

# Quantized electromagnetic field interacting with a classical source: infrared catastrophe and the physical subspace

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## 1 Introduction

We consider the quantized electromagnetic field interacting with a classical given source [5] satisfying the following equation

$$\square A_\mu(x, t) = j_\mu(x, t), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \quad (1.1)$$

where the current density  $j^\mu$  of the source is conserved:

$$\partial^\mu j_\mu(x, t) = 0. \quad (1.2)$$

We construct the quantized radiation field  $A_\mu(x, t)$  and its time derivative  $\dot{A}_\mu(x, t)$  as an operator-valued distributions [9] on  $\mathbb{R}^3$ . Here we assume that the time zero fields  $A_\mu(x) = A_\mu(x, 0)$  and  $\dot{A}_\mu(x) = \dot{A}_\mu(x, 0)$  satisfy the following commutation relations

$$[A_\mu(x), \dot{A}_\nu(y)] = -ig_{\mu\nu}\delta(x - y) \quad (1.3)$$

$$[A_\mu(x), A_\nu(y)] = [\dot{A}_\mu(x), \dot{A}_\nu(y)] = 0 \quad (1.4)$$

with

$$g_{00} = -g_{jj} = 1 \quad (j = 1, 2, 3), \quad g_{\mu\nu} = 0 \quad (\mu \neq \nu).$$

It is well known that the commutation relations require the introduction of an indefinite metric state space  $\mathcal{F}$  in which  $A_\mu(x, t)$  and  $\dot{A}_\mu(x, t)$  act [2, 4, 5, 7]. Hence the usual probabilistic interpretation is not valid in the whole space  $\mathcal{F}$ . According to the Gupta-Bleuler formalism [2, 4], one can select a positive semi-definite subspace  $\mathcal{V}_{\text{phys}} \subset \mathcal{F}$ , called the physical subspace, which is the subspace of all vectors  $\Psi \in \mathcal{F}$  satisfying the Gupta subsidiary condition

$$\partial^\mu A_\mu^{(+)}(x, t)\Psi = 0, \quad (1.5)$$

where  $\partial^\mu A_\mu^{(+)}$  means the positive frequency part of  $\partial^\mu A_\mu$ . Then one can recover the probabilistic interpretation on the physical Hilbert space  $\mathcal{H}_{\text{phys}}$  defined by the quotient space  $\mathcal{V}_{\text{phys}}/\mathcal{V}_0$ , where  $\mathcal{V}_0$  is the set of all neutral vectors in  $\mathcal{V}_{\text{phys}}$  [6, 7].

The solution of (1.1) is uniquely determined by the time zero fields  $A_\mu(x)$  and  $\dot{A}_\mu(x)$ . The time zero fields are given by a representation of the commutation relations (1.3) and (1.4). Thus, via the Gupta subsidiary condition (1.5), the physical subspace  $\mathcal{V}_{\text{phys}}$  depends on the choice of representations of the commutation relations (1.3) and (1.4) for the time zero fields.

Recently, in [8], we characterize the physical subspace  $\mathcal{V}_{\text{phys}}$  in the case where the source is static, i.e.,  $j_0(x, t) = \rho(x)$  is independent of time and  $j_i = 0$  ( $i = 1, 2, 3$ ). We proved that (1) when we take the usual Fock representation as the time zero fields, the physical subspace is positive semi-definite and non-trivial, i.e.,  $\mathcal{V}_{\text{phys}} \neq \{0\}$ , if and only if the infrared regular condition  $|k|^{-3/2}\hat{\rho} \in L^2(\mathbb{R}^3; dk)$  is made and (2) when we choose a non-Fock representation for the time zero fields,  $\mathcal{V}_{\text{phys}}$  is non-trivial even if  $|k|^{-3/2}\hat{\rho} \notin L^2(\mathbb{R}^3; dk)$ . In the case of (1), the physical subspace is trivial, i.e.,  $\mathcal{V}_{\text{phys}} = \{0\}$  if

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$|k|^{-3/2}\hat{\rho} \notin L^2(\mathbb{R}^3; dk)$ . This is a kind of infrared catastrophe. The Hamiltonian  $H$  of this system is given by

$$H = H_f + V,$$

where  $H_f$  is the free Hamiltonian of the photons and  $V$  the interaction Hamiltonian given by

$$V = \int_{\mathbb{R}^3} dx \rho(x) A_0(x).$$

We proved that  $H$  is “self-adjoint” with respect to the indefinite metric mentioned above and that  $H$  leaves  $\mathcal{V}_{\text{phys}}$  invariant, i.e.,  $H(D(H) \cap \mathcal{V}_{\text{phys}}) \subset \mathcal{V}_{\text{phys}}$ , where  $D(H)$  means the domain of  $H$ . Moreover, we showed that, for all  $\Psi, \Psi' \in D(H) \cap \mathcal{V}_{\text{phys}}$ ,

$$\langle \Psi' | H\Psi \rangle = \langle \Psi' | [H_f^T + E_0]\Psi \rangle,$$

where  $H_f^T$  is the free Hamiltonian of the transverse photons and  $E_0 = 1/2 \int |\hat{\rho}(k)|^2/|k|^2 dk$ .

In this paper, we take the Fock representation as the time zero field. Our purpose is to define the physical Hamiltonian  $H_{\text{phys}}$  on the physical Hilbert space  $\mathcal{H}_{\text{phys}}$  from the Hamiltonian  $H$  consistently and prove the self-adjointness of  $H_{\text{phys}}$ . For simplicity, we assume that the source is static. Then, by the result in [8], one can define a reduced Hamiltonian  $H_{\mathcal{V}_{\text{phys}}}$  on  $\mathcal{V}_{\text{phys}}$  in the usual way. In general, for a bounded operator  $T$  on  $\mathcal{V}_{\text{phys}}$ , one can define a bounded operator  $T_{\text{phys}}$  on  $\mathcal{H}_{\text{phys}}$  by  $T_{\text{phys}}[\Psi]_{\text{phys}} = [T\Psi]_{\text{phys}}$  if  $T$  leaves  $\mathcal{V}_0$  invariant. Here we denote by  $[\Psi]_{\text{phys}}$  the element of  $\mathcal{H}_{\text{phys}}$  for a representative  $\Psi \in \mathcal{V}_{\text{phys}}$ . Since the reduced Hamiltonian  $H_{\mathcal{V}_{\text{phys}}}$  is, however, unbounded, the above definition is ill-defined although  $H_{\mathcal{V}_{\text{phys}}}$  leaves  $\mathcal{V}_0$  invariant. Indeed  $\Psi \in D(H_{\mathcal{V}_{\text{phys}}})$  and  $\Psi - \Psi' \in \mathcal{V}_0$  do not necessarily imply  $\Psi' \in D(H_{\mathcal{V}_{\text{phys}}})$ . We give a precise definition of  $H_{\text{phys}}$  and prove the self-adjointness of  $H_{\text{phys}}$ . As a by-product, we find that  $H_{\text{phys}}$  is unitarily equivalent to  $H_f^T + E_0$ .

This paper is organized as follows. In Section 2, we review the usual Boson Fock space and introduce an indefinite metric in the usual way. Through this paper we assume that the time zero fields are given by the Fock representation. Section 3 is devoted to characterize the physical subspace. We first solve the operator-valued Cauchy problem (1.1) in the case where the source depends on time. We define the positive frequency part of  $\partial^\mu A_\mu$  in a rigorous manner and characterize the physical subspace  $\mathcal{V}_{\text{phys}}$ . These results extend the results of [8]. Here again we encounter the infrared catastrophe. In Section 4, we first investigate the properties of the physical Hilbert space  $\mathcal{H}_{\text{phys}}$ . After that we give a precise definition of the physical Hamiltonian  $H_{\text{phys}}$  on  $\mathcal{H}_{\text{phys}}$  and prove the self-adjointness of  $H_{\text{phys}}$ .

## 2 Fock space and representations of the commutation relations

In this section we recall the Boson Fock space with an indefinite metric and define the Fock representation of the commutation relations (1.3) and (1.4) thereon.

In general we denote the inner product and the associated norm of a Hilbert space  $\mathcal{H}$  by  $\langle *, \cdot \rangle_{\mathcal{H}}$  and  $\| \cdot \|_{\mathcal{H}}$ , respectively. The inner product is linear in  $\cdot$  and antilinear in  $*$ . If there is no danger of confusion, we omit the subscript  $\mathcal{H}$  in  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  and  $\| \cdot \|_{\mathcal{H}}$ . For a linear operator  $T$  on  $\mathcal{H}$ , we denote the domain of a linear operator  $T$  by  $D(T)$  and, if  $D(T)$  is dense, the (Hilbert) adjoint of  $T$  by  $T^*$ .

### 2.1 Boson Fock space

We first recall the abstract Boson Fock space and operators therein. The Boson Fock space over  $\mathcal{H}$  is defined by

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \bigotimes_{\mathfrak{s}}^n \mathcal{H} = \left\{ \Psi = \{ \Psi^{(n)} \}_{n=0}^{\infty} \mid \Psi^{(n)} \in \bigotimes_{\mathfrak{s}}^n \mathcal{H}, \sum_{n=0}^{\infty} \| \Psi^{(n)} \|_{\bigotimes_{\mathfrak{s}}^n \mathcal{H}}^2 < \infty \right\},$$

where  $\otimes_s^n \mathcal{H}$  denotes the symmetric tensor product of  $\mathcal{H}$  with the convention  $\otimes_s^0 \mathcal{H} = \mathbb{C}$ .

The creation operator  $a^*(f)$  ( $f \in \mathcal{H}$ ) on  $\mathcal{F}(\mathcal{H})$  is defined by

$$(a^*(f)\Psi)^{(n)} := \sqrt{n} S_n \left( f \otimes \Psi^{(n-1)} \right)$$

with the domain

$$D(a^*(f)) := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \left| \sum_{n=0}^\infty n \left\| S_n \left( f \otimes \Psi^{(n-1)} \right) \right\|_{\otimes_s^n \mathcal{H}}^2 < \infty \right. \right\},$$

where  $S_n$  denotes the symmetrization operator on  $\otimes^n \mathcal{H}$  satisfying  $S_n = S_n^* = S_n^2$  and  $S_n(\otimes^n \mathcal{H}) = \otimes_s^n \mathcal{H}$ . The annihilation operator  $a(f)$  ( $f \in \mathcal{H}$ ) is defined by the adjoint of  $a^*(f)$ , i.e.,  $a(f) := a^*(f)^*$ . By definition,  $a^*(f)$  (resp.  $a(f)$ ) is linear (resp. antilinear) in  $f \in \mathcal{H}$ . As is well known, the creation and annihilation operators leave the finite particle subspace

$$\mathcal{F}_0(\mathcal{H}) = \bigcup_{m=1}^\infty \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^\infty \mid \Psi^{(n)} = 0, \quad n \geq m \right\}$$

invariant and satisfy the canonical commutation relations

$$[a(f), a^*(g)] = \langle f, g \rangle_{\mathcal{H}}, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0.$$

The Fock vacuum  $\Omega_{\mathcal{H}} = \{\Omega_{\mathcal{H}}^{(n)}\} \in \mathcal{F}(\mathcal{H})$  is defined by  $\Omega_{\mathcal{H}}^{(0)} = 1$  and  $\Omega_{\mathcal{H}}^{(n)} = 0$  ( $n \geq 1$ ) and satisfies

$$a(f)\Omega_{\mathcal{H}} = 0, \quad f \in \mathcal{H}. \quad (2.1)$$

It is well known that  $\Omega_{\mathcal{H}}$  is a unique vector satisfying (2.1) up to a constant factor.

Let  $c$  be a contraction operator on  $\mathcal{H}$ , i.e.,  $\|c\| \leq 1$ . We define a contraction operator  $\Gamma(c)$  on  $\mathcal{F}(\mathcal{H})$  by

$$(\Gamma(c)\Psi)^{(n)} = (\otimes^n c)\Psi^{(n)}, \quad \Psi = \{\Psi^{(n)}\}_{n=0}^\infty$$

with the convention  $\otimes^0 c = 1$ . If  $u$  is unitary, i.e.  $u^{-1} = u^*$ , then  $\Gamma(u)$  is also unitary and satisfies  $\Gamma(u)^* = \Gamma(u^*)$  and

$$\Gamma(u)a(f)\Gamma(u)^* = a(uf), \quad \Gamma(u)a^*(f)\Gamma(u)^* = a^*(uf).$$

For a self-adjoint operator  $h$  on  $\mathcal{H}$ , i.e.,  $h = h^*$ ,  $\{\Gamma(e^{ith})\}_{t \in \mathbb{R}}$  is a strongly continuous one-parameter unitary group on  $\mathcal{F}(\mathcal{H})$ . Then, by the Stone theorem, there exists a unique self-adjoint operator  $d\Gamma(h)$  such that

$$\Gamma(e^{ith}) = e^{itd\Gamma(h)}.$$

## 2.2 Indefinite metric space of states

We introduce the indefinite metric space of states in the usual way [9]. The Hilbert space of the one-photon states is given by

$$\mathfrak{h} = \bigoplus^4 L^2(\mathbb{R}^3; dk),$$

where  $k \in \mathbb{R}^3$  is the momentum of a photon. Let

$$\mathcal{F} = \mathcal{F}(\mathfrak{h})$$

be the Hilbert space of the photon field.

We introduce an indefinite metric to  $\mathcal{F}$  as follows. Let us define an operator  $g$  on  $\mathfrak{h}$  by

$$g(f_1, f_2, f_3, f_0) = (-f_1, -f_2, -f_3, f_0)$$

for  $(f_1, f_2, f_3, f_0) \in \mathfrak{h}$ . Then  $g$  is unitary, self-adjoint and hence involution, i.e.,

$$g^* = g^{-1} = g, \quad g^2 = 1.$$

The metric operator is defined by

$$\eta = \Gamma(-g).$$

Then  $\eta$  is also unitary, self-adjoint and involution. We define a metric  $\langle \cdot | \cdot \rangle$  on  $\mathcal{F}$  by

$$\langle \Psi | \Phi \rangle = \langle \Psi, \eta \Phi \rangle$$

for  $\Psi, \Phi \in \mathcal{F}$ . The metric space  $(\mathcal{F}, \langle \cdot | \cdot \rangle)$  is a Krein space (see [3]). We also denote by  $\mathcal{F}$  the Krein space  $(\mathcal{F}, \langle \cdot | \cdot \rangle)$ .

For a densely defined linear operator  $L$  on the Krein space  $\mathcal{F}$ , the adjoin operator  $L^\dagger$  with respect to the metric  $\langle \cdot | \cdot \rangle$  is given by

$$L^\dagger = \eta L^* \eta.$$

We say that  $L$  is  $\eta$ -selfadjoint if  $L^\dagger = L$ .

Let

$$a(f, \mu) = a((\delta_{1\mu} \bar{f}, \delta_{2\mu} \bar{f}, \delta_{3\mu} \bar{f}, \delta_{0\mu} \bar{f})), \quad f \in L^2(\mathbb{R}^3; dk)$$

and set  $a^\dagger(f, \mu) = a(\bar{f}, \mu)^\dagger$ , where  $\bar{f}$  is the complex conjugate of  $f$ . Then  $a(f, \mu)$  and  $a^\dagger(f, \mu)$  leave  $\mathcal{F}_0(\mathfrak{h})$  invariant and satisfy the following commutation relations on  $\mathcal{F}_0(\mathfrak{h})$ :

$$[a(f, \mu), a^\dagger(g, \nu)] = -g_{\mu\nu} \langle \bar{f}, g \rangle_{L^2(\mathbb{R}^3)}, \quad (2.2)$$

$$[a(f, \mu), a(g, \nu)] = [a^\dagger(f, \mu), a^\dagger(g, \nu)] = 0. \quad (2.3)$$

We often use the following symbolic notation by the kernel:

$$a(f, \mu) = \int_{\mathbb{R}^3} dk f(k) a(k, \mu), \quad a^\dagger(f, \mu) = \int_{\mathbb{R}^3} dk f(k) a^\dagger(k, \mu).$$

### 2.3 Free field in the Fock representation

For  $f \in \mathcal{S}(\mathbb{R}^3)$ , we define operator  $A_\mu^{(0)}(f)$  and  $\dot{A}_\mu^{(0)}(f)$  by

$$A_j^{(0)}(f) = \frac{1}{\sqrt{2}} \sum_{i=1}^3 \left[ a^\dagger \left( \frac{e_j^{(i)} \hat{f}}{\sqrt{\omega}}, i \right) + a \left( \frac{e_j^{(i)} \hat{f}(-\cdot)}{\sqrt{\omega}}, i \right) \right]$$

$$A_0^{(0)}(f) = \frac{1}{\sqrt{2}} \left[ a^\dagger \left( \frac{\hat{f}}{\sqrt{\omega}}, 0 \right) + a \left( \frac{\hat{f}(-\cdot)}{\sqrt{\omega}}, 0 \right) \right]$$

and

$$\dot{A}_j^{(0)}(f) = \frac{i}{\sqrt{2}} \sum_{i=1}^3 \left[ a^\dagger \left( \sqrt{\omega} e_j^{(i)} \hat{f}, i \right) - a \left( \sqrt{\omega} e_j^{(i)} \hat{f}(-\cdot), i \right) \right]$$

$$\dot{A}_0^{(0)}(f) = \frac{i}{\sqrt{2}} \left[ a^\dagger \left( \sqrt{\omega} \hat{f}, 0 \right) - a \left( \sqrt{\omega} \hat{f}(-\cdot), 0 \right) \right],$$

where  $\omega(k) = |k|$  is the single photon energy of the wave vector  $k \in \mathbb{R}^3$  and  $e^{(i)}(k) \in \mathbb{R}^3$  ( $i = 1, 2, 3$ ) the polarization vectors satisfying

$$e_j^{(3)}(k) = \frac{k_j}{|k|}, \quad j = 1, 2, 3,$$

$$\sum_{j=1}^3 e_j^{(i)}(k) e_j^{(l)}(k) = \delta_{il}, \quad i, l = 1, 2, 3.$$

It follows from (2.2) and (2.3) that  $\{A_\mu(f), \dot{A}_\mu(f) \mid f \in \mathcal{S}(\mathbb{R}^3)\}$  gives a representation of commutation relations, i.e.,  $A_\mu(f)$  and  $\dot{A}_\mu(f)$  leave  $\mathcal{F}_0(\mathfrak{h})$  invariant and satisfy the following commutation relations on  $\mathcal{F}_0(\mathfrak{h})$ :

$$[A_\mu^{(0)}(f), \dot{A}_\nu^{(0)}(g)] = -ig_{\mu\nu} \langle \bar{f}, g \rangle_{L^2(\mathbb{R}^3)}$$

$$[A_\mu^{(0)}(f), A_\nu^{(0)}(g)] = [\dot{A}_\mu^{(0)}(f), \dot{A}_\nu^{(0)}(g)] = 0.$$

The maps  $\mathcal{S}(\mathbb{R}^3) \ni f \mapsto A_\mu^{(0)}(f)$  and  $\mathcal{S}(\mathbb{R}^3) \ni f \mapsto \dot{A}_\mu^{(0)}(f)$  are operator-valued distributions acting on  $\mathcal{F}_0(\mathfrak{h})$ , i.e., for all  $\Psi, \Phi \in \mathcal{F}_0(\mathfrak{h})$ , the map  $f \mapsto \langle \Psi \mid A_\mu^{(0)}(f, t) \Phi \rangle$  is a tempered distribution. We call the representation  $\{A_\mu(f), \dot{A}_\mu(f) \mid f \in \mathcal{S}(\mathbb{R}^3)\}$  the Fock representation of the abnormal commutation relations. Let

$$A_\mu(f, t) = e^{itH_f} A_\mu(f) e^{-itH_f}, \quad \dot{A}_\mu(f, t) = e^{itH_f} \dot{A}_\mu(f) e^{-itH_f},$$

where  $H_f$  is the free Hamiltonian defined by

$$H_f = d\Gamma \left( \bigoplus^4 \omega \right).$$

**Proposition 2.1.** *The following (i)-(iv) hold:*

(i) *For each  $t \in \mathbb{R}$ , the maps  $\mathcal{S}(\mathbb{R}^3) \ni f \mapsto A_\mu^{(0)}(f, t)$  and  $\mathcal{S}(\mathbb{R}^3) \ni f \mapsto \dot{A}_\mu^{(0)}(f, t)$  are operator-valued distributions acting on  $\mathcal{F}_0(\mathfrak{h})$ , i.e., for all  $\Psi, \Phi \in \mathcal{F}_0(\mathfrak{h})$ , the map  $f \mapsto \langle \Psi \mid A_\mu^{(0)}(f, t) \Phi \rangle$  is a tempered distribution.*

(ii) *For each  $t \in \mathbb{R}$  and  $f, g \in \mathcal{S}(\mathbb{R}^3)$ , the following commutation relations hold on  $\mathcal{F}_0(\mathfrak{h})$ .*

$$[A_\mu^{(0)}(f, t), \dot{A}_\nu^{(0)}(g, t)] = -ig_{\mu\nu} \langle \bar{f}, g \rangle_{L^2(\mathbb{R}^3)},$$

$$[A_\mu^{(0)}(f, t), A_\nu^{(0)}(g, t)] = [\dot{A}_\mu^{(0)}(f, t), \dot{A}_\nu^{(0)}(g, t)] = 0$$

(iii) *For all  $\Psi \in \mathcal{F}_0(\mathfrak{h})$ ,  $A_\mu^{(0)}(f, t)\Psi$  and  $\dot{A}_\mu^{(0)}(f, t)\Psi$  are strongly differentiable and satisfy*

$$\frac{d}{dt} A_\mu^{(0)}(f, t)\Psi = \dot{A}_\mu^{(0)}(f, t)\Psi, \quad \frac{d}{dt} \dot{A}_\mu^{(0)}(f, t)\Psi = A_\mu^{(0)}(\Delta f, t)\Psi.$$

*In particular,  $A_\mu^{(0)}(f, t)\Psi$  is twice differentiable and*

$$\frac{d^2}{dt^2} A_\mu^{(0)}(f, t)\Psi = A_\mu^{(0)}(\Delta f, t)\Psi. \quad (2.4)$$

In the sense of the above proposition, we write symbolically

$$A_\mu^{(0)}(f, t) = \int_{\mathbb{R}^3} dx f(x) A_\mu^{(0)}(x, t).$$

Formally the kernels  $A_\mu^{(0)}(x, t)$  and  $\dot{A}_\mu^{(0)}(x, t)$  are given by

$$A_j^{(0)}(x, t) = \sum_{i=1}^3 \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{dk}{\sqrt{2\omega(k)}} e_j^{(i)}(k) \left( a^\dagger(k, i) e^{i\omega(k)t - ik \cdot x} + a(k, i) e^{-i\omega(k)t + ik \cdot x} \right)$$

$$A_0^{(0)}(x, t) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \frac{dk}{\sqrt{2\omega(k)}} \left( a^\dagger(k, 0) e^{i\omega(k)t - ik \cdot x} + a(k, 0) e^{-i\omega(k)t + ik \cdot x} \right)$$

and

$$\dot{A}_j^{(0)}(x, t) = i \sum_{i=1}^3 \int_{\mathbb{R}^3} dk \sqrt{\frac{\omega(k)}{2(2\pi)^3}} e_j^{(i)}(k) \left( e^{i\omega(k)t - ik \cdot x} a^\dagger(k, i) - e^{-i\omega(k)t + ik \cdot x} a(k, i) \right),$$

$$\dot{A}_0^{(0)}(x, t) = i \int_{\mathbb{R}^3} dk \sqrt{\frac{\omega(k)}{2(2\pi)^3}} \left( e^{i\omega(k)t - ik \cdot x} a^\dagger(k, 0) - e^{-i\omega(k)t + ik \cdot x} a(k, 0) \right).$$

Thus we have the unique solution  $A_\mu^{(0)}(x, t)$  of the following Cauchy problem in the operator-valued distribution sense:

$$\begin{aligned} \frac{d^2}{dt^2} A_\mu^{(0)}(x, t) - \Delta A_\mu^{(0)}(x, t) &= 0 \\ A^{(0)}(x, 0) &= A^{(0)}(x) \\ \dot{A}^{(0)}(x, 0) &= \dot{A}^{(0)}(x). \end{aligned}$$

We set  $A_\mu^{(0)}(x) = A_\mu^{(0)}(x, 0)$  and  $\dot{A}_\mu^{(0)}(x) = \dot{A}_\mu^{(0)}(x, 0)$ .

### 3 Interaction field and the physical subspace

#### 3.1 Interaction field

For the technical simplicity, we assume that  $j_\mu \in \mathcal{S}(\mathbb{R}^4)$ . Our first task is to solve the following operator-valued Cauchy problem:

$$\begin{aligned} \frac{d^2}{dt^2} A_\mu(x, t) - \Delta A_\mu(x, t) &= j_\mu(x, t) \\ A_\mu(x, 0) &= A_\mu^{(0)}(x, 0) \\ \frac{d}{dt} A_\mu(x, t) \Big|_{t=0} &= \dot{A}_\mu^{(0)}(x, 0). \end{aligned}$$

**Proposition 3.1.** *The unique solution of the above Cauchy problem is given explicitly by*

$$A_\mu(x, t) = A_\mu^{(0)}(x, t) + A_\mu^{(\text{cl})}(x, t),$$

where

$$A_\mu^{(\text{cl})}(x, t) = \int_0^t d\tau \frac{\sin(t - \tau)\omega}{\omega} j_\mu(\tau, x)$$

and  $\omega$  is the self-adjoint operator satisfying  $\omega^2 = -\Delta$ .

Let

$$\dot{A}_\mu(x, t) = \dot{A}_\mu^{(0)}(x, t) + \dot{A}_\mu^{(\text{cl})}(x, t),$$

where

$$\dot{A}_\mu^{(\text{cl})}(x, t) = \int_0^t d\tau \cos(t - \tau)\omega j_\mu(\tau, x).$$

**Remark 3.1.**  $A_\mu(x, t)$  and  $\dot{A}_\mu(x, t)$  satisfy the Heisenberg equations

$$\begin{aligned}\frac{d}{dt}A_\mu(x, t) &= i[H(t), A_\mu(x, t)] = \dot{A}_\mu(x, t) \\ \frac{d}{dt}\dot{A}_\mu(x, t) &= i[H(t), \dot{A}_\mu(x, t)]\end{aligned}$$

on  $\mathcal{F}_0 \cap D(H(t))$ , where  $H(t)$  is the Hamiltonian of this system given formally by

$$\begin{aligned}H(t) &= \frac{1}{2} \int_{\mathbb{R}^3} dx : \left[ \sum_{j=1}^3 \dot{A}_j(x, t)^2 + (\nabla A_j(x, t))^2 \right. \\ &\quad \left. - \dot{A}_0(x, t)^2 - (\nabla A_0(x, t))^2 \right] : + \int_{\mathbb{R}^3} dx j^\mu(x, t) A_\mu(x, t).\end{aligned}$$

Here  $: \dots :$  denotes the Wick ordering.

### 3.2 Positive frequency part and Physical subspace

In what follows, we solve the Gupta subsidiary condition for the interaction field  $A_\mu$  and characterize the physical subspace. To this end, we define the positive frequency part of  $\partial^\mu A_\mu$  in a rigorous manner. Let

$$B(f, t) := \dot{A}_0(f, t) + \sum_{j=1}^3 A_j(\partial_{x_j} f, t), \quad \dot{B}(f, t) := \frac{d}{dt} B(f, t)$$

and set

$$\begin{aligned}b(h) &:= i \left( \dot{B}(g_s, s) - B(\dot{g}_s, s) \right) \\ b^\dagger(h) &:= b(\bar{h})^\dagger,\end{aligned}$$

where the function  $g_s$  is defined by

$$\hat{g}_s(k) = \frac{h(-k)}{2\omega(k)} e^{is\omega(k)} \quad (3.1)$$

and  $\dot{g}_s$  denotes the derivative of  $g_s$  with respect to  $s$ . Since, by the charge conservation law  $\partial^\mu j_\mu = 0$ ,  $B$  is a free field, i.e.,

$$\frac{d^2}{dt^2} B(f, t) \Psi = B(\Delta f, t) \Psi, \quad \Psi \in \mathcal{F}_0(\mathfrak{h}),$$

$b(h)$  and  $b^\dagger(h)$  are independent of  $s \in \mathbb{R}$ . It follows from direct calculation that

$$B(f, t) = b(e^{-it\omega} \hat{f}(-\cdot)) + b^\dagger(e^{it\omega} \hat{f}).$$

We call  $b(e^{-it\omega} \hat{f}(-\cdot))$  (resp.  $b^\dagger(e^{it\omega} \hat{f})$ ) the positive frequency part (resp. the negative frequency part) of  $B(f, t)$  and write

$$B^{(+)}(f, t) = b(e^{-it\omega} \hat{f}(-\cdot)), \quad B^{(-)}(f, t) = b^\dagger(e^{it\omega} \hat{f}).$$

We define the physical subspace by

$$\mathcal{V}_{\text{phys}} = \{ \Psi \in \mathcal{F} \mid b(h)\Psi = 0, h \in \mathcal{S}(\mathbb{R}^3) \}.$$

Then we have

$$\langle \Psi \mid B(f, t) \Phi \rangle = 0, \quad \Psi, \Phi \in \mathcal{V}_{\text{phys}}.$$

To characterize  $\mathcal{V}_{\text{phys}}$ , we define a unitary operator  $U$  by

$$U = \exp \left[ -\frac{1}{\sqrt{2}} \left( a^\dagger \left( \frac{\hat{j}_0(\cdot, 0)}{\omega^{3/2}}, 3 \right) - a \left( \frac{\hat{j}_0(\cdot, 0)}{\omega^{3/2}}, 3 \right) \right) \right] W,$$

where  $W$  is a unitary operator defined by the following relations:

$$\begin{aligned} W\Omega &= \Omega, \\ Wa(f, j)W &= a(f, j), \quad j = 1, 2, \\ Wa(f, 3)W &= \frac{1}{\sqrt{2}}[a(f, 3) + a(f, 0)], \\ Wa(f, 0)W &= \frac{1}{\sqrt{2}}[a(f, 3) - a(f, 0)]. \end{aligned}$$

**Lemma 3.2.** *Let  $\omega^{-3/2}\hat{j}_0(0, \cdot) \in L^2(\mathbb{R}^3)$ . Then  $U$  For all  $h \in \mathcal{S}(\mathbb{R}^3)$ , the following holds:*

$$U^{-1}b(h)U = ia(\sqrt{\omega}h, 0).$$

*Proof.* By direct calculation, we have

$$b(h) = \frac{i}{\sqrt{2}} \left[ a(\sqrt{\omega}h, 3) - a(\sqrt{\omega}h, 0) + \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} dk h(k) \frac{\hat{j}_0(k, 0)}{\omega(k)} \right],$$

where we have used the equation (1.2) and integration by parts. □

Similar as in paper [8], we have the following theorem.

**Theorem 3.3.** (i) *If  $\omega^{-3/2}\hat{j}_0(\cdot, 0) \in L^2(\mathbb{R}^3; dk)$ , then  $\mathcal{V}_{\text{phys}}$  is positive semi-definite and*

$$\mathcal{V}_{\text{phys}} = U\mathcal{F}_{\text{TL}},$$

where  $\mathcal{F}_{\text{TL}} = \mathcal{F}(\oplus^3 L^2(\mathbb{R}^3)) \otimes \{\alpha\Omega \mid \alpha \in \mathbb{C}\}$ .

(ii) *If  $\omega^{-3/2}\hat{j}_0(\cdot, 0) \notin L^2(\mathbb{R}^3; dk)$ , then  $\mathcal{V}_{\text{phys}}$  is trivial,*

$$\mathcal{V}_{\text{phys}} = \{0\}.$$

## 4 Physical Hilbert space and the physical Hamiltonian

### 4.1 Physical Hilbert space

In the rest of this section, we assume that  $\omega^{-3/2}\hat{\rho} \in L^2(\mathbb{R}^3; dk)$ . Then, by Theorem 3.3, the physical subspace  $\mathcal{V}_{\text{phys}}$  is closed and given by  $\mathcal{V}_{\text{phys}} = U\mathcal{F}_{\text{TL}} \neq \{0\}$ .

Let  $\mathcal{F}_{\text{T}} = \mathcal{F}(\oplus^2 L^2(\mathbb{R}^3)) \otimes \{\alpha\Omega_{\oplus^2 L^2(\mathbb{R}^3)} \mid \alpha \in \mathbb{C}\}$ . Then the physical subspace  $\mathcal{V}_{\text{phys}}$  is decomposed into

$$\mathcal{V}_{\text{phys}} = \mathcal{V}_1 \oplus \mathcal{V}_0, \tag{4.1}$$

where

$$\mathcal{V}_1 = U\mathcal{F}_{\text{T}}, \quad \mathcal{V}_0 = U[\mathcal{F}_{\text{T}}^\perp \cap \mathcal{F}_{\text{TL}}].$$



**Lemma 4.1.** (1)  $\mathcal{V}_1$  is closed and satisfies

$$\langle \Psi_1 | \Psi'_1 \rangle = \langle \Psi_1, \Psi'_1 \rangle, \quad \Psi_1, \Psi'_1 \in \mathcal{V}_1.$$

In particular,  $\mathcal{V}_1$  is positive definite.

(2)  $\mathcal{V}_0$  is closed and

$$\mathcal{V}_0 = \{ \Psi_0 \in \mathcal{V}_{\text{phys}} \mid \langle \Psi_0 | \Psi'_0 \rangle = 0 \}.$$

*Proof.* See the proof of [8, Theorem 2.17]. □

For subspaces  $\mathcal{U}$ ,  $\mathcal{V}$  and  $\mathcal{X}$  in  $\mathcal{F}$ ,  $\mathcal{X} = \mathcal{V} [+ ] \mathcal{U}$  stands for the orthogonal direct sum with respect to the metric  $\langle \cdot | \cdot \rangle$ , i.e., the following hold: (1) for all  $x \in \mathcal{X}$ , there exist unique vectors  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$  such that  $x = u + v$ , (2) for all  $u \in \mathcal{U}$  and  $v \in \mathcal{V}$ ,  $\langle u | v \rangle = 0$  holds.

**Lemma 4.2.** It follows that

$$\mathcal{V}_{\text{phys}} = \mathcal{V}_1 [+ ] \mathcal{V}_0.$$

*Proof.* Since  $\mathcal{V}_{\text{phys}}$  is positive semi-definite with respect to the metric  $\langle \cdot | \cdot \rangle$ , we have, by the Schwarz inequality, for all  $\Psi_1 \in \mathcal{V}_1$  and  $\Psi_0 \in \mathcal{V}_0$ ,  $|\langle \Psi_1 | \Psi_0 \rangle|^2 \leq \langle \Psi_1 | \Psi_1 \rangle \langle \Psi_0 | \Psi_0 \rangle = 0$ . Hence we have  $\langle \Psi_1 | \Psi_0 \rangle = 0$ , which, together with (4.1), we obtain the desired result. □

Let us define the physical Hilbert space  $\mathcal{H}_{\text{phys}}$  by the quotient space  $\mathcal{V}_{\text{phys}}/\mathcal{V}_0$ . We denote by  $[\Psi]_{\text{phys}}$  the element of  $\mathcal{H}_{\text{phys}}$  for the representative  $\Psi \in \mathcal{V}_{\text{phys}}$  and by  $\langle \cdot, \cdot \rangle_{\text{phys}}$  the inner product of  $\mathcal{H}_{\text{phys}}$ :

$$\langle [\Psi]_{\text{phys}}, [\Psi']_{\text{phys}} \rangle_{\text{phys}} = \langle \Psi | \Psi' \rangle, \quad \Psi, \Psi' \in \mathcal{V}_{\text{phys}}.$$

**Lemma 4.3.**  $\mathcal{H}_{\text{phys}}$  is unitarily equivalent to  $\mathcal{V}_1$ .

*Proof.* The map  $T_1 : \mathcal{V}_1 \longrightarrow \mathcal{H}_{\text{phys}}$  defined by

$$T_1 \Psi_1 = [\Psi_1]_{\text{phys}}, \quad \Psi_1 \in \mathcal{V}_1$$

is isometrically isomorphism, i.e.,  $T_1$  is bijective and satisfies, for all  $\Psi_1, \Psi'_1 \in \mathcal{V}_1$ ,

$$\langle \Psi_1 | \Psi'_1 \rangle = \langle T_1 \Psi_1, T_1 \Psi'_1 \rangle_{\text{phys}}. \quad (4.2)$$

By Lemma 4.1 (1), the left hand side in the above equation is equal to  $\langle \Psi_1, \Psi'_1 \rangle$ . Thus the proof is complete. □

## 4.2 Physical Hamiltonian

In the rest of this section, mainly for simplicity, we assume that the external source is static, i.e.,  $j_0(x, t) = \rho(x)$  and  $j_i \equiv 0$ . Then the Hamiltonian of this system is time independent and given by

$$H = H_f + A_0^{(0)}(\rho).$$

In [8], we prove that  $H$  is  $\eta$ -self-adjoint on  $D(H) = D(H_f)$ . Our task is to define the physical Hamiltonian  $H_{\text{phys}}$  on the physical Hilbert space  $\mathcal{H}_{\text{phys}} = \mathcal{V}_{\text{phys}}/\mathcal{V}_0$  consistently and prove its self-adjointness. Let  $P_{\text{phys}}$  be the orthogonal projection onto  $\mathcal{V}_{\text{phys}}$ , i.e.,  $P_{\text{phys}} = P_{\text{phys}}^* = P_{\text{phys}}^2$ . We first note that,  $H$  is not reduced by  $\mathcal{V}_{\text{phys}}$ , i.e.,  $P_{\text{phys}}H \not\subset HP_{\text{phys}}$ . The following lemma is a good starting point for our problem.

**Lemma 4.4.** (1)  $P_{\text{phys}}$  leaves  $D(H)$  invariant, i.e.,  $P_{\text{phys}}D(H) \subset D(H)$ .

(2)  $H$  leaves  $\mathcal{V}_{\text{phys}}$  invariant, i.e.,  $H(D(H) \cap \mathcal{V}_{\text{phys}}) \subset \mathcal{V}_{\text{phys}}$ .

*Proof.* Let  $P_{\text{TL}} = \Gamma(1 \oplus 1 \oplus 1 \oplus 0)$  be the orthogoanl projection onto  $\mathcal{F}_{\text{T}}$ . Then  $P_{\text{phys}}$  is given by

$$P_{\text{phys}} = UP_{\text{TL}}U^{-1}.$$

Since  $U, U^{-1}, P_{\text{TL}}$  leaves  $D(H_{\text{f}})$  invariant and  $D(H) = D(H_{\text{f}})$ , we conclude (1).

Let  $\Psi \in D(H) \cap \mathcal{V}_{\text{phys}}$ . Then there exists a vector  $\Phi \in \mathcal{F}_{\text{TL}}$  such that  $\Phi = U^{-1}\Psi \in D(H)$ . Hence we have

$$\begin{aligned} H\Psi &= HP_{\text{phys}}\Psi = U \cdot U^{-1}HU \cdot P_{\text{TL}}U^{-1}\Psi \\ &= U[H_{\text{f}} - a^*(\hat{\rho}/\sqrt{\omega}, 3) - a(\hat{\rho}(\cdot)/\sqrt{\omega}, 0) + E_0]\Phi \\ &= U\hat{H}\Phi \in \mathcal{V}_{\text{phys}}, \end{aligned} \quad (4.3)$$

where

$$\hat{H} = H_{\text{f}} - a^*(\hat{\rho}/\sqrt{\omega}, 3) + E_0, \quad (4.4)$$

$$E_0 = \frac{1}{2} \int dk \frac{|\hat{\rho}(k)|^2}{\omega(k)^2}. \quad (4.5)$$

Thus we have the desired result.  $\square$

By the above lemma, one can define a reduced operator  $H_{\mathcal{V}_{\text{phys}}}$  of  $H$  on  $\mathcal{V}_{\text{phys}}$  as follows:

$$\begin{aligned} D(H_{\mathcal{V}_{\text{phys}}}) &= D(H) \cap \mathcal{V}_{\text{phys}} \\ H_{\mathcal{V}_{\text{phys}}}\Psi &= H\Psi, \quad \Psi \in D(H_{\mathcal{V}_{\text{phys}}}). \end{aligned}$$

Since  $H$  is densely defined and closed on  $\mathcal{F}$ , it follows from Lemma 4.4 that  $H_{\mathcal{V}_{\text{phys}}}$  is also densely defined and closed on  $\mathcal{V}_{\text{phys}}$ . The following lemma follows from (4.3).

**Lemma 4.5.** *The operator  $\hat{H}$  defined by (4.4) is closed on  $D(\hat{H}) = D(H_{\text{f}})$  and satisfies*

$$H_{\mathcal{V}_{\text{phys}}}\Psi = U\hat{H}U^{-1}\Psi, \quad \Psi \in D(H_{\mathcal{V}_{\text{phys}}}). \quad (4.6)$$

We consider the resolvent of  $H_{\mathcal{V}_{\text{phys}}}$ . In general we denote by  $\rho(A)$  the resolvent set of a linear operator  $A$ . Since the creation operators are infinitesimally small with respect to  $H_{\text{f}}$ , we observe that  $\rho(\hat{H}) \neq \emptyset$ . By (4.6) we have the following lemma:

**Lemma 4.6.** *It follows that  $\rho(\hat{H}) \subset \rho(H_{\mathcal{V}_{\text{phys}}})$  and that the resolvent of  $H_{\mathcal{V}_{\text{phys}}}$  at  $z \in \rho(\hat{H})$  is given by*

$$(H_{\mathcal{V}_{\text{phys}}} - z)^{-1} = U(\hat{H} - z)^{-1}U^{-1}.$$

Let

$$\mathcal{R} = \{z \in \rho(H_{\text{f}} + E_0) \mid 2\epsilon + b(\epsilon)/|E_0 - z| < 1 \text{ with some } \epsilon > 0\},$$

where

$$b(\epsilon) = \frac{\|\hat{\rho}/\omega\|}{2\epsilon} + \frac{\|\hat{\rho}/\sqrt{\omega}\|}{\sqrt{2}}.$$

For all  $z \in \mathcal{R}$ , it follows that  $\|a^*(\hat{\rho}/\sqrt{\omega}, 3)(H_{\text{f}} + E_0 - z)^{-1}\| < 1$  and hence that  $z \in \rho(\hat{H})$  and

$$(\hat{H} - z)^{-1} = \sum_{n=0}^{\infty} (H_{\text{f}} + E_0 - z)^{-1} [a^*(\hat{\rho}/\sqrt{\omega}, 3)(H_{\text{f}} + E_0 - z)^{-1}]^n. \quad (4.7)$$

**Lemma 4.7.** *Let  $z \in \mathcal{R}$ . Then  $(H_{\mathcal{V}_{\text{phys}}} - z)^{-1}$  leaves  $\mathcal{V}_0$  invariant.*

*Proof.* By Lemma 4.6 and the equation (4.7), we have, for all  $\Psi_0 = U\Phi_0 \in \mathcal{V}_0$ ,

$$(H_{\mathcal{V}_{\text{phys}}} - z)^{-1}\Psi_0 = U \sum_{n=0}^{\infty} (H_{\text{f}} + E_0 - z)^{-1} [a^*(\hat{\rho}/\sqrt{\omega}, 3)(H_{\text{f}} + E_0 - z)^{-1}]^n \Phi_0 \in \mathcal{V}_0.$$

The proof is complete.  $\square$

Let us fix  $z_0 \in \mathcal{R}$  and set

$$R_0 := (H_{\mathcal{V}_{\text{phys}}} - z_0)^{-1}.$$

Then, by the above lemma, the following operator  $[R_0]_{\text{phys}}$  on  $\mathcal{H}_{\text{phys}}$  is well-defined:

$$[R_0]_{\text{phys}}[\Psi]_{\text{phys}} = [R_0\Psi]_{\text{phys}}, \quad [\Psi]_{\text{phys}} \in \mathcal{H}_{\text{phys}}.$$

**Lemma 4.8.** (1)  $[R_0]_{\text{phys}}$  is bounded and  $\|[R_0]_{\text{phys}}\| \leq \|R_0\|$ .

(2)  $[R_0]_{\text{phys}}$  is injective and  $[R_0]_{\text{phys}}^{-1}$  is closed.

*Proof.* (1) follows from Lemma 4.1 and the equation (4.2). By the boundedness of  $[R_0]_{\text{phys}}$ ,  $[R_0]_{\text{phys}}^{-1}$  is closed if  $[R_0]_{\text{phys}}$  is injective. We need only to prove the injectivity on  $[R_0]_{\text{phys}}$ . Let  $[R_0]_{\text{phys}}[\Psi]_{\text{phys}} = 0$ . Then  $\langle R_0\Psi | R_0\Psi \rangle = 0$  and hence  $R_0\Psi \in \mathcal{V}_0$ . Hence there exists a vector  $\Phi_0 \in \mathcal{F}_{\text{T}}^{\perp} \cap \mathcal{F}_{\text{TL}}$  such that  $U^{-1}R_0\Psi = \Phi_0$ . By (4.6), we have

$$\Psi = (H_{\mathcal{V}_{\text{phys}}} - z_0)R_0\Psi = U \sum_{n=0}^{\infty} (H_{\text{f}} + E_0 - z_0)^{-1} [a^*(\hat{\rho}/\sqrt{\omega}, 3)(H_{\text{f}} + E_0 - z_0)^{-1}]^n \Phi_0 \in \mathcal{V}_0,$$

which implies that  $[\Psi]_{\text{phys}} = 0$ . Thus the proof is complete.  $\square$

Let us define the physical Hamiltonian  $H_{\text{phys}}$  by

$$H_{\text{phys}} := z_0 + [R_0]_{\text{phys}}^{-1}.$$

By Lemma 4.8,  $H_{\text{phys}}$  is closed on  $\mathcal{V}_{\text{phys}}$ . Our next task is to prove that the definition of  $H_{\text{phys}}$  is independent of the choice of  $z_0 \in \mathcal{R}$ . To this end, we seek another expression of  $H_{\text{phys}}$ . Let  $P_1$  be the orthogonal projection onto  $\mathcal{V}_1$  given by

$$P_1 = UP_{\text{T}}U^{-1},$$

where  $P_{\text{T}} = \Gamma(1 \oplus 1 \oplus 0 \oplus 0)$  is the orthogonal projection onto  $\mathcal{F}_{\text{T}}$ . We set

$$\mathcal{D}_1 := \{[\Psi]_{\text{phys}} \in \mathcal{H}_{\text{phys}} \mid P_1\Psi \in D(H_{\mathcal{V}_{\text{phys}}})\}.$$

Since  $[\Psi]_{\text{phys}} = [\Psi']_{\text{phys}}$  implies that  $P_1\Psi = P_1\Psi'$ , the right hand side of the above equation is independent of the choice of the representative. Since  $P_1$  leaves  $D(H_{\mathcal{V}_{\text{phys}}})$  invariant, it follows that  $\mathcal{D}_1$  is dense in  $\mathcal{H}_{\text{phys}}$ . The following proposition implies that  $H_{\text{phys}}$  is densely defined and independent of the choice of  $z_0 \in \mathcal{R}$ .

**Proposition 4.9.** (1)  $D(H_{\text{phys}}) = \mathcal{D}_1$ .

(2) For all  $[\Psi]_{\text{phys}} \in D(H_{\text{phys}})$ ,  $H_{\text{phys}}[\Psi]_{\text{phys}} = [H_{\mathcal{V}_{\text{phys}}}P_1\Psi]_{\text{phys}}$ .

*Proof.* Let  $[\Psi]_{\text{phys}} \in \mathcal{D}_1$ . Then we have  $P_1\Psi \in D(H_{\mathcal{V}_{\text{phys}}})$  and set  $\Phi := (H_{\mathcal{V}_{\text{phys}}} - z_0)P_1\Psi \in D((H_{\mathcal{V}_{\text{phys}}} - z_0)^{-1})$ . By direct calculation, we have

$$[\Psi]_{\text{phys}} = [P_1\Psi]_{\text{phys}} = [(H_{\mathcal{V}_{\text{phys}}} - z_0)^{-1}\Phi]_{\text{phys}} = [R_0]_{\text{phys}}[\Phi]_{\text{phys}} \in D([R_0]_{\text{phys}}^{-1})$$

and hence  $[\Psi]_{\text{phys}} \in D(H_{\text{phys}})$ . Conversely, setting  $[\Psi]_{\text{phys}} \in D(H_{\text{phys}}) = D([R_0]_{\text{phys}}^{-1})$ , there exists a vector  $[\Phi]_{\text{phys}} \in \mathcal{H}_{\text{phys}}$  such that  $[\Psi]_{\text{phys}} = [R_0]_{\text{phys}}[\Phi]_{\text{phys}} = [R_0\Phi]_{\text{phys}}$ . By the fact  $P_1$  leaves  $D(H_{\mathcal{V}_{\text{phys}}})$  invariant, we have  $P_1\Psi = P_1R_0\Phi \in D(H_{\mathcal{V}_{\text{phys}}})$  and hence  $[\Psi]_{\text{phys}} \in \mathcal{D}_1$ . Thus we conclude (1).

Let  $[\Psi]_{\text{phys}} \in D(H_{\text{phys}}) = \mathcal{D}_1$ . Then it follows from the above discussion that  $P_1\Psi \in D(H_{\mathcal{V}_{\text{phys}}})$  and that there exists a vector  $[\Phi]_{\text{phys}} \in \mathcal{H}_{\text{phys}}$  such that  $[\Psi]_{\text{phys}} = [R_0]_{\text{phys}}[\Phi]_{\text{phys}}$ . Since we have

$$H_{\text{phys}}[\Psi]_{\text{phys}} - [H_{\mathcal{V}_{\text{phys}}}P_1\Psi]_{\text{phys}} = [z_0\Psi + \Phi - H_{\mathcal{V}_{\text{phys}}}P_1\Psi]_{\text{phys}},$$

we need only to prove  $z_0\Psi + \Phi - H_{\mathcal{V}_{\text{phys}}}P_1\Psi \in \mathcal{V}_0$ . Indeed, by  $P_1\Psi - R_0\Phi \in \mathcal{V}_0 \cap D(H_{\mathcal{V}_{\text{phys}}})$ , we have

$$z_0\Psi + \Phi - H_{\mathcal{V}_{\text{phys}}}P_1\Psi = z_0(1 - P_1)\Psi - (H_{\mathcal{V}_{\text{phys}}} - z_0)(P_1\Psi - R_0\Phi) \in \mathcal{V}_0.$$

Thus the proof is complete.  $\square$

As is shown in Lemma 4.3, the physical Hilbert space  $\mathcal{H}_{\text{phys}}$  is unitarily equivalent to  $\mathcal{V}_1$  and  $T_1\mathcal{V}_1 = \mathcal{H}_{\text{phys}}$ . In order to prove that the self-adjointness of the physical Hamiltonian  $H_{\text{phys}}$ , we prepare the following lemma:

**Lemma 4.10.** *The following operator equation holds:*

$$H_{\text{phys}} = T_1P_1H_{\mathcal{V}_{\text{phys}}}P_1T_1^{-1}.$$

*Proof.* Let  $[\Psi]_{\text{phys}} \in D(H_{\text{phys}})$ . Then we have  $P_1\Psi = T_1^{-1}[P_1\Psi]_{\text{phys}} = T_1^{-1}[\Psi]_{\text{phys}} \in D(H_{\mathcal{V}_{\text{phys}}})$  and

$$H_{\text{phys}}[\Psi]_{\text{phys}} = [P_1H_{\mathcal{V}_{\text{phys}}}P_1\Psi]_{\text{phys}} = T_1P_1H_{\mathcal{V}_{\text{phys}}}P_1T_1^{-1}[\Psi]_{\text{phys}}.$$

Thus we obtain  $H_{\text{phys}} \subset T_1P_1H_{\mathcal{V}_{\text{phys}}}P_1T_1^{-1}$ . Conversely, setting  $[\Psi]_{\text{phys}} \in D(T_1P_1H_{\mathcal{V}_{\text{phys}}}P_1T_1^{-1})$ , we have  $P_1\Psi = P_1T_1^{-1}[\Psi]_{\text{phys}} \in D(H_{\mathcal{V}_{\text{phys}}})$  and hence  $[\Psi]_{\text{phys}} \in D(H_{\text{phys}})$ . Then we have

$$T_1P_1H_{\mathcal{V}_{\text{phys}}}P_1T_1^{-1}[\Psi]_{\text{phys}} = [P_1H_{\mathcal{V}_{\text{phys}}}P_1\Psi]_{\text{phys}} = H_{\text{phys}}[\Psi]_{\text{phys}}.$$

Thus we obtain the desired result.  $\square$

By the above lemma, we observe that  $H_{\text{phys}}$  is self-adjoint if and only if  $P_1H_{\mathcal{V}_{\text{phys}}}P_1$  is self-adjoint. Indeed, by direct calculation, we have

$$P_1H_{\mathcal{V}_{\text{phys}}}P_1 = P_1U\hat{H}U^{-1}P_1 = U[H_f^T + E_0]P_TU^{-1},$$

where  $H_f^T = d\Gamma(\oplus^2\omega) \otimes I$ . Thus we have the following theorem:

**Theorem 4.11.** (1)  $H_{\text{phys}}$  is self-adjoint and bounded below.

(2)  $H_{\text{phys}}$  has a unique ground state  $[U\Omega]_{\text{phys}}$  with the ground state energy  $E_0$ :

$$H_{\text{phys}}[U\Omega]_{\text{phys}} = E_0[U\Omega]_{\text{phys}}.$$

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## References

- [1] A. Arai, *Fock spaces and quantum fields*, Nippon-Hyouronsya, Tokyo (2000) (in Japanese).
- [2] K. Bleuler, Eine neue methode zur Behandlung der longitudinalen und skalaren photonen, *Helv. Phys. Acta.* **23** (1950), 567-586.
- [3] J. Bogner, *Indefinite inner product space*, Springer-Verlag, Berlin (1973).
- [4] S. N. Gupta, Theory of longitudinal photon in quantum electrodynamics, *Proc. Phys. Soc.* **A63** (1950), 681-691.
- [5] C. Itzykson and j.-B. Zuber, *Quantum field Theory*, McGraw-Hill, New York (1980).
- [6] T. Kugo and I. Ojima, *Local covariant operator formalism of non-abelian gauge theories and quark confinement problem*, *Prog. Theor. Phys. Suppl.* **66** (1979), 1-130.
- [7] N. Nakanishi, Indefinite-metric quantum field theory, *Prog. Theor. Phys. Suppl.* **51** (1972), 1-95.
- [8] A. Suzuki, Physical subspace in a model of the quantized electromagnetic field coupled to an external field with an indefinite metric (to appear in *J. Math. Phys.*).
- [9] A. S. Wightman and L. Gårding, Field as operator-valued distributions in quantum field theory, *Ark. Fys.* **28** (1964), 129-184.