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ON THE CONNECTIVITY OF SOME COMPLETE INTERSECTIONS

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ABSTRACT. We show that the complement of a degree d hypersurface in a projective complete intersection, whose defining equations have degrees strictly larger than d, has a rational connectivity higher than expected. The key new feature is that a positivity result replaces the usual transversality conditions needed to get such connectivity results.

1. INTRODUCTION

Let V be an n-dimensional complex complete intersection in $\mathbb{P}^N$, defined by the equations $f_1 = f_2 = \cdots = f_k = 0$. Here $N = n + k$ and each polynomial $f_j \in \mathbb{C}[X_0, \cdots, X_N]$ is homogeneous of some degree d_j. We assume that the singular set V_{sing} is finite. Let $H \subset \mathbb{P}^N$ be a reduced hypersurface given by $h = 0$, for a homogeneous polynomial h of degree d. We define the tangency set of V and H to be the set

$$T(V,H) = \{x \in V_{reg} \cap H_{reg} \mid T_xV \subset T_xH \}$$

where $V_{reg} = V \setminus V_{sing}$ is the smooth part of V and H_{reg} has a similar meaning. The main result of this note is the following.

Theorem 1.1. *Assume that $V_{sing} \cap H = \emptyset$ and $d < \min_j d_j$. Then the following hold.*

(i) The tangency set $T(V,H)$ is finite. In particular, if in addition $H_{sing} \cap V$ is a finite set, then the complete intersection $W = V \cap H$ has at most isolated singularities.

(ii) Assume that the Milnor fiber $F_h = \{x \in \mathbb{C}^{N+1} \mid h(x) = 1 \}$ is s-connected, for some $s \leq n-1$. Then the Milnor fiber of the function germ $g : (CV, 0) \to (\mathbb{C}, 0)$, namely the n-dimensional affine complete intersection

$$F = \{x \in \mathbb{C}^{N+1} \mid f_1(x) = f_2(x) = \cdots = f_k(x) = 0,\ h(x) = 1 \}$$

is also s-connected. In particular, if $s = n-1$ e.g. if $\dim H_{sing} < k$, then F is a bouquet of n-dimensional spheres.

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Remark 1.2. Without any assumption on the degrees, if W is smooth, then the second claim above is classical, see Hamm [11] when V is smooth or refer to Looijenga [13], p. 75 for the general case. Similarly, when W has at most isolated singularities, by taking a generic hyperplane section through the origin and using [12], we get that F is $(n-2)$-connected. The whole point of the paper is that the assumption $d < \min_j d_j$ replaces in a misterious way the transversality of H with respect to V and leads to stronger connectivity results than the general ones just described in this Remark. It seems that Theorem 1.1 cannot be obtained by using just the general properties of the perverse sheaf of vanishing cycles as presented for instance in [6].

Example 1.3. Assume that $k = 1$ and $d = 1$, i.e. V is a hypersurface having at most isolated singularities and H is a hyperplane avoiding these singularities. Both claims are then already known, see for instance [2], p. 205 or [1] for the first claim and [3] for the second (this case is in fact just a reformulation of A. Némethi results on quasi-tame polynomials in [14], [15]). More generally, for a smooth complete intersection which is non-degenerate in the sense that $d_j > 1$ for all $j = 1, \cdots, k$ and a hyperplane, the first claim is just Remark 7.5 in [9], see also [1] where the claim (i) is obtained for $k = 1$.

Example 1.4. Assume that V and H are smooth hypersurfaces of the same degree. Then it is easy to see that the singular locus of W can be arbirarily large. Indeed, for

$$f_1 = X_0^2 + X_1^2 + \cdots + X_N^2,\ h = 2X_0^2 + X_1^2 + \cdots + X_N^2$$

W is not even reduced, while for

$$f_1 = X_0^2 + X_1^2 + \cdots + X_N^2,\ h = 2X_0^2 + 2X_1^2 + X_2^2 + \cdots + X_N^2$$

W is reduced, with a codimension one singular set.

Moreover, even when W has at most isolated singularities, this does not imply that F is a bouquet of n-dimensional spheres. The following example was kindly provided by A. Némethi. When $N = 5$, consider the equations

$$f_1 = X_0^2 + X_1^2 + \cdots + X_5^2,\ h = X_2^2 + X_3^2 + 2X_4^2 + 2X_5^2.$$

Then $H_3(F) = \mathbb{Z}^3$ even though $n = 4$ in this case. Of course, F is 2-connected by our Remark 1.2. To obtain the same drop in connectivity in the case $d_1 < d$, it is enough to take $d_1 = 1$ and note that F is in this case just the complement $M = V \setminus W = \mathbb{P}^n \setminus W$ of a projective hypersurface having at most isolated singularities. Examples of such complements with $H_{n-1}(M) \neq 0$ can be easily obtained, see for instance [4], Chapter 6.

Exactly as in the case $k = 1$ treated in [3] we get from Theorem 1.1 the following. Recall the convention $\dim \emptyset = -1$.

Corollary 1.5. *Let X be an n-dimensional complete intersection in $\mathbb{C}^N$ and let V be the projective closure of X in $\mathbb{P}^N$. If H_∞ denotes the hyperplane at infinity $\mathbb{P}^N \setminus \mathbb{C}^N$ and $\delta = \dim(V_{sing} \cap H_\infty)$, then X is $(n - \delta - 2)$-connected.*

The following consequence is also obvious, compare to [4], p.146.

Corollary 1.6. *With the notation and assumptions from Theorem 1.1, there is an s-equivalence $\phi : M := V \setminus W \to K(\mu_d, 1)$, where μ_d is the cyclic group of d-roots of unity. In particular, the reduced rational cohomology of M vanishes in degrees $\leq s$.*

The reader would perhaps benefit by reading the section (5.1) from the paper [7] alongside with this note. In fact Proposition 5.4 in [7] is proved using Theorem 1.1 above.

2. Tangency sets

In this section we prove the first claim in Theorem 1.1. The method is already present in [1], but for the reader's convenience we recall it here. This idea plays also in key role in proving the topological claim in Theorem 1.1 in the next section.

Assume that $\dim T(V, H) > 0$. Then we can find, by taking repeated hyperplane sections if necessary, an irreducible algebraic set C of dimension 1 such that $C \subset T(V, H)$. For $x \in \mathbb{C}^{N+1}$, $x \neq 0$, we denote by $[x]$ the corresponding point in $\mathbb{P}^N$. For $[x] \in C$ we have

$$d_x h \in Span(d_x f_1, \cdots, d_x f_k). \tag{2.1}$$

Moreover, if $[x_n] \in C$ is a sequence of points in C converging to $[y] \in \overline{C}$, then by continuity and using the assumption $V_{sing} \cap H = \emptyset$, we get

$$d_y h \in Span(d_y f_1, \cdots, d_y f_k). \tag{2.2}$$

For $i = 0, \cdots, N$, let $U_i = \{[x] \in \mathbb{P}^N \mid x_i \neq 0 \}$. Let $D = \max_j d_j$ and define the following homogeneous polynomials (all of degree D): $h^i = X_i^{D-d} h(X)$ and $f_j^i = X_i^{D-d_j} f_j(X)$ for $j = 1, \cdots, k$. For $[x] \in \overline{C} \cap U_i$, it follows that

$$d_x h^i = \sum_j \lambda_j^i d_x f_j^i$$

where the complex numbers λ_j^i depend only on the point $[x]$, and not on the chosen representative x. For $[x] \in \overline{C} \cap U_i \cap U_j$, we get

$$\frac{\lambda_a^i}{x_i^{d_a - d}} = \frac{\lambda_a^j}{x_j^{d_a - d}} \tag{2.3}$$

for all $a = 1, \cdots, k$. Therefore, the collection of regular functions $(\lambda_a^i)_i$ defines a regular section λ_a of the line bundle $\mathcal{O}(d - d_a)$ over the irreducible projective curve $\overline{C}$. Since $d - d_a < 0$, it follows that this section is identically zero on $\overline{C}$. Applying this to all sections λ_a, we get that $C \subset H_{sing}$, a contradiction. Indeed, by definition $C \subset T(V,H)$ and $T(V,H) \cap H_{sing} = \emptyset$.

3. Homotopy types of Milnor fibers

In this section we prove the second claim in Theorem 1.1. The proof is divided in several steps, so that the reader may easily follow the argument.

3.1. **Step 1: finding good equations for V.** Recall the Bertini Theorem, saying that the general member of a linear system is smooth outside the base locus, see for instance [10], p. 137. Assume that the degrees d_j of the equations defining V are ordered such that $d_1 \leq d_2 \leq \cdots \leq d_k$. Then using Bertini Theorem, we can replace each equation f_j by a linear combination

$$f'_j = f_j + \sum_{i=1,j-1} a_i f_i$$

where a_i is a generic homogeneous polynomial of degree $d_j - d_i$, such that if we define

$$Z_j = V(f'_j, f'_{j+1}, ..., f'_k)$$

then Z_j is a complete intersection of dimension $(n+j-1)$ whose singular locus is contained in the finite set V_{sing}, for all $j = 1, 2, \cdots, k$. To keep the notation simple, we assume in the sequel that the original polynomials $f_1, ..., f_k$ already satisfy this property.

3.2. **Step 2: non-proper Morse Theory and plan of proof.** The main technical tool in proving our claim is the following result of Hamm, see Proposition 3 in [12] for a more general version, with our addition concerning the condition $(c0)$ in Lemma 3 and Example 2 in [5]. See also Proposition 11 in [8].

Proposition 3.3. *Let A be a locally complete intersection in $\mathbb{C}^p$ with $\dim A = e$. Let $g_1, ..., g_p$ be polynomials in $\mathbb{C}[X_1, ..., X_p]$ such that the singular locus of A is contained in $A_1 = \{z \in A \mid g_1(z) = 0\}$. For $1 \leq j \leq p$, denote by Σ_j the set of critical points of the mapping $(g_1, ..., g_j) : A \setminus A_1 \to \mathbb{C}^j$ and let $\overline{\Sigma}_j$ denote the closure of Σ_j in A. Assume that the following conditions hold.*

1. *(c0) The set $\{z \in A \mid |g_1(z)| \leq a_1, ..., |g_p(z)| \leq a_p\}$ is compact for any positive real numbers a_j, $j = 1, ..., p$.*
2. *(c1) The critical set Σ_1 is finite.*
3. *(cm) (for $m = 2, ..., p$) The map $(g_1, ..., g_{m-1}) : \overline{\Sigma}_m \to \mathbb{C}^{m-1}$ is proper.*

Then A has the homotopy type of a space obtained from A_1 by attaching e-dimensional cells.

The main difference of the use of this result here versus its uses in [5] and [8] is that the former transversality (or genericity) assumptions are replaced with the assumptions in Theorem 1.1.

Our plan of proving Theorem 1.1 is the following. Let

$$F_j = \{x \in \mathbb{C}^{N+1} \mid f_j(x) = f_{j+1}(x) = \cdots = f_k(x) = 0,\ h(x) = 1\ \}$$

with the convention $F_{k+1} = F_h$, the Milnor fiber of h. In the next step we show that each of the pairs (F_{j+1}, F_j) are in the situation of the pair (A, A_1) in the above Proposition. It follows that F_{j+1} has the homotopy type of a space obtained from F_j by attaching $(n+j)$-cells. Since $s \leq n-1$, it follows that the s-connectivity of F_{j+1} implies the s-connectivity of F_j, e.g. by using the homotopy exact sequence of the pair (F_{j+1}, F_j). This proves the second claim in Theorem 1.1 modulo the following.

3.4. **Step 3: the study of the pairs (F_{j+1}, F_j).** As explained above, in this step we take $A = F_{j+1} = \{x \in \mathbb{C}^{N+1} \mid f_{j+1}(x) = \cdots = f_k(x) = 0,\ h(x) = 1\ \}$, $g_1 = f_j$ and show that we can find generic linear forms $g_2, ..., g_p$ with $p = N+1$ satisfying all the assumptions in Proposition 3.3. More precisely, let $\mathcal{S}_1$ be a Whitney regular stratification of the complete intersection $W_1 = Z_j \cap H$, having as open stratum the regular part $S_0 = W_{1,reg}$ of W_1. We choose g_2 such that the hyperplane $H_2 : g_2 = 0$ is transverse to any stratum of the stratification $\mathcal{S}_1$ and $\dim(V(g_1) \cap H_2) = N-2$. Then there is an induced Whitney regular stratification $\mathcal{S}_2$ of $W_2 = W_1 \cap H_2$, whose strata are the non-empty intersections of the strata in $\mathcal{S}_1$ with the hyperplane H_2. Choose g_3 such that the hyperplane $H_3 : g_3 = 0$ is transverse to any stratum of the stratification $\mathcal{S}_2$ and $\dim(V(g_1) \cap H_2 \cap H_3) = N-3$. Continue in the same way to define $g_4, ..., g_p$.

The condition (c0) in Proposition 3.3 is clearly satisfied, since $\dim(V(g_1) \cap H_2 \cap \ldots \cap H_p) = N - p = -1$ implies that the mapping $(g_1, g_2, ..., g_p) : \mathbb{C}^p \to \mathbb{C}^p$ is finite, in particular it is proper.

Assume that the condition (c1) fails. Then, exactly as in the proof in the previous section, we may find an irreducible curve $C \subset \Sigma_1$ and $g_1|C$ is constant, say equal to $c \in \mathbb{C}^*$. Note that $x \in \Sigma_1$ implies that there is a relation of the following type

$$d_x g_1 = \lambda d_x h + \sum_{i=j+1,k} \lambda_i d_x f_i.$$

Since $x \in A \setminus A_1$, we have $g_1(x) \neq 0$ and the Euler relation shows that $\lambda \neq 0$. Therefore the above relation can be re-written as

$$d_x h = \sum_{i=j,k} \lambda_i d_x f_i$$

with $\lambda_j \neq 0$.

Let $[C] = \{[x] \in \mathbb{P}^n \mid x \in C\}$ and assume that $[x_n]$ is a sequence of points in $[C]$ converging to a point $[y] \in \overline{[C]}$. Since $x_n \in C$, it follows that $h(x_n) = 1$, and hence the sequence x_n is bounded away from the origin. Two cases are possible. The first case, when the sequence of norms $|x_n|$ is bounded, leads to a convergent subsequence x_{n_m} converging to a point $y \in \overline{C} \subset A$. Using the Euler relation we get $d = \lambda_j(y) \cdot d_j \cdot c$, i.e. λ_j is regular and non-zero at $[y]$.

The second case, when the sequence of norms $|x_n|$ is unbounded, leads to a subsequence $y_m = x_{n_m}$ such that $|y_m| \to +\infty$ and the sequence $z_m = \frac{y_m}{|y_m|}$ converges to some point z. It follows that $g_1(z) = \lim f_j(y_m) \cdot |y_m|^{-d_j} = \lim c \cdot |y_m|^{-d_j} = 0$ and similarly $h(z) = 0$. It follows that $[z] \notin Z_{j,sing}$. In particular $d_z f_j \neq 0$, and hence λ_j does not have a pole at $[z] = [y]$.

In conclusion, we get exactly as in the previous section, a regular section λ_j without any zeroes of the line budle $\mathcal{O}(d - d_j)$ over the irreducible projective curve $\overline{[C]}$, which is a contradiction.

Finally, assume that $m \geq 2$ and that the condition $(c(m-1))$ holds but the condition (cm) fails, i.e. the map

$$(g_1, \ldots, g_{m-1}) : \overline{\Sigma}_m \to \mathbb{C}^{m-1}$$

is not proper. This means that there is a sequence of points $p_n \in \overline{\Sigma}_m$ such that $|p_n| \to +\infty$ and $g_a(p_n) \to c_a$ for $1 \leq a \leq m-1$. Since Σ_m is dense in $\overline{\Sigma}_m$, we may assume that $p_n \in \Sigma_m$. Moreover Σ_{m-1} is clearly contained in Σ_m, and $p_n \in \Sigma_m \setminus \Sigma_{m-1}$ (at least for n large enough), since the condition $(c(m-1))$ holds. It follows that the differentials

$$d_{p_n} f_{j+1}, \ldots, d_{p_n} f_k, d_{p_n} h, d_{p_n} g_1, \ldots, d_{p_n} g_{m-1}$$

are linearly independent and $d_{p_n} g_m$ is a linear combination of them. Let $q_n = \frac{p_n}{|p_n|}$ and assume that this sequence converges to some point q. Then exactly as above we get $[q] \in W_1 \cap H_2 \cap \ldots \cap H_{m-1}$, $[p_n] = [q_n] \in W_{1,reg} \cap H_2 \cap \ldots \cap H_{m-1}$ and $\lim [q_n] = [q]$. It follows, using once again the Euler relation, that $[q] \in W_1 \cap H_2 \cap \ldots \cap H_{m-1} \cap H_m$ and the hyperplane H_m contains the limit of tangent spaces $T = \lim T_{[q_n]}(W_{1,reg} \cap H_2 \cap \ldots \cap H_{m-1})$. By the Whitney (a)-regularity condition, this limit T contains the tangent space T' at $[q]$ to the stratum in the stratification $\mathcal{S}_{m-1}$ of $W_1 \cap H_2 \cap \ldots \cap H_{m-1}$ that contains the point $[q]$. Therefore $T' \subset H_m$, which contradicts the fact that H_m is transverse to the stratification $\mathcal{S}_{m-1}$. This completes the verification of the assumptions in Proposition 3.3 and hence the proof of Theorem 1.1.

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