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ON INJECTIVITY OF TAME MAPPINGS

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In this short note, we give a criterion for the injectivity of tame mappings. This was part of the talk given by the third named author at the second Australian-Japanese meeting on real and complex singularities, held in Kyoto in November 2007. For a more comprehensive study and also for a list of relevant articles on this topic, we send the reader to our paper [1].

Let U be a convex open set of $\mathbf{R}^m$ and $f : U \to \mathbf{R}^n$ a tame map. Let B denote the set where f is not C^1, and let $\widehat{B}$ denote the set where f is not a C^1-immersion. For a subset V of U we put

$$\mathcal{C}(f,V) = \text{the convex hull of } \{df(x) : x \in V - B\}.$$

We denote its closure by $\overline{\mathcal{C}}(f,V)$, and define $\mathcal{CC}(f,V)$ by

$$\mathcal{CC}(f,V) = \text{the cone of } \mathcal{C}(f,V) \text{ with vertex } 0.$$

We denote its closure by $\overline{\mathcal{CC}}(f,V)$. We will also use the following notations, $\mathcal{C}(f) = \mathcal{C}(f,U)$, $\mathcal{CC}(f) = \mathcal{CC}(f,U)$, etc., for shortness.

Take $x, x' \in U$, $x \neq x'$, and put $v = \frac{x'-x}{|x'-x|}$. We denote the segment connecting x and x' by $[x,x']$. If f is tame, then we have the following well known facts,

- $[x,x'] \cap \widehat{B}$ is a finite set, or
- a sub-segment of $[x,x']$ is subset of $\widehat{B}$.

Setting $g(t) = f(t + tv)$, we have

$$g'(t) = df(x+tv)v, \qquad \text{whenever } x + tv \notin B.$$

Let Σ denote the set of singular matrices and Σ_v denote the set of singular matrices annihilating v.

Theorem. A tame map $f : U \to \mathbf{R}^n$ is injective, if the following conditions hold:

- $\mathcal{C}(f, U - \widehat{B})$ does not contain singular matrices.
- If $\widehat{B}$ contains a segment with direction v, then f is not constant on this segment, and Σ_v is an extremal set of $\overline{\mathcal{CC}}(f)$.

It is not hard to see that the theorem follows from Lemmas 1 and 3 below.

Lemma 1. Assume that $[x,x'] \cap \widehat{B}$ is a finite set. If $\mathcal{C}(f, U - \widehat{B})$ does not contain singular matrices, then $f(x) \neq f(x')$.

Proof. By supposition, the set $\widehat{C} = \{t \in [0,1] : x + tv \in \widehat{B}\}$ is finite. If $\mathcal{C}(f, U - \widehat{B}) \cap \Sigma = \emptyset$, then $\mathcal{C}(f, [x,x'] - \widehat{B}) \cap \Sigma_v = \emptyset$. Then we obtain that

$$\mathcal{C}(g, [0,1] - \widehat{C}) = \mathcal{C}(f, [x,x'] - \widehat{B}) \cdot v \not\ni 0.$$

We employ the following lemma to complete the proof. □

Lemma 2. Let $g : [a, b] \to \mathbf{R}^n$ be a tame map and let $\widehat{C}$ be a finite subset of $[a, b]$. If $\mathcal{C}(g, [a, b] - \widehat{C})$ does not contain 0, then $g(a) \neq g(b)$.
Proof. The proof is similar to the proof of Lemma 5.3, in [1], and we omit it. □
Lemma 3. Assume that $[x, x'] \cap \widehat{B}$ contains a sub-segment with direction v. If Σ_v is an extremal set of the closure of $\mathcal{CC}(f)$, then $f(x) \neq f(x')$, or $f|_{[x,x']}$ is constant.
Proof. Since f is tame, we may choose a vector u such that

$$C_s = \{t \in [0,1] : x + tv + su \in B\}$$

are finite sets when $0 < s < \varepsilon$. Set $g_s(t) = f(x + tv + su)$. We remark that

$$g_s'(t) \in \mathcal{C}(g_s, [0,1] - C_s) = \mathcal{C}(f, [x + su, x' + su] - B) \cdot v \subset \mathcal{C}(f) \cdot v.$$

We assume that $g_0(t)$ is not constant. This means that $g_0'(t)$ is not zero on some subinterval of $[0,1]$. We then obtain that $g_s(t)$ is not constant for sufficiently small $s > 0$, and $\langle w, g_s'(t)\rangle > 0$ for some $t \in [0,1]$. We thus have

$$\langle w, g_s(1)\rangle - \langle w, g_s(0)\rangle = \int_0^1 \langle w, g_s'(t)\rangle dt > 0.$$

When $s \to 0$, we obtain

$$\langle w, g_0(1)\rangle - \langle w, g_0(0)\rangle = \int_0^1 \langle w, g_0'(t)\rangle dt \geq 0.$$

Assuming that the equality holds, we will have that $\langle w, g_0'(t)\rangle = 0$ for almost all t. We will conclude that $g_0'(t) = 0$, which is a contradiction. By Lemma 4.3 in [1], we have

$$(*) \qquad g_0'(t) = \lim_{s \to 0} g_s'(t) = \lim_{s \to 0} df(x + tv + su) \cdot v$$

and this is in the closure of $\mathcal{C}(f) \cdot v$. Remark that $df(x + tv + su)$ goes to infinity, even though the right hand side of $(*)$ stays in a compact set.

We now remark that the closure of $\mathcal{C}(f) \cdot v$ is the image of the closure of the cone of $\mathcal{C}(f)$ by the map defined by $A \mapsto Av$, that is,

$$\overline{\mathcal{C}(f) \cdot v} = \overline{\mathcal{CC}}(f) \cdot v.$$

Since Σ_v is an extremal set of $\overline{\mathcal{CC}}(f)$, 0 is an extremal point of $\overline{\mathcal{C}(f) \cdot v}$. This means that $\langle w, z\rangle \geq 0$ for any $z \in \overline{\mathcal{C}}(f) \cdot v$ and the equality holds only if $z = 0$. Since $\langle w, g_0'(t)\rangle = 0$ for almost all t, we conclude that $g_0'(t) = 0$ for almost all t. □
Remark. If f is locally Lipschitz, one can replace the assumption in Lemma 3 by the following condition:
Σ_v is an extremal set of the closure of $\mathcal{CC}(f)$ (since $\mathcal{C}(f)$ is a bounded set).

References

[1] T. Fukui, K. Kurdyka and L. Paunescu, Tame nonsmooth inverse mapping theorems, http://front.math.ucdavis.edu/0712.2476.

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