = {}^t(n, \boldsymbol{P})$ as independent variables, the evolution equations can be cast into a Hamiltonian form

$$\partial_t \begin{pmatrix} n \\ \boldsymbol{P} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -(\nabla\times\boldsymbol{P})\times \end{pmatrix} \begin{pmatrix} \partial_n \mathcal{H}_F \\ \partial_{\boldsymbol{P}} \mathcal{H}_F \end{pmatrix} \tag{21}$$

with the conventional Hamiltonian (total energy)

$$\mathcal{H}_F = \int H_F d^3x = \int n \left[\frac{(\boldsymbol{P} - e\boldsymbol{A}/c)^2}{2m} + e\phi + \varepsilon(n) \right] d^3x$$

(see (7)). We may write (21) as $\partial_t \boldsymbol{u} = \mathcal{A}\partial_{\boldsymbol{u}}\mathcal{H}_F$. The anti-symmetric operator $\mathcal{A}$ defines the Poisson bracket $\{\mathcal{F}, \mathcal{G}\} = \langle \mathcal{A}\partial_{\boldsymbol{u}}\mathcal{F}, \partial_{\boldsymbol{u}}\mathcal{G}\rangle$. One observes that the "helicity" defined by

$$C = \int (\nabla \times \boldsymbol{P}) \cdot \boldsymbol{P} d^3x$$

is a Casimir invariant (the total particle number $N = \int n d^3x$ is also a Casimir invariant).

Using the Clebsch parameterization (15) of the canonical momentum $\boldsymbol{P}$ (induced by our Lagrangian (14)) in the Hamiltonian (21), however, we obtain a canonical system governing the Clebsch variables:

$$\begin{cases} \partial_t n = \partial_S \mathcal{H}_F \\ \partial_t S = -\partial_n \mathcal{H}_F \\ \partial_t \Lambda^j = \partial_{\sigma_j} \mathcal{H}_F \\ \partial_t \sigma_j = \partial_{\Lambda^j} \mathcal{H}_F \end{cases} \tag{22}$$

where $\Lambda^j = n\lambda^j$ and $j = 1, \cdots, \nu$ (ν may be an arbitrary integer). This system is equivalent to (17)-(20).

In the next section, we will reveal the physical meaning of the Clebsch variables to explain why the topological defect of the non-canonical system is removed by the constraints in (14).

5 Unification of the Lagrangian and Eulerian Views

From (15), we see that the scalar fields λ^j and σ_j may be seen as providing "parameterization" of the vector field $\boldsymbol{P}$ as a 1-form. If the index j runs over the space dimension (= 3), then the vectors $\{\nabla\sigma_j\}$ may no longer be linear independent. Henceforth, we assume both λ^j and σ_j to be the components of three-vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}$. This interpretation of λ^j and σ_j will now be exploited to relate the Lagrangian and Eulerian formulations of the Lagrangians.

Using the identity $\boldsymbol{V} \equiv D_t\boldsymbol{x}$ (x_j is the Cartesian parameterization), one may write

$$\boldsymbol{P}\cdot\boldsymbol{V} - \boldsymbol{\lambda}\cdot D_t\boldsymbol{\sigma} = \boldsymbol{P}\cdot D_t\boldsymbol{\xi} - \boldsymbol{\mu}\cdot D_t\boldsymbol{\sigma},$$

where $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{\sigma}$ and $\boldsymbol{\mu} = \boldsymbol{\lambda} - \boldsymbol{P}$. Then, (14) transforms to

$$\mathcal{L}_F = (\boldsymbol{P}\cdot D_t\boldsymbol{\xi} - H_F - D_tS - \boldsymbol{\mu}\cdot D_t\boldsymbol{\sigma})\,\rho. \tag{23}$$

From this form, the equivalence between the Eulerian and the previous Lagrangian (5) formalism will be established when we connect $D_t\boldsymbol{\xi}$ with $\dot{\boldsymbol{Q}}$.

In (23), the variation of $\boldsymbol{\mu}$ forces $D_t\boldsymbol{\sigma} = 0$ implying

$$D_t\boldsymbol{\xi} = D_t\boldsymbol{x} \equiv \boldsymbol{V}. \tag{24}$$

This Eulerian representation of the flow velocity is compared with the Lagrangian representation (6) that invoked the diffeomorphism $\boldsymbol{Q}(\boldsymbol{x}_0, t)$. The time derivative $\dot{}$ of $\boldsymbol{Q}(\boldsymbol{x}_0, t)$ is taken for fixed $\boldsymbol{x}_0$ (the initial points of every streamlines). The convective D_t of $\boldsymbol{\xi}$, on the other hand, must be evaluated under the condition $D_t\boldsymbol{\sigma} = 0$. We will now show that such a $\boldsymbol{\sigma}$ (in its simplest choice) is nothing but $\boldsymbol{x}_0$ (hence, $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{\sigma}$ is the "displacement"). The constraint $D_t\boldsymbol{\sigma} = 0$ (i.e., fixing $\boldsymbol{x}_0$), then, allows the identification $\dot{\boldsymbol{Q}} = D_t\boldsymbol{\xi}$.

To show $\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{x}_0 = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$, let us first note that $D_t\boldsymbol{\sigma} = 0$ implies that $\boldsymbol{\sigma}$ is constant along every streamline defined by $\boldsymbol{V}$ and, therefore, must be a function only of the "initial condition" because the initial condition cannot change during the evolution. The simplest expression of such a $\boldsymbol{\sigma}$ is the initial condition itself. Formally, we can construct $\boldsymbol{\sigma}$ by solving $D_t\boldsymbol{\sigma} = 0$ for a given initial condition $\boldsymbol{\sigma}(\boldsymbol{x}, 0) = \boldsymbol{\sigma}_0(\boldsymbol{x})$. The corresponding "characteristics" equation is

$$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{V}(\boldsymbol{x}, t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{25}$$

Solving (25) for every initial value $\boldsymbol{x}_0$, we may construct the diffeomorphism $\boldsymbol{x} = \boldsymbol{Q}(\boldsymbol{x}_0, t)$. Using the inverse map $\boldsymbol{Q}^{-1}(\boldsymbol{x}, t)$, we may write $\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{\sigma}_0(\boldsymbol{Q}^{-1}(\boldsymbol{x}, t))$. The simplest initial condition $\boldsymbol{\sigma}_0(\boldsymbol{x}) = \boldsymbol{x}$, i.e., $\boldsymbol{\sigma}_0$ is the identity, leads to

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{Q}^{-1}(\boldsymbol{x}, t) \equiv \boldsymbol{x}_0(\boldsymbol{x}, t). \tag{26}$$

We already mentioned that when $\boldsymbol{\sigma} = \boldsymbol{x}_0$, $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{\sigma}$ is the Eulerian representation of the "displacement" so that $D_t\boldsymbol{\xi} = \boldsymbol{V} = \dot{\boldsymbol{Q}}$, establishing the equivalence of the Eulerian and Lagrangian descriptions.

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