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Singularities of Semiconformal Mappings

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A smooth mapping $\phi : M \to \Sigma$ between Riemannian manifolds is said to be semiconformal if $d\phi$ is conformal orthogonal to the fibres wherever $d\phi$ is non-zero. If $\Sigma = \mathbb{R}^2$ with its standard flat metric and we write $\phi = (f, g)$ for smooth functions f and g, then ϕ is semiconformal if and only if

$$\|\nabla f\| = \|\nabla g\| \quad \text{and} \quad \langle \nabla f, \nabla g\rangle = 0. \tag{1}$$

In fact, if Σ is 2-dimensional and we view ϕ in isothermal coördinates, then this is the general form of a semiconformal ϕ. If $M = \mathbb{R}^2$ as well, then (1) holds if and only if

$$\left.\begin{array}{l}\partial f/\partial x = \pm\,\partial g/\partial y\\ \partial f/\partial y = \mp\,\partial g/\partial x\end{array}\right\}\quad \text{the (anti-)Cauchy-Riemann equations}$$

so ϕ is (anti-)holomorphic. If $\dim\Sigma \geq 3$, then semiconformal mappings are quite rigid so let us suppose from now on that $\dim\Sigma = 2$ and $\dim M = 3$. It is the first non-trivial case beyond holomorphic mappings. In fact, for simplicity, let us suppose $M \subseteq \mathbb{R}^3$ and $\Sigma = \mathbb{R}^2$. As far as the local behaviour of semiconformal mappings goes, this is already a sufficiently difficult yet interesting case and much remains unknown.

So from now one we wish to study pairs of smooth functions f and g on $M \subseteq \mathbb{R}^3$ such that (1) holds where ∇ is the usual gradient operator on $\mathbb{R}^3$. Also, let us call f and g a pair of conjugate functions.

Question When does a smooth function f on $M \subseteq \mathbb{R}^3$ admit a conjugate?

Locally, a smooth function on $\mathbb{R}^2$ admits a conjugate if and only if f is harmonic (and this has strong consequences). In [2] we showed that there is a (rather complicated) partial differential equation that must be satisfied if f on $M \subseteq \mathbb{R}^3$ is to admit a conjugate. In this article we shall investigate some of the simple conformally invariant conditions that we can impose on f (simpler that the partial differential equation found in [2]).

Question What happens if we impose, additionally, that f and g be harmonic?

In this case $\phi : M \to \mathbb{R}^2$ is precisely a harmonic morphism [5] meaning that it pulls back locally defined harmonic functions on $\mathbb{R}^2$ to harmonic functions on M. (That harmonic morphisms satisfy (1) was observed by Jacobi in 1848.) Notice, however, that $\phi : M \to \Sigma$ being semiconformal is a conformally invariant concept both on M and Σ whence

$$\underset{\boxed{\text{holomorphic on } \mathbb{R}^2}}{h} \circ \phi\Bigg(\underbrace{\frac{x + (1/2\lambda)\|x\|^2 r}{\lambda + \langle r, x\rangle + (1/4\lambda)\|r\|^2\|x\|^2}}_{\boxed{\text{inversion on } \mathbb{R}^3}}\Bigg) \qquad \begin{array}{c} r \in \mathbb{R}^3 \\ \lambda \in \mathbb{R}_{>0}\end{array}$$

is semiconformal whenever ϕ is. An inversion generally destroys harmonicity.

Examples of conjugate pairs on $M^{\text{open}} \subseteq \mathbb{R}^3 \ni (x_1, x_2, x_3)$

1. $f = x_1$, $g = x_2$ ($\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is orthographic projection).
2. $f = {x_1}^2 - {x_2}^2 - {x_3}^2$, $g = 2x_1\sqrt{{x_2}^2 + {x_3}^2}$ (away from the x_1-axis).
3. $f = x_2 \dfrac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2}$, $g = x_3 \dfrac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2}$ (away from the x_1-axis).
4. $f = \dfrac{(1 - \|x\|^2)x_2 + 2x_1x_3}{{x_2}^2 + {x_3}^2}$, $g = \dfrac{(1 - \|x\|^2)x_3 - 2x_1x_2}{{x_2}^2 + {x_3}^2}$ (Hopf, see below).
5. $f = \log\|x\|$, $g = \arccos\dfrac{x_1}{\|x\|}$.

Example #4 is the Hopf fibration $S^3 \longrightarrow S^2$, which is a harmonic morphism, viewed in (conformal) stereographic coördinates $\mathbb{R}^n \hookrightarrow S^n$. Viewed like this, in the flat Euclidean metrics, the result is no longer a harmonic morphism.

Counterexamples (from [2]). The following do not admit conjugates, even locally.

- $f = x_1x_2x_3$.
- $f = {x_1}^3 + {x_2}^3 + {x_3}^3$.

Singularities? For holomorphic functions $\mathbb{C} \longrightarrow \mathbb{C}$ we can have

- essential singularities, e.g. $z \longmapsto e^{1/z}$ at $z = 0$.
- poles, e.g. $z \longmapsto 1/z$ or $z \longmapsto 1/z^2$ at $z = 0$.
- critical points, e.g. $z \longmapsto z^2$ at $z = 0$.
- branching, e.g. $z \longmapsto \sqrt{z}$ at $z = 0$.

We can ask about similar behaviour for semiconformal mappings. Certainly, we may compose a semiconformal submersion (no critical points) with a holomorphic function. For example

$$\text{Example \#1 composed with } z \longmapsto z^2 \quad \Longrightarrow \quad f = {x_1}^2 - {x_2}^2,\ g = 2x_1x_2$$

and $\phi = (f, g)$ has critical points along the x_3-axis.

Question Can a semiconformal mapping have an isolated critical point? Although the function $f = {x_1}^2 - {x_2}^2 - {x_3}^2$ in Example #2 has an isolated critical point at the origin, we shall soon see that its conjugate is forced to be $\pm 2x_2\sqrt{{x_2}^2 + {x_3}^2} +$ constant and so is not smooth along the x_1-axis. The answer to this question is currently unknown.

Singularities? Instead of a critical point where $d\phi = 0$, we can find examples where $\|d\phi\| \longrightarrow \infty$, equivalently $\|\nabla f\| \longrightarrow \infty$. The semiconformal map

$$\phi(x) = \frac{1}{\|x\| - x_1}(x_2, x_3) \quad (x = (x_1, x_2, x_3) \in \mathbb{R}^3)$$

shows that isolated singularities can occur. This map is projection $x \mapsto x/\|x\|$ to the 2-sphere along geodesics passing through the origin followed by stereographic projection and has an isolated singularity at the origin. It is also harmonic and for a harmonic morphism, up to translation, this is the only kind of isolated singularity that can occur [4]. In particular for a harmonic morphism from a connected open subset of $\mathbb{R}^3$, there can be at most one such singularity.

The map

$$\phi(x) = \frac{1}{1 - |x|^2 + \sqrt{(|x|^2-1)^2 + 4({x_2}^2 + {x_3}^2)}}(x_2, x_3)$$

is semiconformal and has two isolated singularities at the points $(\pm 1, 0, 0)$. A cursory glance suggests that the whole x_1-axis is singular, but in fact away from the points $(\pm 1, 0, 0)$, ϕ extends smoothly to a map with values in the Riemann sphere, taking on the values 0 or ∞, depending on whether $|x_1| < 1$ or $|x_1| > 1$, respectively. In this case ϕ is projection $S^3 \to S^2$ along geodesics passing through a common point (and so also passing through the antipodal point) to the orthogonal equatorial 2-sphere viewed in stereographic coördinates (whose point at infinity differs from the point and its antipodal point defining the projection).

Does there exist a semiconformal map from a domain of $\mathbb{R}^3$ with more than two isolated singular points?

Branching? More complicated singularities can occur along one-dimensional algebraic subsets. In fact, there are harmonic morphisms constructed in [1], which are branched along knots in $\mathbb{R}^3$. Specifically, it is easily verified that for local holomorphic functions $h(z)$ and $k(z)$, the equation

$$(k(z)^2 - 1)x_1 - i(k(z)^2 + 1)x_2 + 2k(z)x_3 + 2h(z) = 0 \tag{2}$$

where it implicitly defines z as a smooth function of (x_1, x_2, x_3), is a harmonic morphism onto $\mathbb{C} = \mathbb{R}^2$ and it is shown in [3] that this is the general local harmonic morphism. If $h(z)$ and $k(z)$ are polynomials, then (2) is itself a polynomial in z. Viewed as such, it defines a covering branched along the zero locus of its discriminant. For example

$$\begin{aligned} h(z) = iz, k(z) = z \quad &\Longrightarrow (x_1 - ix_2)z^2 + 2(x_3 + i)z - (x_1 + ix_2) = 0 \\ &\text{discriminant} = 4({x_1}^2 + {x_2}^2 + {x_3}^2 - 1) + 8ix_3 \end{aligned}$$

so this one is branched over the unit circle in the (x_1, x_2)-plane. In [1] it is shown that the pair of polynomials $h(z) = z^5 + iz^3$, $k(z) = z^3$ gives rise to branching over a trefoil knot and $h(z) = z^7 + iz^5$, $k(z) = z^5$ over a cinquofoil. In each case, however, there seems to be an extra component to the branching locus going to infinity. It is unknown which knots can occur nor whether this extra component can be eliminated.

Conformal Invariants There are certain polynomials in the derivatives of a smooth function $f = f(x_1, x_2, x_3)$ that merely scale under conformal transformation. An obvious example is

$$J \equiv \|\nabla f\|^2 = \eta^{ij} f_i f_j = f_i f^i \quad \text{(summation convention)} \qquad \text{where } f_i \equiv \nabla_i f,$$

η^{ij} is the flat metric on $\mathbb{R}^3$, and $f^i \equiv \eta^{ij} f_j$ (as is usual in differential geometry), because

$$\eta_{ij} \mapsto \hat{\eta}_{ij} = \Omega^2 \eta_{ij} \implies \hat{\eta}^{ij} = \Omega^{-2}\eta^{ij} \implies \hat{J} = \Omega^{-2} J.$$

We say that J is a conformal invariant of weight -2. A less obvious example is

$$Z \equiv f^{ij}f_if_j + f^if_if^j{}_j \qquad \text{where } f_{ij} \equiv \nabla_i\nabla_j f \quad \text{(Hessian)},$$

which we claim to be a conformal invariant of weight -4. We may check this by using the following well-known formula for the Levi-Civita connection $\hat{\nabla}_i$ of $\hat{\eta}_{ij} \equiv \Omega^2\eta_{ij}$.

$$\hat{\nabla}_i\omega_j = \nabla_i\omega_j - \Upsilon_i\omega_j - \Upsilon_j\omega_i + \Upsilon^k\omega_k\eta_{ij}, \text{ where } \Upsilon_i \equiv \nabla_i \log\Omega.$$

It follows that

$$\hat{f}_{ij} = \hat{\nabla}_if_j = \nabla_if_j - \Upsilon_if_j - \Upsilon_jf_i + \Upsilon^kf_k\eta_{ij} = f_{ij} - \Upsilon_if_j - \Upsilon_jf_i + \Upsilon^kf_k\eta_{ij}$$

whence

$$\hat{f}^{ij}\hat{f}_i\hat{f}_j = \Omega^{-4}[f^{ij}f_if_j - \Upsilon^if^jf_if_j] \qquad \hat{f}^j{}_j = \Omega^{-2}[f^j{}_j + \Upsilon^jf_j]$$

and

$$\hat{Z} = \hat{f}^{ij}\hat{f}_i\hat{f}_j + \hat{f}^i\hat{f}_i\hat{f}^j{}_j = \Omega^{-4}[f^{ij}f_if_j + f^if_if^j{}_j] = \Omega^{-4}Z,$$

as required. Local solutions of the partial differential equation $Z = 0$ are called 3-harmonic and it follows that 3-harmonic functions are preserved by Möbius transformations on $\mathbb{R}^3$. Conversely, this property characterises the Möbius transformations [8].

Another simple conformal invariant is

$$X \equiv 2f_i{}^jf_jf^{ik}f_k - f^if_if^{jk}f_{jk} + f^if_i(f^j{}_j)^2 \quad \text{of weight } -6.$$

This one has a good geometric interpretation related to conjugate functions, which we now explain. Suppose that a smooth f defined on $M \subseteq \mathbb{R}^3$ admits a conjugate g. Writing (1) more explicitly as

$$f_if^i = g^ig_i \quad \text{and} \quad f^ig_i = 0$$

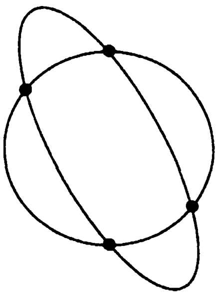
we can differentiate again to conclude that

$$f^{ij}g_ig_j + f^{ij}f_if_j = \nabla^j(f^ig_i)g_j + \tfrac{1}{2}\nabla^j(f_if^i - g_ig^i) = 0$$

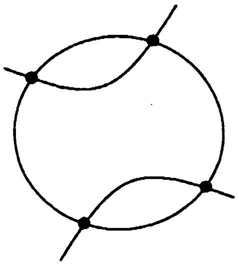
and now consider the three equations together

$$\text{(a) } f^ig_i = 0 \qquad \text{(b) } g^ig_i = f_if^i \qquad \text{(c) } f^{ij}g_ig_j + f^{ij}f_if_j = 0 \tag{3}$$

at any chosen basepoint as algebraic equations for the vector g_i. Let us suppose that J is non-zero at this basepoint. Equation (a) says that g_i lies in the plane orthogonal to f_i and then (b) adds that it lies on the circle in this plane of radius $\sqrt{J}$. Equation (c) tells us that g_i satisfies another quadratic equation on this plane. The invariant X controls how these two quadrics meet. Specifically, it is easy algebra (done in [2]) to check that these quadrics meet if and only if $X \leq 0$. When $X < 0$, they meet in four points:–



or



So $X \leq 0$ is a necessary condition in order that f admit a conjugate and this is already sufficient to obstruct many functions. For example,

$$f = x_1x_2x_3 \Longrightarrow X = 6(x_1x_2x_3)^2 \quad \text{and} \quad f = 1/\|x\| \Longrightarrow X = 2/\|x\|^{10}$$

so neither of these functions admit a local conjugate. Also, where $X < 0$ the equations (3) already constrain g quite tightly. Specifically, it is clear from the definition (1) that if (f, g) is a conjugate pair then so is $(f, \pm g + \text{constant})$ but where $X < 0$ there is at most one other pair of solutions for g_i. In this case, we can find this other pair quite explicitly as follows. If (f, g) is a conjugate pair and we set

$$F_i \equiv f_i/\sqrt{J} \qquad G_i \equiv g_i/\sqrt{J} \qquad \vec{H} \equiv \vec{F} \times \vec{G},$$

then

$$M \equiv \begin{pmatrix} F_1 & G_1 & H_1 \\ F_2 & G_2 & H_2 \\ F_3 & G_3 & H_3 \end{pmatrix}$$

is an orthogonal matrix. If we now define a, b, c according to

$$M^t \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} M = \begin{pmatrix} . & . & . \\ . & a & b \\ . & b & c \end{pmatrix},$$

and set

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \equiv \pm\sqrt{\frac{J}{(a-c)^2 + 4b^2}}\, M \begin{pmatrix} 0 \\ a-c \\ 2b \end{pmatrix},$$

then it is easily verified that

$$\text{(a) } f^i\omega_i = 0 \qquad \text{(b) } g^i\omega_i = f_if^i \qquad \text{(c) } f^{ij}\omega_i\omega_j + f^{ij}f_if_j = 0.$$

If $b = 0$ then $\vec{\omega} = \pm\vec{g}$ but, otherwise, this is a distinct pair of solutions to (3). It corresponds locally to an alternative conjugate function for f if and only if $\vec{\omega}$ is closed as a 1-form. In the case of Example #4 arising from the Hopf fibration,

$$f = \frac{(1 - \|x\|^2)x_2 + 2x_1x_3}{x_2{}^2 + x_3{}^2} \Longrightarrow X = -8\frac{(1 + \|x\|^2)^2}{(x_2{}^2 + x_3{}^2)^4} < 0$$

and we can follow through the procedure above to obtain

$$\begin{aligned}
\omega_1 &= \frac{2(x_1x_2 - x_3)(x_3(\|x\|^2 - 1) + 2x_1x_2)}{(x_2{}^2 + x_3{}^2)\sqrt{x_2{}^2(1 + \|x\|^2)^2 - 4(x_1x_2 - x_3)(x_1x_2 + x_3\|x\|^2)}} \\
\omega_2 &= \frac{2(x_1x_2 - x_3)((2x_3{}^2 - x_1x_2x_3)\|x\|^2 + 3x_1x_2x_3 - x_2{}^2(x_1{}^2 - x_2{}^2 - x_3{}^2 - 1))}{(x_2{}^2 + x_3{}^2)^2\sqrt{x_2{}^2(1 + \|x\|^2)^2 - 4(x_1x_2 - x_3)(x_1x_2 + x_3\|x\|^2)}} \\
\omega_3 &= \frac{\left[\begin{array}{l} x_2\|x\|^2((x_2{}^2 + x_3{}^2)^2 + x_1{}^2(x_2{}^2 - x_3{}^2)) - 2x_1x_3(x_2{}^4 + x_1{}^2(3x_2{}^2 - x_3{}^2) - x_3{}^4) \\ \quad + 2x_2(x_2{}^4 + 3x_2{}^2x_3{}^2 + 2x_3{}^4 - x_1{}^2x_2{}^2 + 5x_1{}^2x_3{}^2) \\ \quad\quad + 2x_1x_3(3x_2{}^2 - x_3{}^2) + x_2(x_2{}^2 - x_3{}^2) \end{array}\right]}{(x_2{}^2 + x_3{}^2)^2\sqrt{x_2{}^2(1 + \|x\|^2)^2 - 4(x_1x_2 - x_3)(x_1x_2 + x_3\|x\|^2)}}.
\end{aligned}$$

We compute

$$\frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2} = \frac{2(1+\|x\|^2)\left[\begin{array}{c} x_2((x_1{}^2+x_2{}^2)^2 - x_3{}^4) - 2x_1x_3(x_1{}^2+3x_2{}^2+x_3{}^2) \\ +\,2x_2(x_2{}^2 - x_1{}^2 + 2x_3{}^2) + 2x_1x_3 + x_2 \end{array}\right]}{(x_2{}^2(1+\|x\|^2)^2 - 4(x_1x_2 - x_3)(x_1x_2 + x_3\|x\|^2))^{3/2}}$$

and conclude that $\vec{\omega}$ is not closed. Therefore, the only local conjugate function to f is $\pm g +$ constant, where g the smooth function given in Example #4.

When $X = 0$ but $Z \neq 0$, the quadric defined by (a) and (c) together from (3) intersects the circle defined by (a) and (b) together in exactly two points. Indeed, this conclusion is easily reached by making an orthogonal change of coördinates at the basepoint so as to normalise $f_1 = f_2 = f_{12} = 0$ in which case

$$X = 2f_3{}^2(f_{11} + f_{33})(f_{22} + f_{33}) \quad \text{and} \quad Z = f_3{}^2(f_{11} + f_{22} + 2f_{33}).$$

Effecting this normalisation can be quite difficult but computing X and Z directly is easy enough. In Examples #2 and #3 we find that $X \equiv 0$ but

$$f = x_1{}^2 - x_2{}^2 - x_3{}^2 \Rightarrow Z = -16(x_2{}^2 + x_3{}^2) \quad \text{and} \quad f = \frac{x_2\|x\|^2}{x_2{}^2 + x_3{}^2} \Rightarrow Z = 2\frac{x_2\|x\|^6}{(x_2{}^2 + x_3{}^2)^4},$$

respectively, neither of which vanishes on an open set. Hence, there are no extraneous conjugates to either of these functions f.

There are higher order invariants too, the simplest of which is the cubic invariant

$$\mathfrak{S} \equiv f_{ij}f^{ij} + 3(f_i{}^i)^2 + 4f^i f_{ij}{}^j \quad \text{of weight } -4$$

from [7], reminiscent of the projectively invariant Schwarzian. The rôle of this particular invariant, if any, in the theory of semiconformal mappings is unclear.

What if X and Z both vanish? Our simplest example of a semiconformal mapping is #1. Here,

$$f(x_1, x_2, x_3) = x_1 \implies J \equiv 1$$

but, clearly, all higher invariants vanish. In particular, X and Z are both zero. The same is true of

$$F = \frac{x_1 + x_1{}^2 + x_2{}^2 + x_3{}^2}{1 + 2(x_1 + x_2 + x_3) + 3(x_1{}^2 + x_2{}^2 + x_3{}^2)}.$$

A direct verification is quite tedious. Better to notice that this example is obtained from the case $f = x_1$ by a simple inversion:–

$$F(x_1, x_2, x_3) = f\Big(\frac{x + (1/2\lambda)\|x\|^2 r}{\lambda + \langle r, x\rangle + (1/4\lambda)\|r\|^2\|x\|^2}\Big) \quad \text{for} \quad \begin{array}{l} \lambda = 1 \\ r = (2,2,2). \end{array}$$

Also

$$f = \log\|x\| \implies X = 0, \ Z = 0, \ \mathfrak{S} = \frac{8}{(x_1{}^2 + x_2{}^2 + x_3{}^2)^2}.$$

Remarkably, it is possible to classify the smooth functions f for which X and Z both vanish.

Theorem *Suppose f is a smooth function defined on a connected open subset of $\mathbb{R}^3$ with vanishing X and Z invariants. Then, up to scale and Möbius transformations (that is to say, conformal automorphisms of S^3 viewed in stereographic coördinates $\mathbb{R}^3 \hookrightarrow S^3$), precisely one of the following holds*

(i) $f = \text{constant}$
(ii) $f = x_1 + \text{constant}$
(iii) $f = \log\|x\| + \text{constant}$
(iv) $f = \arctan(x_3/x_2) + \text{constant}.$

Proof. Firstly, notice that f is constant if and only if $J \equiv 0$. Also,

$$f = \arctan(x_3/x_2) \implies X = 0,\ Z = 0,\ \mathfrak{S} = \frac{2}{({x_2}^2 + {x_3}^2)^2}.$$

Therefore, it is clear that cases (i)–(iv) are distinct. Furthermore, by discarding case (i), we may consider f near a point where J is non-vanishing. There we set $V_i \equiv J^{-1} f_i$ and claim that

$$\nabla_i V_j + \nabla_j V_i - \tfrac{2}{3}(\nabla^k V_k)\eta_{ij} = 0. \tag{4}$$

To see this, let us compute

$$\nabla_i V_j = J^{-1} f_{ij} - 2J^{-2} {f_i}^k f_k f_j$$

whence

$$(\nabla^i V^j)\nabla_i V_j = J^{-2} f^{ij} f_{ij} \qquad (\nabla^i V^j)\nabla_j V_i = J^{-2} f^{ij} f_{ij} - 4J^{-3} {f_i}^j f_j f^{ik} f_k + 4J^{-4}(f^{ij} f_i f_j)^2$$
$$\nabla^k V_k = J^{-1} {f^j}_j - 2J^{-2} f^{ij} f_i f_j.$$

Hence, if we denote the tensor on the left hand side of (4) by K_{ij}, then

$$\begin{aligned}\|K\|^2 \equiv K^{ij}K_{ij} &= \left(\nabla^i V^j + \nabla^j V^i - \tfrac{2}{3}(\nabla^l V_l)\eta^{ij}\right)\left(\nabla_i V_j + \nabla_j V_i - \tfrac{2}{3}(\nabla^k V_k)\eta_{ij}\right)\\ &= 2(\nabla^i V^j)\nabla_i V_j + 2(\nabla^i V^j)\nabla_j V_i - \tfrac{4}{3}(\nabla^k V_k)^2\\ &= 4J^{-2} f^{ij} f_{ij} - 8J^{-3} {f_i}^j f_j f^{ik} f_k + 8J^{-4}(f^{ij} f_i f_j)^2 - \tfrac{4}{3}\left(J^{-1} {f^j}_j - 2J^{-2} f^{ij} f_i f_j\right)^2\\ &= J^{-4}\left[4J^2 f^{ij} f_{ij} - 8J {f_i}^j f_j f^{ik} f_k + \tfrac{8}{3}(f^{ij} f_i f_j)^2 - \tfrac{4}{3}J^2({f^j}_j)^2 + \tfrac{16}{3} J f^{ij} f_i f_j {f^k}_k\right]\\ &= J^{-4}\left[\tfrac{8}{3}Z^2 - 4JX\right].\end{aligned}$$

Therefore, if X and Z both vanish then (4) follows. This is a familiar equation known as the conformal Killing equation. In any case, it is easily verified (as in [6]) to be equivalent to the following system

$$\nabla_i V_j = \mu_{ij} + \nu\eta_{ij} \qquad \text{where } \mu_{ij} \text{ is skew}$$
$$\nabla_i \mu_{jk} = \eta_{ij}\rho_k - \eta_{ik}\rho_j \qquad \nabla_i \nu = -\rho_i$$
$$\nabla_i \rho_j = 0,$$

which can be solved explicitly:–

$$\rho_j = -r_j$$
$$\mu_{jk} = r_j x_k - r_k x_j + m_{jk} \qquad \nu = r_j x^j + \lambda$$
$$V_j = r_k x^k x_j - \tfrac{1}{2} x^k x_k r_j + \lambda x_j - m_{jk} x^k - s_j,$$

where r_j, μ_{jk}, λ, and s_j are constant with m_{jk} skew. The slightly peculiar signs are chosen so that the Lie bracket of vector fields $V^i\nabla_i$ coincides with the Lie bracket of the following corresponding matrices

$$M = \begin{pmatrix} \lambda & r_j & 0 \\ s_i & m_{ij} & -r_i \\ 0 & -s_j & -\lambda \end{pmatrix}. \tag{5}$$

Notice that matrices of this form are precisely those such that GM is skew where

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \delta_i{}^j & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and } \delta_i{}^j \text{ is the } 3 \times 3 \text{ identity matrix.}$$

In other words, if we realise the Lie group $\mathrm{O}(4,1)$ as real 5×5 matrices A such that $A^tGA = G$, then the Lie algebra $\mathfrak{o}(4,1)$ is realised as matrices of the form (5). But the group $\mathrm{O}(4,1)$ acts as Möbius transformations on S^3 and so we can normalise the conformal Killing fields on $\mathbb{R}^3$ (i.e. solutions of (4)) by normalising matrices of the form (5) under the Adjoint action of $\mathrm{O}(4,1)$. The similar task of normalising a skew matrix under orthogonal similarity is well-known and one need only be careful to take proper care of null eigenvectors in order to conclude that M can be thrown into one of the following canonical forms.

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{pmatrix} \quad \begin{pmatrix} 0 & \kappa & 0 & 0 & 0 \\ -\kappa & 0 & 0 & 0 & -\kappa \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ 0 & \kappa & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

The first of these gives the conformal Killing field

$$\lambda\Big(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\Big) + \mu\Big(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\Big)$$

but if $V_i = f_i/(f^jf_j)$, then $f_i = V_i/(V^jV_j)$ whence

$$f_i\,dx^i = \frac{\lambda x_1\,dx_1 + (\lambda x_2 - \mu x_3)\,dx_2 + (\lambda x_3 + \mu x_2)\,dx_3}{\lambda^2 x_1{}^2 + (\lambda x_2 - \mu x_3)^2 + (\lambda x_3 + \mu x_2)^2}.$$

However, the exterior derivative of this 1-form is

$$2\lambda\mu x_1\frac{(\lambda x_3 + \mu x_2)\,dx_1 \wedge dx_2 + \lambda x_1\,dx_2 \wedge dx_3 + (\lambda x_2 - \mu x_3)\,dx_3 \wedge dx_1}{(\lambda^2 x_1{}^2 + (\lambda x_2 - \mu x_3)^2 + (\lambda x_3 + \mu x_2)^2)^2},$$

which vanishes if and only if $\lambda = 0$ or $\mu = 0$. Clearly we cannot have $\lambda = \mu = 0$. Hence, if $\lambda = 0$ we can take $f = \frac{1}{\mu}\arctan(x_3/x_2)$ and if $\mu = 0$ we can take $f = \frac{1}{\lambda}\log\|x\|$. Thus, we encounter cases (iii) and (iv). The second canonical form in (6) gives

$$V = \kappa\Big(\big(1 + \tfrac{1}{2}(x_1{}^2 - x_2{}^2 - x_3{}^2)\big)\,\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2} + x_1x_3\frac{\partial}{\partial x_3}\Big) + \mu\Big(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\Big)$$

and similar reasoning leads to $f_i\,dx^i$ being closed if and only if one of κ or μ vanishes. The case $\kappa = 0$ reverts to one already considered. If $\mu = 0$ instead, then

$$f_i\,dx^i = 2\frac{(2+{x_1}^2-{x_2}^2-{x_3}^2)\,dx_1 + 2x_1x_2\,dx_2 + 2x_1x_3\,dx_3}{\kappa((2+{x_1}^2-{x_2}^2-{x_3}^2)^2 + 4{x_1}^2{x_2}^2 + 4{x_1}^2{x_3}^2)}$$

and we can take $f = \frac{1}{\sqrt{2\kappa}}\arctan(y_3/y_2)$, where

$$y_1 = \frac{2\sqrt{2}x_3}{2+2\sqrt{2}x_2+\|x\|^2} \qquad y_2 = \frac{2\sqrt{2}x_1}{2+2\sqrt{2}x_2+\|x\|^2} \qquad y_3 = \frac{\|x\|^2-2}{2+2\sqrt{2}x_2+\|x\|^2}. \tag{7}$$

But (7) is a Möbius transformation so we have arrived at case (iv) again. Finally, let us consider the third canonical form in (6). We obtain

$$V = \frac{\partial}{\partial x_1} + \mu\Big(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\Big)$$

and hence

$$f_i\,dx^i = \frac{dx_1 - \mu x_3\,dx_2 + \mu x_2\,dx_3}{1+\mu^2({x_2}^2+{x_3}^2)}$$

with exterior derivative

$$2\mu\frac{\mu x_2\,dx_1\wedge dx_2 + dx_2\wedge dx_3 - \mu x_3\,dx_3\wedge dx_1}{(1+\mu^2({x_2}^2+{x_3}^2))^2},$$

forcing $\mu = 0$. We encounter case (ii) and our analysis is complete. ☐

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