数理解析研究所講究録
第 1609 巻 2008 年 24-35

Mathematical Theory of Time Operators in Quantum Physics

Asao Arai (新井朝雄)*
Department of Mathematics, Hokkaido University
Sapporo, 060-0810
Japan
E-mail:arai@math.sci.hokudai.ac.jp

Abstract

Some aspects of mathematical theory of time operators in quantum physics are reviewed.

Keywords: time operator, Hamiltonian, time-energy uncertainty relation, spectrum, canonical commutation relation, Weyl representation, weak Weyl relation, generalized weak Weyl relation.

Mathematics Subject Classification (2000). 81Q10, 47N50.

1 Introduction

This paper is a short review on mathematical theory of time operators in quantum physics [2, 6, 7, 8, 9, 10, 12, 13]. There are some types or classes of time operators which are not necessarily equivalent each other. We first recall the definitions of them with comments.

Let $\mathcal{H}$ be a complex Hilbert space. We denote the inner product and the norm of $\mathcal{H}$ by $\langle\cdot,\cdot\rangle$ (antilinear in the first variable) and $\|\cdot\|$ respectively. For a linear operator A on a Hilbert space, $D(A)$ denotes the domain of A. Let H be a self-adjoint operator on $\mathcal{H}$ and T be a symmetric operator on $\mathcal{H}$.

*This work is supported by the Grant-in-Aid No.17340032 for Scientific Research from Japan Society for the Promotion of Science (JSPS).

The operator T is called an *ordinary time operator* of H if there is a dense subspace $\mathcal{D}$ of $\mathcal{H}$ such that $\mathcal{D} \subset D(TH) \cap D(HT)$ and the canonical commutation relation (CCR)

$$[T, H] := (TH - HT) = i$$

holds on $\mathcal{D}$ (i.e., $[T, H]\psi = i\psi$, $\forall\psi \in \mathcal{D}$), where i is the imaginary unit. In this case, T is called a *canonical conjugate* to H too.

The name "time operator" for the operator T comes from the quantum theoretical context where H is taken to be the Hamiltonian of a quantum system and the heuristic classical-quantum correspondence based on the structure that, in the classical relativistic mechanics, time is a canonical conjugate variable to energy in each Lorentz frame of coordinates. We remark, however, that this name is somewhat misleading, because time is not an observable in the usual quantum theory, but just a parameter assigning the time when a quantum event is observed. But we follow convention in this respect. By the same reason as just remarked, T is not necessarily (essentially) self-adjoint. But this does not mean that it is "unphysical" [2, 13].

From a representation theoretic point of view, the pair (T, H) is a symmetric representation of the CCR with one degree of freedom [3, Chapter 3]. But one should remember that, as for this original form of representation of the CCR, the von Neumann uniqueness theorem ([3, Theorem 3.23], [14], [15, Theorem VIII.14]) does not necessarily hold. In other words, (T, H) is not necessarily unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom. Indeed, for example, it is obvious that, if H is semi-bounded (i.e., bounded below or bounded above), then (T, H) cannot be unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

A classification of pairs (T, H) with T being bounded (hence the case where T is a bounded self-adjoint operator) has been done by G. Dorfmeister and J. Dorfmeister [11].

A weak form of time operator is defined as follows. We say that a symmetric operator T is a *weak time operator* of H if there is a dense subspace $\mathcal{D}_{\mathrm{w}}$ of $\mathcal{H}$ such that $\mathcal{D}_{\mathrm{w}} \subset D(T) \cap D(H)$ and

$$\langle T\psi, H\phi\rangle - \langle H\psi, T\phi\rangle = \langle \psi, i\phi\rangle \,, \quad \psi, \phi \in \mathcal{D}_{\mathrm{w}},$$

i.e., (T, H) satisfies the CCR in the sense of sesquilinear form on $\mathcal{D}_{\mathrm{w}}$. Obviously an ordinary time operator T of H is a weak time operator of H. But the converse is not true[1].

In contrast to the weak form of time operator, there is a strong form. We say that T is a *strong time operator* of H if, for all $t \in \mathbb{R}$, $e^{-itH}D(T) \subset D(T)$ and

$$Te^{-itH}\psi = e^{-itH}(T + t)\psi, \quad \psi \in D(T). \tag{1.1}$$

[1]It is easy to see, however, that, if T is a weak time operator of H and $D(TH) \cap D(HT)$ is dense in $\mathcal{H}$, then T is an ordinary time operator with $\mathcal{D} = D(TH) \cap D(HT)$.

We call (1.1) the *weak Weyl relation* [2]. From a representation theoretic point of view, we call a pair (T, H) obeying the weak Weyl relation a *weak Weyl representation* of the CCR. This type of representation of the CCR was extensively studied by Schmüdgen [17, 18]. It is shown that a strong time operator of H is an ordinary time operator of H [13]. But the converse is not true.

Relations among different types of time operators are shown as follows:

$$\begin{aligned}\{\text{strong time operators}\} &\subsetneqq \{\text{ordinary time operators}\} \\ &\subsetneqq \{\text{weak time operators}\}. \end{aligned} \tag{1.2}$$

There is a generalized version of strong time operator [2]. We say that T is a *generalized time operator* of H if, for each $t \in \mathbb{R}$, there is a *bounded self-adjoint* operator $K(t)$ on $\mathcal{H}$ with $D(K(t)) = \mathcal{H}$, $e^{-itH}D(T) \subset D(T)$ and a *generalized weak Weyl relation* (GWWR)

$$Te^{-itH}\psi = e^{-itH}(T + K(t))\psi \quad (\forall \psi \in D(T)) \tag{1.3}$$

holds. In this case, the bounded operator-valued function $K(t)$ of $t \in \mathbb{R}$ is called the *commutation factor* of the GWWR under consideration.

In what follows, we present fundamental results on time operators.

2 Weak Time Operators

An important aspect of a weak time operator T of H is that a *time-energy uncertainty relation* is naturally derived. Indeed, one can prove that, for all unit vectors ψ in $\mathcal{D}_{\mathrm{w}} \subset D(T) \cap D(H)$,

$$(\Delta T)_\psi (\Delta H)_\psi \geq \frac{1}{2}, \tag{2.1}$$

where, for a linear operator A on $\mathcal{H}$ and $\phi \in D(A)$ with $\|\phi\| = 1$,

$$(\Delta A)_\phi := \|(A - \langle \phi, A\phi \rangle)\phi\|,$$

called the *uncertainty of A in the vector ϕ*. Note that, by (1.2), (2.1) holds also in the case where T is a strong time operator or an ordinary time operator of H.

3 Galapon Time Operator

As an important example of ordinary time operator, we describe a time operator introduced by Galapon [12] (see also [10]).

Let $\mathcal{H}$ be a complex Hilbert space and H be a self-adjoint operator on $\mathcal{H}$ which has the following properties (H.1) and (H.2):

(H.1) The spectrum of H, denoted $\sigma(H)$, is purely discrete with $\sigma(H) = \{E_n\}_{n=1}^{\infty}$, where each eigenvalue E_n of H is simple and $0 < E_n < E_{n+1}$ for all $n \in \mathbb{N}$ (the set of positive integers).

(H.2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{E_n{}^2} < \infty.$

By (H.1), H has a complete orthonormal system (CONS) of eigenvectors $\{e_n\}_{n=1}^{\infty}$: $He_n = E_n e_n$, $n \in \mathbb{N}$. Using $\{e_n\}_{n=1}^{\infty}$, one can define a linear operator T on $\mathcal{H}$ as follows:

$$D(T) := \left\{ \psi \in \mathcal{H} \middle| \sum_{n=1}^{\infty} \sum_{m \neq n}^{\infty} \left| \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2 < \infty \right\} \tag{3.1}$$

$$T\psi := i \sum_{n=1}^{\infty} \left(\sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right) e_n, \quad \psi \in D(T). \tag{3.2}$$

We denote by $\mathcal{D}_0$ the subspace algebraically spanned by the set $\{e_n\}_{n=1}^{\infty}$. It follows from (H.2) that $\mathcal{D}_0 \subset D(T)$. Moreover we have:

Proposition 3.1 The operator

$$T_1 := T|\mathcal{D}_0 \tag{3.3}$$

(the restriction of T to $\mathcal{D}_0$) is symmetric.

Let $\mathcal{D}_c$ be the subspace algebraically spanned by $\{e_n - e_m \in \mathcal{H} | n, m \geq 1\})$. Then it is easy to see that $\mathcal{D}_c$ is dense in $\mathcal{H}$ and

$$\mathcal{D}_c \subset \mathcal{D}_0.$$

The next theorem shows that T_1 is an ordinary time operator of H:

Theorem 3.2 [12] *It holds that*

$$\mathcal{D}_c \subset D(T_1 H) \cap D(HT_1) \tag{3.4}$$

and

$$[T_1, H]\psi = i\psi, \qquad \psi \in \mathcal{D}_c. \tag{3.5}$$

We call T_1 the *Galapon time operator*. Detailed properties of T and T_1 have been investigated in [10]. Here we only mention a result on boundedness of T_1.

Theorem 3.3 [10, Theorem 4.5] *Suppose that there exist constants $\alpha > 1, C > 0$ and $a > 0$ such that*

$$E_n - E_m \geq C(n^\alpha - m^\alpha), \quad n > m > a.$$

Then T_1 is a bounded self-adjoint operator with $D(T_1) = \mathcal{H}$ and $T_1 = T$.

This result is striking in a sense, because there has been a "belief" or a "folklore" among physicists that there are no self-adjoint canonical conjugates to a Hamiltonian which is bounded below. Theorem 3.3 clearly shows that this belief is an illusion. We also remark that the Galapon time operator T_1 is not a strong time operator. This follows from directly calculating $(T_1 e^{-itH} - e^{-itH}T_1)e_n$ ($t \in \mathbb{R}$) or a property of strong time operators (see Theorem 4.1 below).

Remark 3.1 Theorem 3.3 does not cover the case where $E_n = \varepsilon_n := a(n-1) + b$, $n \in \mathbb{N}$ ($a > 0, b > 0$ are constants), i.e., the case where $\{E_n\}_n$ is the spectrum of a one-diemsional quanutm harmonic oscillator. But, by using another method, one can prove that T_1 with $E_n = \varepsilon_n$ is a bounded self-adjoint operator on $\mathcal{H}$ and $T_1 = T$ ([10, Theorem 4.6]). Putting $\hat{\theta} := aT_1$ and $\hat{N} := a^{-1}H - b$ (the number operator), we have $[\hat{\theta}, \hat{N}] = i$ on $\mathcal{D}_c$ and $\sigma(\hat{\theta}) = [-\pi, \pi]$. This allows one to interpret $\hat{\theta}$ a *quantum phase operator*. For the details, see [10, Example 4.2].

4 Strong Time Operators

Suppose that a self-adjoint operator H has a strong time operator T. A basic property of H is given in the next theorem:

Theorem 4.1 [13] *The operator H is purely absolutely continuous (hence H has no eigenvalues).*

This theorem implies that, for all $\psi, \phi \in \mathcal{H}$, $\lim_{t\to\pm\infty} \langle \psi, e^{-itH}\phi\rangle = 0$ [3, Theorem 7.5].

Hence one can ask how fast the transition probability amplitude $\langle \psi, e^{-itH}\phi\rangle$ decays as $t \to \pm\infty$. The strong time operator T controls it in some way:

Theorem 4.2 *Let $n \in \mathbb{N}$. Then, for all $\phi, \psi \in D(T^n)$ and $t \in \mathbb{R}\setminus\{0\}$,*

$$|\langle \phi, e^{-itH}\psi\rangle| \leq \frac{d_n^T(\phi,\psi)}{|t|^n}, \tag{4.1}$$

where $d_n^T(\phi,\psi)$ is as follows:

$$d_1^T(\phi,\psi) := \|T\phi\|\|\psi\| + \|\phi\|\|T\psi\|,$$

$$d_n^T(\phi,\psi) := \|T^n\phi\|\|\psi\| + \|\phi\|\|T^n\psi\| + \sum_{r=1}^{n-1} \frac{n!}{(n-r)!r!} d_{n-r}^T(\phi, T^r\psi), \quad n \geq 2.$$

The theorem with $n = 1$ (resp. $n \geq 2$) was proved by Miyamoto [13] (resp. the present author [2, Theorem 8.5]). Note that, in estimate (4.1), the order n of decay in $|t|$ is exactly equal to the order of the domain in T to which ϕ and ψ belong and the constant $d_n^T(\phi, \psi)$ is determined by n, T, ϕ and ψ. In this way the strong time operator T has a connection to quantum dynamics, independently of whether it is (essentially) self-adjoint or not.

As for properties of the strong time operator T we have the following theorem:

Theorem 4.3 ([13], [2, Theorem 2.8]) *If H is semi-bounded (i.e., bounded below or bounded above), then T is not essentially self-adjoint.*

This theorem combined with a general theorem ([3, p.117, Appendix C], [16, Theorem X.1]) implies that, in the case where H is semi-bounded, the spectrum $\sigma(T)$ of T is one of the following three sets :

(i) $\mathbb{C}$.

(ii) $\overline{\Pi}_+$, the closure of the upper half-plane $\Pi_+ := \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$.

(iii) $\overline{\Pi}_-$, the closure of the lower half-plane $\Pi_- := \{z \in \mathbb{C} | \mathrm{Im} z < 0\}$.

From this point of view, it is interesting to examine which one is realized, depending on properties of H. In this respect we have the following theorem:

Theorem 4.4 [6, Theorem 2.1] *The following* (i)—(iii) *hold:*

(i) *If H is bounded below, then $\sigma(T)$ is either $\mathbb{C}$ or $\overline{\Pi}_+$.*

(ii) *If H is bounded above, then $\sigma(T)$ is either $\mathbb{C}$ or $\overline{\Pi}_-$.*

(iii) *If H is bounded, then $\sigma(T) = \mathbb{C}$.*

Example 4.1 Let Δ be the n-dimensional generalized Laplacian acting in $L^2(\mathbb{R}_x^n)$ ($n \in \mathbb{N}$), where $\mathbb{R}_x^n := \{x = (x_1, \cdots, x_n) | x_j \in \mathbb{R}, j = 1, \cdots, n\}$, and

$$H_0 := -\frac{\Delta}{2m} \tag{4.2}$$

with a constant $m > 0$. In the context of quantum mechanics, H_0 represents the free Hamiltonian of a free nonrelativistic quantum particle with mass m in the n-dimensional space $\mathbb{R}_x^n$. It is well known that H_0 is a nonnegative self-adjoint operator. We denote by $\hat{x}_j$ the multiplication operator on $L^2(\mathbb{R}_x^n)$ by the j-th variable $x_j \in \mathbb{R}_x^n$ and set

$$\hat{p}_j := -iD_j \tag{4.3}$$

with D_j being the generalized partial differential operator in the variable x_j on $L^2(\mathbb{R}^n_x)$. It is easy to see that $\hat{x}_j$ and $\hat{p}_j$ are injective. For each $j = 1, \cdots, n$, one can define a linear operator on $L^2(\mathbb{R}^n_x)$ by

$$T_j := \frac{m}{2}\left(\hat{x}_j\hat{p}_j^{-1} + \hat{p}_j^{-1}\hat{x}_j\right) \tag{4.4}$$

with domain

$$D(T_j) := \{f \in L^2(\mathbb{R}^n_x) | \hat{f} \in C_0^\infty(\Omega_j)\}, \tag{4.5}$$

where $\hat{f}$ is the Fourier transform of f and $\Omega_j := \{k = (k_1, \cdots, k_n) \in \mathbb{R}^n_k | k_j \neq 0\}$. One can show that T_j is a strong time operator of H_0 ([2, 13]). The time operator T_j is called the *Aharonov-Bohm time operator* [1]. One can prove that

$$\sigma(T_j) = \overline{\Pi}_+, \quad j = 1, \cdots, n.$$

For proof, see [6, §4.1].

Example 4.2 A Hamiltonian of a free relativistic spinless particle with mass $m \geq 0$ moving in $\mathbb{R}^n_x$ is given by

$$H(m) := \sqrt{-\Delta + m^2} \tag{4.6}$$

acting in $L^2(\mathbb{R}^n_x)$. It is shown that the operator

$$T_j(m) := H(m)\hat{p}_j^{-1}\hat{x}_j + \hat{x}_j H(m)\hat{p}_j^{-1} \tag{4.7}$$

with $D(T_j(m)) := D(T_j)$ is a strong time operator of $H(m)$ [2, Example 11.4]. Moreover one can prove the following fact [6, §4.2]:

$$\sigma(T_j(m)) = \overline{\Pi}_+, \quad j = 1, \cdots, n.$$

5 A Class of Generalized Time Operators

A general theory of generalized time operators including various examples has been developed in [2]. Here we only describe a special class of generalized time operators. Let H be a self-adjoint operator on a complex Hilbert space $\mathcal{H}$ and T be a symmetric operator on $\mathcal{H}$. We call the operator T a *generalized strong time operator* of H if $e^{-itH}D(T) \subset D(T)$ for all $t \in \mathbb{R}$ and there exists a bounded self-adjoint operator $C \neq 0$ on $\mathcal{H}$ with $D(C) = \mathcal{H}$ such that

$$Te^{-itH}\psi = e^{-itH}(T + tC)\psi, \quad \psi \in D(T). \tag{5.1}$$

We call C the *noncommutative factor* for (H, T). The pair (H, T) with T a generalized strong time operator has properties similar to those of (H, T) with T a strong time operator, but in weakened forms.

Theorem 5.1 [2] *Let T be a generalized strong time operator of H with noncommutative factor C. Then:*

(i) *Let H be semi-bounded and*

$$CT \subset TC. \tag{5.2}$$

Then T is not essentially self-adjoint .

(ii) *H is reduced by $\overline{\mathrm{Ran}(C)}$ and the reduced part $H|\overline{\mathrm{Ran}(C)}$ to $\overline{\mathrm{Ran}(C)}$ is purely absolutely continuous.*

(iii) *Let H be bounded below. Then, for all $\beta > 0$, $e^{-\beta H}D(\overline{T}) \subset D(\overline{T})$ and*

$$\overline{T}e^{-\beta H}\psi - e^{-\beta H}\overline{T}\psi = -i\beta e^{-\beta H}C\psi, \quad \psi \in D(\overline{T}). \tag{5.3}$$

For (T, H) with T a generalized strong time operator, Theorem 4.2 is generalized as follows:

Theorem 5.2 [2, Theorem 8.9] *Let T be a generalized strong time operator of H with noncommutative factor C. Then, for each $n \in \mathbb{N}$, there exists a subspace $\mathcal{D}_n(T, C)$ such that, for all $\phi \in D(T^n)$ and $\psi \in \mathcal{D}_n(T, C)$ $(\phi, \psi \neq 0)$,*

$$|\langle \phi, e^{-itH}C^n\psi \rangle| \leq \frac{d_n(\phi, \psi)}{|t|^n}, \quad t \in \mathbb{R} \setminus \{0\} \tag{5.4}$$

where $d_n(\phi, \psi) > 0$ is a constant independent of t.

6 A Mapping on the Space of Weak Weyl Representations and Construction of Weyl Representations

One can consider the set of all weak Weyl representations:

$$\mathrm{WW}(\mathcal{H}) := \{(T, H)|(T, H) \text{ is a weak Weyl representation}\}. \tag{6.1}$$

Let $(T, H) \in \mathrm{WW}(\mathcal{H})$. Then, by Theorem 4.1, one can define, via functional calculus,

$$L(H) := \log |H|, \tag{6.2}$$

which is self-adjoint. One can also show that the operator

$$D(T, H) := \frac{1}{2}(TH + \overline{HT}) \tag{6.3}$$

is densely defined and symmetric.

By direct computations, one can show that the following commutation relations hold:

$$[T, D(T, H)] = iT \quad \text{on } D(T^2H) \cap D(HT^2) \cap D(THT), \tag{6.4}$$
$$[H, D(T, H)] = -iH \quad \text{on } D(H^2T) \cap D(TH^2) \cap D(HTH), \tag{6.5}$$
$$[H, L(H)] = 0 \quad \text{on } D(HL(H)) \cap D(L(H)H). \tag{6.6}$$

This implies that, if there is a domain $\mathcal{D} \subset D(T) \cap D(H)$ such that $T\mathcal{D} \subset \mathcal{D}$ and $H\mathcal{D} \subset \mathcal{D}$, then $\{T, H, D(T, H)\}$ generates a Lie subalgebra of $\mathfrak{L}(\mathcal{D})$ (the vector space of all linear operators on $\mathcal{D}$). If we introduce

$$A := -iT, \quad B := H, \quad C := -iD(T, H),$$

then we have

$$[C, A] = -A, \quad [C, B] = B, \quad [A, B] = 1$$

on $\mathcal{D}$. This is the same set of commutation relations as that defining the harmonic oscillator Lie algebra generated by three elements a, $a^\dagger$ and $a^\dagger a$ obeying $[a, a^\dagger] = 1$ (the correspondence is: $a \to A, a^\dagger \to B, a^\dagger a \to C$). In other words, $\{A, B, C\}$ gives a representation of the harmonic oscillator Lie algebra. But this representation is somewhat unusual in the sense that B is not the adjoint of A and C is antisymmerric.

We can prove the following theorem:

Theorem 6.1 [9, Theorem 2.4] $(D(T, H), L(H)) \in \mathrm{WW}(\mathcal{H})$.

By this theorem, we can define a mapping $f : \mathrm{WW}(\mathcal{H}) \to \mathrm{WW}(\mathcal{H})$ by

$$f(T, H) := (D(T, H), L(H)), \quad (T, H) \in \mathrm{WW}(\mathcal{H}). \tag{6.7}$$

Thus, starting from each weak Weyl representation $(T, H) \in \mathrm{WW}(\mathcal{H})$, we have a set $\{f^n(T, H)\}_{n=1}^{\infty}$ of weak Weyl representations which may be an infinite set.

The quantity

$$E_0(H) := \inf \sigma(H),$$

the infimum of the spectrum of $\sigma(H)$, is called the lowest energy of H. The following theorem is concerned with unitary equivalence between (T, H) and $f(T, H)$ ($(T, H) \in \mathrm{WW}(\mathcal{H})$).

Theorem 6.2 *If* $\inf_{\lambda \in \sigma(H)} \log |\lambda| \neq E_0(H)$, *then* (T, H) *is not unitarily equivalent to* $f(T, H)$.

Proof. By the spectral mapping theorem, we have $\sigma(L(H)) = \{\log |\lambda| \,|\, \lambda \in \sigma(H)\}$. If (T, H) is unitarily equivalent to $f(T, H)$, then $\sigma(H) = \sigma(L(H))$. This implies that $E_0(H) = \inf_{\lambda \in \sigma(H)} \log |\lambda|$. But this contradicts the present assumption. □

Corollary 6.3 *If $H \geq 0$, then (T, H) is not unitarily equivalent to $f(T, H)$.*

Proof. If $H \geq 0$, then $E_0(H) \geq 0$. Hence, if $E_0(H) > 0$, then $\inf_{\lambda \in \sigma(H)} \log |\lambda| = \log E_0(H) \neq E_0(H)$. If $E_0(H) = 0$, then $\inf_{\lambda \in \sigma(H)} \log |\lambda| = -\infty$. Thus the assumption of Theorem 6.2 is satisfied. □

Investigations towards a complete classification of $\{f^n(T, H)\}_{n=1}^{\infty}$ are still in progress.

It also is interesting to know when $(\overline{D(T, H)}, L(H))$ becomes a Weyl representation of the CCR. As for this aspect, we have the following result:

Theorem 6.4 [9, Corollary 2.6] *Suppose that $D(T, H)$ is essentially self-adjoint. Then, for all $s, t \in \mathbb{R}$,*

$$e^{is\overline{D(T,H)}} e^{itL(H)} = e^{-ist} e^{itL(H)} e^{is\overline{D(T,H)}}.$$

Namely $(\overline{D(T, H)}, L(H))$ is a Weyl representation of the CCR.

Example 6.1 In the case where $H = H_0$ and $T = T_j$ (Example 4.1), we can prove that $D(T_j, H_0)$ is essentially self-adjoint. Hence, by Theorem 6.4, $(\overline{D(T_j, H_0)}, \log H_0)$ is a Weyl representation. Therefore, by the von Neumann uniqueness theorem ([14], [3, Theorem 3.23]), we can conclude that $(\overline{D(T_j, H_0)}, \log H_0)$ is unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

Example 6.2 Let us consider Example 4.2. In this example, there is a big difference between the case $m = 0$ and the case $m > 0$. Indeed, we can prove the following facts:

(i) If $m = 0$, then $D(T_j(0), H(0))$ is essentially self-adjoint. Hence, by Theorem 6.4, $(\overline{D(T_j(0), H(0))}, \log H(0))$ is unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

(ii) If $m > 0$, then $D(T_j(m), H(m))$ is not essentially self-adjoint and

$$\sigma(D(T_j(m), H(m))) = \overline{\Pi}_+.$$

In particular, $(\overline{D(T_j(m), H(m))}, \log H(m))$ is not unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

These mathematical structures are interesting in the sense that it gives a representation theoretic meaning to the mass m.

7 Concluding Remark

Finally we would like to give a remark on Theorem 6.4 from a view-point of natural philosophy or *quantum-mathematical cosmology*[2] (*not* physics). Suppose that $\mathcal{H}$ is separable. Let (T, H) be a pair obeying the weak Weyl relation such that $D(T, H)$ is essentially self-adjoint. Then, by Theorem 6.4 and the von Neumann uniqueness theorem, $(\overline{D(T, H)}, L(H))$ is unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom. On the other hand, a direct sum of the Schrödinger representation of the CCR with one degree of freedom describes a set of external degrees of freedom associated with the usual macroscopic perception of space. Hence, in this representation theoretic scheme, one can infer that a pair (T, H) obeying the weak Weyl relation "creates" a set of external degrees which is a basis for quantum mechanics associated with the usual (daily-life) space-time picture that the humanbeing has. In this sense, a pair (T, H) obeying the weak Weyl relation may be more fundamental in ontological structures or orders (cosmos). Thus an important thing is to how to interpret philosophically, in a proper way, a pair (T, H) obeying the weak Weyl relation such that $D(T, H)$ is esssentially self-adjoint. A possible view-point for this is that T is a fundamental "time" and H is a fundamental "energy" in the metaphysical sense that they produce a "phase" or a "rank" in the metaphysical dimension of existence which is more directly connected with the usual picture of space-time in the physical or sensorial-phenomenal dimension. In connection with this philosophical view-point, we are now considering the problem of uniqueness of weak Weyl representations [8].

References

[1] Y. Aharonov and D. Bohm, Time in the quantum theory and the uncertainty relation for time and energy, *Phys. Rev.* **122** (1961), 1649–1658.

[2] A. Arai, Generalized weak Weyl relation and decay of quantum dynamics, *Rev. Math. Phys.* **17** (2005) , 1071–1109 .

[3] 新井朝雄 (A. Arai), 量子現象の数理 (Mathematical Principles of Quantum Phenomena), 朝倉書店 (Asakura-shoten), 2006.

[4] 新井朝雄 (A. Arai), 量子現象によって開示される存在論的構造 (Ontological structures revealed by quantum phenomena), シェリング年報 (Schelling Annals) **14** (2006) （日本シェリング協会 Schelling-Gesellschaft Japan） , 95–103. http://eprints.lib.hokudai.ac.jp/dspace/handle/2115/27964.

[2]This is not a standard terminology. Basic elements of the philosophy mentioned here are outlined in [4, 5].

[5] A. Arai, Fundamental symmetry principles in quantum mechanics and its philosophical phases, *Symmetry: Culture and Science* **17** (2006), 141–157.

[6] A. Arai, Spectrum of time operators, *Lett. Math. Phys.* **80** (2007), 211–221.

[7] A. Arai, Some aspects of time operators, in *Quantum Bio-Informatics* (Ed. L. Accardi, W. Freudenberg and M. Ohya), 2008, pp.26–35.

[8] A. Arai, On the uniqueness of weak Weyl representations of the canonical commutation relation, mp_arc 08-27.

[9] A. Arai and Y. Matsuzawa, Construction of a Weyl representation from a weak Weyl representation of the canonical commutation relation, *Lett. Math. Phys.* **83** (2008), 201–211.

[10] A. Arai and Y. Matsuzawa, Time operators of a Hamiltonian with purely discrete spectrum, mp_arc 08-24.

[11] G. Dorfmeister and J. Dorfmeister, Classification of certain pairs of operators (P, Q) satisfying $[P, Q] = -i\mathrm{Id}$, *J. Funct. Anal.* **57** (1984), 301–328.

[12] E. A. Galapon, Self-adjoint time operator is the rule for discrete semi-bounded Hamiltonians, *Proc. R. Soc. Lond. A* **458** (2002), 2671–2689.

[13] M. Miyamoto, A generalized Weyl relation approach to the time operator and its connection to the survival probability, *J. Math. Phys.* **42** (2001), 1038–1052.

[14] J. von Neumann, Die Eindeutigkeit der Schrödingerschen Operatoren, *Math. Ann.* (1931) **104**, 570–578.

[15] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis* (Academic Press, New York, 1972).

[16] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness* (Academic Press, New York, 1975).

[17] K. Schmüdgen, On the Heisenberg commutation relation. I, *J. Funct. Anal.* **50** (1983), 8–49.

[18] K. Schmüdgen, On the Heisenberg commutation relation. II, *Publ. RIMS, Kyoto Univ.* **19** (1983), 601–671.